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Fixed Point Theorem in Fuzzy Metric Space

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In this present work, we prove fixed point theorem for contractive mapping $F : X \to X$ in fuzzy metric spaces that have a nonempty $F-$ invariant complete subspace E, then prove the uniqueness the fixed point in E. Though many theorems in fuzzy metric space in this case, our theorem is a new type of these theorems as it is we proved that a unique fixed point there exists in $F-$ invariant complete subset E in X. Finally, we give an interesting example in complete fuzzy metric space that satisfies in the conditions of our theorem and prove the uniqueness fixed point of F.

1. Introduction

The concept of fuzzy sets was introduced initially by Zadeh [13] in 1965. To use this concept in topology and analysis, many authors have extensively developed the theory of fuzzy sets and applications. George and Veeramani [3] modified the concept of fuzzy metric space introduced by Kramosil and Michalk [7]. Vasuki [12] obtained the fuzzy version of common fixed- point theorems which had extra condition, in fact, he proved a fuzzy common fixed- point theorem by a strong definition of the Cauchy sequence. Bhaskar and Lakshmikantham [2] introduced the notion of a coupled fixed point of a mapping of two variables. Later some authors obtained some results on coupled fixed-point theorems in metric and cone metric spaces (see, e.g., [4, 5]). Recently in [9] several fixed- point theorems in the fuzzy *b*-metric spaces have been studied and sufficient condition for a sequence to be Cauchy in the fuzzy *b*-metric space has been given. Ansari et al. [1], using fixed point methods, proved the fuzzy orthogonally $*-n-$ derivation on orthogonally fuzzy C^* -algebra for the functional equation. In [8] authors introduced the interval-valued intuitionistic fuzzy set and proved some frank aggregation operators based on the interval-valued intuitionistic fuzzy numbers. In this work, first, we review the basic required definitions and corollary of fuzzy metric spaces and then we prove our main results.

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2. Preliminaries

Definition 1 [11]**.** A binary operation $*:[0,1]^2 \to [0,1]$ is called a continuous t −norm if $([0,1],*)$ is an abelian topological monoid ; i.e.,

(1) $*$ is associative and commutative,
(2) $*$ is continuous,
(3) $a*1=a$ for all $a\in[0,1]$,
(4) $a*b\le c*d$ whenever $a\le c$ and $b\le d$, for each $a, b, c, d\in[0,1]$.

Two typica example of continuous t − norms are $a*1b = ab,\ a*2b = max\{a+b-1,0\}$.

Definition 2. The 3-typle $(X,M,*)$ is called a fuzzy metric space if X is an arbitrary non-empty set, * is a continuous t-norm and M is a fuzzy set on $X^2\times[0,\infty)$ satisfying the following condition [3],
for each $x,y,z\in X$ and $t, s>0$
$(FM-1)$ $M(x,y,t)>0,$
$(FM-2)$ $M(x,y,t)=1$ if and only if $x=y$,
$(FM-3)$ $M(x,y,t)=M(y,x,t)$,
$(FM-4)$ $M(x,y,t)*M(y,z,s)\le M(x,z,t+s)$,
$(FM-5)$ $M(x,y,.):(0,\infty)\to[0,1]$ is continuous.

Example 1. Let (X,d) be an ordinary metric space and ψ be an increasing and continuous function from R^+ into $(0, 1)$ such that $lim_{t\to\infty}\psi(t)=1$. For example $\psi(t)=e^{(-1/x)}$ [12] .
Let $a*b\le ab$ for all $a, b\in[0,1]$ and for each $t\in(0,\infty)$, define $M(x,y,t)=[\psi(t)]^{d(x,y)}$ for all $x,y\in X$. Then $(X,M,*)$ is a fuzzy metric space.

Definition 3. Let $(X,M,*)$ be a fuzzy metric space [3]. Then,

(i) A sequence x_n in X is said to be convergent to a point $x\in X$ if $lim_{n\to\infty}M(x_n,x,t)=1$ for all $t>0$.
(ii) A sequence x_n in X is called the Cauchy sequence if for each $0<\varepsilon<1$ and $t>0$, there exists $n_0\in N$ such that $M(x_n,x_m,t)>1-\varepsilon$ for each $n,m>n_0$.
(iii) A fuzzy metric space in which every Cauchy sequence is convergent is said to be complete.

Definition 4. Let $(X,M,*)$ be a fuzzy metric space. A sequence x_n in X is said to be Cauchy if $lim_{n\to\infty}M(x_{n+p},x_n,t)=1$ for all $t>0$ and $p>0$ [10].

Lemma 1. For all $x,y\in X$, $M(x,y,.)$ is a non-decreasing function [6].

Definition 5. Let $(X,M,*)$ be a fuzzy metric space. M is said to be continuous on sequence on $X^2\times(0,\infty)$, if $lim_{n\to\infty}M(x_n,x,t)=M(x,y,t)$ whenever (x_n,y_n,t_n) is a sequence in $X^2\times(0,\infty)$ which converges to a point $(x,y,t)\in X^2\times(0,\infty)$; i.e., $lim_{n\to\infty}M(x_n,x,t)=lim_{n\to\infty}M(y_n,y,t)=1$ and $lim_{n\to\infty}M(x,y,t_n)=M(x,y,t)$ [12].

Lemma 2. M is a continuous function on $X^2\times(0,\infty)$ [6].

Definition 6. Let $(X,M,*)$ be a fuzzy metric space. M is said to satisfies the n - property on $X^2\times(0,\infty)$ if $lim_{n\to\infty}[M(x,y,k^n t)]^{(np)}=1$ where $x,y\in X, k>0, p>0$ [12].

Example 2. Let (X,d) be an ordinary metric space, $a*b=ab$ for all $a,b\in[0,1]$ and $M(x,y,t)=e^{-d(x,y)/t}$ for every $x,y\in X$ and $t>0$ [12]. Then we have

$$lim_{n\to\infty} [M(x,y,k^n t)]^{np} = lim_{n\to\infty} (e^{-d(x,y)/k^n})^{np} lim_{n\to\infty} e^{-d(x,y)np/k^n} = 1.$$

Therefore, M is satisfied the n-property on $X^2 \times (0,\infty)$.

Lemma 3. Let $(X,M,*)$ be a fuzzy metric space, $a*b \geq ab$ for all $a, b \in [0,1]$ and M satisfies $n-$property. suppose $\{x_n\}$ is a sequence in X such that for all $n \in N$, $M(x_n, x_{n+1}, kt) \geq M(x_{n-1}, x_n, t)$ for every $0 < k < 1$, then the sequence $\{x_n\}$ is a Cauchy sequence [12].

Definition 7. [2] An element $(x,y) \in X \times X$ is called a coupled fixed point of the mapping $F : X \times X \to X$ if $F(x,y) = x, F(y,x) = y$ [2].

Theorem 1. (Fuzzy Banach Contraction theorem [12]). Let $(X,M,*)$ be a complete fuzzy metric space such that $lim_{t\to\infty} M(x,y,t) = 1;$ for all $x,y \in X.$ and let $T : X \to X$ be a mapping satisfying $M(T(x),T(y),kt) \geq M(x,y,t)$ for all $x,y \in X$ where $0 < k < 1$. Then T has a unique fixed point.

Corollary 1. Let $a*b \geq ab$ for all $a, b \in [0,1]$ and let $(X,M,*)$ be a complete fuzzy metric space such that M has n -property. Assume there is function $F : X \times X \to X$ such that $M(F(x,y),F(u,v),kt) \geq M(x,u,t) * M(y,v,t)$ for all $x,y,u,v \in X$, where $0 < k < 1$. Then there exists a unique $x \in X$ such that $x = F(x,x)$ [11].

3. Main results

We give the theorem and example for fixed point in fuzzy metric space. Though there are thousands fixed point theorems in fuzzy metric space, our theorem is a new type of theorem because we prove unique fixed point is in $F-$ invariant complete subspace of fuzzy metric space $(X,M,*)$.

Theorem 2. Let $(X,M,*)$ be a fuzzy metric space and $F : X \to X$ be a mapping. Assume that there exists $k \in [0,1)$ and a nonempty $F-$ invariant complete subset E of X such that, for $t > 0$; and $lim_{t\to\infty} M(x,y,t) = 1$ for all $x,y \in E$ where $0 < k < 1$ .and let $M(F(u),F(v),t) \geq M(u,v,t)$ for all $u,v \in X, u \neq v,$ Then, there exists a unique $x \in X$ such that $x = f(x)$.

Proof. Since E is a nonempty F $-$invariant complete subset of X, so $F|E : E \to E$ is a mapping and $(E,M,*)$ is a complete fuzzy metric space. Now, from the fuzzy Banach Contraction theorem (Theorem 1), there exists a unique fixed point in E. Let $u \in X - E$ be another fixed point for F. Then $F(u) = u$. Also $u \neq x$ then we have

$$M(u,x,t) = M(F(u),F(x),t) > M(u,x,t)$$

This is a contradiction. Thus, all points except x are fixed point of F .☐

In this example we show that F has a unique fixed point and for all $\{F^n(x)\}$ does not necessarily converge to the fixed point.

Example 3. Put $X = \{0\} \cup \{1 + 1/n; n \in N\}$ and define a metric d on X by $d(x,y) = |x| + |y|$, $\forall x, y \in X$ with $y \neq x$. Denote $a*b = ab$ for all $a,b \in [0,1]$. Then $(X,M,*)$ is a complete fuzzy metric space, where $M(x,y,t) = e^{-d(x,y)/t}$ for all $x,y \in X$. Define a mapping F on X by

$$f(x) = \begin{cases} 0 & if\, x = 0 \\ 1 + \dfrac{1}{n+1} & if\, x = 1 + \dfrac{1}{n} \end{cases}$$

Then F satisfies the assumption in Theorem 1, however $\{F^n(2)\}$ does not converge.

Proof. From Example 1 it is obvious that $(X,M,*)$ is a complete fuzzy metric space and 0 is a unique fixed point of F. Put $E=\{0\}$. Now, if $x,y\in E$ then $x=y=0$. so, $M(F(x),F(y),kt)=M(x,y,t)=1$ and $lim_{t\to\infty}M(x,y,t)=\lim_{t\to\infty}M(0,0,t)=1$.

Let $x,y\in X$, $y\neq x$, then one of these states occur:

(i) $\exists n\in N$ such that $y=1+1/n$, $x=0$.

(ii) $\exists n\in N$ such that $x=1+1/n$, $y=0$.

(iii) $\exists m,n\in N$ such that m ¹ n, $x=1+\dfrac{1}{n}$, $y=1+\dfrac{1}{m}$.

If (*i*) is satisfied then,

$$M(F(x),F(y),t)=e^{\frac{-\left(1+\frac{1}{n+1}\right)}{t}}=e^{\frac{-1}{t}}.e^{\frac{-1}{t(n+1)}}>e^{\frac{-1}{t}}.e^{\frac{-1}{tn}}=e^{\frac{-\left(1+\frac{1}{n}\right)}{t}}=M(x,y,t).$$

Thus, we have $M(F(x),F(y),t)>M(x,y,t)$.

Similarly, if (*ii*) is satisfied, we have $M(F(x),F(y),t)>M(x,y,t)$.

Finally, if (*iii*) is satisfied then,

$$M(F(x),F(y))=e^{\frac{-\left(1+\frac{1}{n+1}+1+\frac{1}{m+1}\right)}{t}}=e^{\frac{-\left(1+\frac{1}{n+1}\right)}{t}}.e^{\frac{-\left(1+\frac{1}{m+1}\right)}{t}}>e^{\frac{-\left(1+\frac{1}{n}\right)}{t}}.e^{\frac{-\left(1+\frac{1}{m}\right)}{t}}=e^{\frac{-\left(1+\frac{1}{n}+1+\frac{1}{m}\right)}{t}}=M(x,y,t).$$

From above, F satisfies the assumption in Theorem T. Since $F^n(2)=1+\dfrac{1}{n+1}$, thus, $F^n(2)\to 1$.s $n\to\infty$ but $1\notin X$, i.e., $\{F^n(2)\}$ dose not converge in X .☐

4. Conclusion

In this study, we presented the fixed- point theorem for contractive mapping $F:X\to X$ in fuzzy metric spaces that have a nonempty $F-$invariant complete subspace E. Also, we proved the uniqueness the fixed point in E. The results of our work are very interesting and results are given in an example showing the mapping F has the unique fixed point in $F-$ invariant complete subset E in X.

In future, we will introduce generalized forms of fuzzy set and will prove some coupled fixed point and coincidence point theorems for contractive mapping in new fuzzy metric spaces and will mention its results and applications.

Conflict of interest: The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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