FUZZY OPTIMIZATION AND MODELLING 2(4) (2021) 52-64
E-ISNN: 2676-7007
Contents lists available at FOMJ
Fuzzy Optimization and Modelling
Journal homepage: http://fomj.qaemiau.ac.ir/

Paper Type: Research Paper

Stochastic Sensitivity Analysis in Data Envelopment Analysis

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A R T I C L E I N F O

Article history:
Received 29 November 2021
Revised 25 December 2021
Accepted 26 December 2021
Available online 26 December2021

Keywords:
Data envelopment analysis
Efficiency sensitivity analysis
Stability region
Stochastic data

A B S T R A C T

Data Envelopment Analysis (DEA) is an impeccable approach based on mathematical programming for the efficiency measurement of homogeneous Decision-Making Units (DMUs). One of the topics of interest in data envelopment analysis (DEA) is the sensitivity and stability analysis of a specific DMU that determines ranges within which all data may be altered for any DMU before a reclassification from efficient to inefficient status (or vice versa) happens. In many real-world applications, the managers to estimate the under supervision DMUs encounter stochastic data and require a way to deal with the sensitivity analysis of DMUs with this special data. In DEA, efficient DMUs are of primary importance as they define the efficient frontier. The intent of this paper is to present the sensitivity analysis with stochastic data for efficient DMUs when inputs and outputs are stochastic and variations in the data are simultaneously considered for all DMUs. The models explained in this paper for treating sensitivity analysis in DEA are expanded by according them chance-constrained programming formulations. The ordinary route used in chance-constrained programming is followed here by replacing these stochastic models with their deterministic equivalents. The optimal solution of these models leads to allowable input/ output variations.

1. Introduction

Analyzing and managing the efficiency of entities is the main responsibility of the top-level management team that can be carried out by different methods. Data envelopment analysis (DEA) introduced by Charnes et al. [10] is identified as a successful tool in evaluating the relative performance of entities and organizations, and based on various production process assumptions, several different models have been developed [39, 41, 45, 55, 57, 60]. Since DEA is data-based, it is significant to assess possible input/output changes (data perturbation) of a DMU such that its obtained efficiency classification stays fixed. In the context of DEA, sensitivity analysis has been one of the remarkable issues which express to what extent perturbations in the input/output data are tolerable before changing DEA efficiency. Numerous studies have addressed to this topic,

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DOI: 0.30495/FOMJ.2021.1946159.1049

see, for example, Ahn and Seiford [1], Smith [51], Sexton et al. [50], Charnes and Neralic [12], Seiford and Zhu [48, 49], Charnes et al. [11, 13], Thompson et al. [53, 54], Jahanshahloo et al. [26], Neralić and Wendell [38], Hladík [23], Khalili-Damghani and Taghavifard [29] and He et al. [22] among others.

Charnes et al. [8] discussed analytical methods of DEA sensitivity analysis. They presented an algorithmic approach by updating the inverse of the optimal basis matrix and obtained the stability radius in the direction of one output for preserving the efficiency of efficient DMU. They also noted that standard methods of linear programming sensitivity analysis are not being used in DEA. Neralić [37] developed sufficient conditions for the preservation of efficiency classification of all DMUs in the additive model of DEA using an approximate inverse of the perturbed optimal basis matrix.

Lotfi and Jahanbakhsh [33] examined the efficiency and effectiveness simultaneously in a three-stage process using a unified model. Zhou et al. [58] estimated environmental performance which is the basis for analyzing environment policy and decision making. Kang et al. [28] evaluated the efficiency of the emergency department. Tavakoli et al. [52] assessed organizational performance concerning human capital management by fuzzy DEA. Mostafaee and Soleimani-Damaneh [35] studied the anchor points in DEA and the main results of their paper led to a new relationship between DEA and sensitivity analysis in linear programming. Hosseinzadeh et al. [24] reviewed ranking articles in DEA, which categorized ranking methods into seven groups. Liu and Wang [23] studied the sensitivity analysis of profit based on system dynamics. Lotfi et al. [34] estimated return-to-scale sensitivity analysis in supply chain management. Emrouznejad and Yang [20] surveyed scientific studies during the first 40 years of DEA (1978-2018). Charnes and Neralić [12] introduced sensitivity analysis of the additive model for an efficient unit and provided sufficient conditions for simultaneous variations of all inputs and all outputs such that its obtained efficiency classification does not change.

Jahanshahloo et al. [26] considered DEA interval models to define the stability radius of each unit in the presence of interval data in such a way that the efficiency classification remains unchanged. Thompson et al. [53, 54] used the strong complementary slackness condition (SCSC) to analyze the stability of the CCR model in a situation where the data for all efficient DMUs and all inefficient DMUs simultaneously changed in opposite directions but to the same ratio. Charnes et al. [11] and Charnes et al. [13] used the super-efficiency model for sensitivity analysis of each DMU. In this approach, they considered a deteriorating scenario for efficient DMUs and an improving scenario for inefficient DMUs. Zhu [59] amended the work of Charnes et al. [13] to identify permissible variations in every input and output for each DMU before an alteration occurs in status for the DMU under evaluation. Jahanshahloo et al. [25] suggested a new approach to sensitivity analysis of a DMU under test. They extend a stability region for DMU_o by using the supporting hyperplanes which pass through DMU_o and the new frontier which is constructed by eliminating DMU_o from the observations set.

Boljunčić [7] employed an iterative procedure. He achieved possible input/output changes by using the optimal simplex tableau and applying parametric programming (input/output changes as parameters). Mozaffari et al. [36] provided a method for sensitivity and stability analysis of all DMUs with interval data using the MOLP approach. Jahanshahloo et al. [27] examined the sensitivity analysis of the inefficient DMUs, their technique yielded an exact necessary change region in which the efficiency score of a specific inefficient DMU changes to a defined efficiency score. Daneshvar et al. [16] and Ghazi et al. [21] developed the stability region by supporting hyperplanes of the PPS. Dar et al. [17] studied the sensitivity of performance classification and the returns to scale (CRS, IRS, and DRS) of DMUs based on input and output slacks. Of late, Khoveyni and Eslami [31] investigated the internal structures of DMUs to detect their efficiency stability regions. Their proposed method finds the stability regions of an extreme network-efficient two-stage production process when its inputs increase, its intermediate products and final outputs decrease, and the data of the other two-stage production processes remain fixed. Also, Arabjazi et al. [3] expanded the largest performance stability region for an extreme efficient DMU whose data can be changed in all directions of input/output space, including both directions of improving the situation and worsening the situation such that under these changes the efficiency classification of all extreme DMUs will be preserved. Moreover, they found the largest symmetric cell to the center of the extremely efficient DMU under evaluation, leading to an efficiency stability radius.

In recent years, studies on performance sensitivity analysis have been implemented with the presence of special data. Sanei et al. [46], Wen et al. [56], Khalili-Damghani and Taghavifard [29] investigated the sensitivity analysis with fuzzy data. Banihashemi et al. [4] obtained the stability region of efficient and inefficient units with integer data. Khodabakhshi et al. [30] extended sensitivity analysis of the super-efficiency of DMUs based on input relaxation super-efficiency measure. He et al. [22] determined the stability radius with bounded uncertainty.

The methodologies discussed above developed sensitivity analysis methods for the situation in which data variations are applied to only the specific DMU that is being evaluated, and the data for the remaining DMUs are assumed to be fixed. This assumption may not be appropriate because data variations may exist in each input and output of all of the DMUs. Also in many real-world applications, the managers are faced with stochastic data and they require evaluating input/output changes in the presence of this particular data. In DEA, efficient DMUs are of primary importance as they define the efficient frontier. Therefore, in this paper, we present the sensitivity analysis of efficiency for efficient DMUs when inputs and outputs are stochastic and variations in the data are considered to not only all of the DMUs but also the input and output subsets of interest.

The rest of this study is organized as follows. In Section 2, some basic concepts about DEA models and the sensitivity analysis method proposed by Seiford and Zhu [49] will be introduced. In Section 3, we will present sensitivity analysis with stochastic data. Finally, the conclusions and some suggestions for future research are given in Section 4.

2. Preliminaries

Suppose that there are n homogenous decision-making units $\text{DMU}_j\ (j=1,\dots,n)$ that convert m inputs $x_{ij}\ (i=1,\dots,m)$ into s outputs $y_{rj}\ (r=1,\dots,s)$, and DMU_o is the DMU under evaluation. The production possibility set T_v is defined by:

$$T_v=\left\{(\mathrm{X},\mathrm{Y})\,\middle|\, \mathrm{X}\geq\sum_{j=1}^{n}\lambda_j \mathrm{X}_j,\ \mathrm{Y}\leq\sum_{j=1}^{n}\lambda_j \mathrm{Y}_j,\ \sum_{j=1}^{n}\lambda_j=1,\ \lambda_j\geq 0,\ j=1,\dots,n\right\}. \tag{1}$$

The above definition implies that the BCC model of Banker et al. [5] is as follows:

$$\begin{aligned}
\theta^{o*}=\min\quad & \theta^o\\
\text{s.t.}\quad & \sum_{j=1}^{n}\lambda_j x_{ij}\leq\theta^o x_{io}, \quad i=1,\dots,m\\
& \sum_{j=1}^{n}\lambda_j y_{rj}\geq y_{ro}, \quad r=1,\dots,s\\
& \sum_{j=1}^{n}\lambda_j=1,\\
& \lambda_j\geq 0, \quad j=1,\dots,n.
\end{aligned} \tag{2}$$

Moreover, the following linear programming problem is the additive model:

$$
\begin{aligned}
Max \quad & \sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+ \\
s.t. \quad & \sum_{j=1}^{n} \lambda_j x_{ij} + s_i^- = x_{io} \qquad i = 1,\dots,m \\
& \sum_{j=1}^{n} \lambda_j y_{rj} - s_r^+ = y_{ro} \qquad r = 1,\dots,s \\
& \sum_{j=1}^{n} \lambda_j = 1 \\
& \lambda_j \ge 0,\ s_i^- \ge 0,\ s_r^+ \ge 0,\ j = 1,\dots,n,\ i = 1,\dots,m,\ r = 1,\dots,s.
\end{aligned} \tag{3}
$$

In linear programming problem (3), Max $\left(\sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+\right) = \sum_{i=1}^{m} s_i^{-*} + \sum_{r=1}^{s} s_r^{+*} = 0$ if and only if DMU_o is Parato-Koopmans Efficient (for details see [15]).

We know that, in most models of DEA, the efficiency score of the best performers is one. To discriminate between these efficient DMUs, many methods have been suggested. One of the most important models for ranking extreme efficient units was proposed by Andersen and Petersen (AP) [2]. This model is:

$$
\begin{aligned}
AP: \quad & Min \quad \theta_o \\
s.t. \quad & \sum_{j=1, j\ne o}^{n} \lambda_j x_{ij} \le \theta x_{io} \qquad i = 1,\dots,m \\
& \sum_{j=1, j\ne o}^{n} \lambda_j y_{rj} \ge y_{ro} \qquad r = 1,\dots,s \\
& \sum_{j=1, j\ne o}^{n} \lambda_j = 1 \\
& \lambda_j \ge 0 \quad , \qquad j = 1,\dots,n,\ j \ne o.
\end{aligned} \tag{4}
$$

Definition 1. (Reference Set). For a DMU_o, we define its reference set E_o to be: $E_o = \{j \mid \lambda_j^* > 0\}$ in some optimal solution to (2).

Definition 2. (Pareto–Koopmans Efficiency). A DMU is fully efficient, if and only if it is not possible to improve any input or output without worsening some other input or output [15].

Definition 3. A DMU_o is extreme efficient, if and only if it satisfies the following two conditions:
(i) It is efficient (Pareto–Koopmans Efficient).
(ii) $|E_o| = 1$.

Definition 4. A DMU_o is non-extreme efficient, if and only if it satisfies the following two conditions:
(i) It is efficient (Pareto–Koopmans Efficient).
(ii) $|E_o| > 1$ (that is the CCR envelopment model corresponding DMU_o has alternate optimal).

Seiford and Zhu [49] provided a linear programming problem, a modified DEA model, to study the sensitivity of efficiency classifications in the additive model for simultaneous data changes in all DMUs where absolute changes in the data were of interest. The absolute data variation can be expressed as follows:

For efficient DMU_o

$$\begin{cases}\hat{x}_{io}=x_{io}+\alpha_i, & \alpha_i\geq 0,\ i\in I\\ \hat{x}_{io}=x_{io}, & i\notin I\end{cases} \quad and \quad \begin{cases}\hat{y}_{ro}=y_{ro}-\beta_r, & \beta_r\geq 0,\ r\in o\\ \hat{y}_{ro}=y_{ro}, & r\notin o\end{cases} \tag{5}$$

For DMU_j $(j\neq o)$

$$\begin{cases}\hat{x}_{ij}=x_{ij}-\tilde{\alpha}_i, & \tilde{\alpha}_i\geq 0,\ i\in I\\ \hat{x}_{ij}=x_{ij}, & i\notin I\end{cases} \quad and \quad \begin{cases}\hat{y}_{rj}=y_{rj}+\tilde{\beta}_r, & \tilde{\beta}_r\geq 0,\ r\in o\\ \hat{y}_{rj}=y_{rj}, & r\notin o\end{cases} \tag{6}$$

where (^) represents adjusted data. Moreover the data changes defined above are not only applied to all DMUs, but also different in various inputs and outputs. Based upon the above data variations, Seiford and Zhu [49] provided the following model which studies the sensitivity of additive DEA models.

$$\begin{aligned}
\text{Min}\quad & \sum_{i\in I}u_i^- + \sum_{r\in O}u_r^+ \\
\text{s.t.}\quad & \sum_{j=1,j\neq o}^{n}\lambda_j x_{ij}\leq x_{io}+u_i^- && i\in I\\
& \sum_{j=1,j\neq o}^{n}\lambda_j x_{ij}\leq x_{io} && i\notin I\\
& \sum_{j=1,j\neq o}^{n}\lambda_j y_{rj}\geq y_{ro}-u_r^+ && r\in O\\
& \sum_{j=1,j\neq o}^{n}\lambda_j y_{rj}\geq y_{ro} && r\notin O\\
& \sum_{j=1,j\neq o}^{n}\lambda_j =1 \\
& \mathrm{u}_i^-,\mathrm{u}_r^+,\lambda_j\,(j\neq o)\geq 0\quad , && j=1,\ldots,n
\end{aligned} \tag{7}$$

Based upon model (7), we have:

Theorem 1. Suppose DMU_o is a frontier point, if $0\leq\alpha_i+\tilde{\alpha}_i\leq u_i^{-*}$ $(i\in I)$, $0\leq\beta_r+\tilde{\beta}_r\leq u_r^{+*}$ $(r\in O)$, then DMU_o remains as a frontier point, where $u_i^{-*}(i\in I)$ and $u_r^{+*}(r\in O)$ are optimal values in (7).

For the proof and details, see Seiford and Zhu [49].

3. Stochastic Sensitivity Analysis

Assume that $\tilde{X}_j=(\tilde{x}_{1j},\ldots,\tilde{x}_{mj})^T$ and $\tilde{Y}_j=(\tilde{y}_{1j},\ldots,\tilde{y}_{sj})^T$ are the stochastic input and output vectors. These components have been considered to be normally distributed. Also, let $X_j=(x_{1j},\ldots,x_{mj})^T$ and $Y_j=(y_{1j},\ldots,y_{sj})^T$ be the mean input and output vector, therefore the inputs and outputs have a normal distribution as follows.

$$\tilde{x}_{ij}=N(x_{ij},\delta_{ij}^2),\qquad \tilde{y}_{rj}=N(y_{rj},\delta_{rj}^2)$$

Suppose all input and output components to be jointly normally distributed in the following chance-constrained version of a stochastic DEA model:

$$\begin{aligned} Max \quad & \sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+ \\ \text{s.t.} \quad & P\left\{\sum_{j=1}^{n} \lambda_j x_{ij} + s_i^- = x_{io}\right\} \geq 1\text{-}\alpha \qquad i = 1,\dots,m \\ & P\left\{\sum_{j=1}^{n} \lambda_j y_{rj} - s_r^+ = y_{ro}\right\} \geq 1\text{-}\alpha \qquad r = 1,\dots,s \ , \\ & \sum_{j=1}^{n} \lambda_j = 1, \\ & \lambda_j \geq 0,\ s_i^- \geq 0,\ s_r^+ \geq 0,\ j = 1,\dots,n,\ i = 1,\dots,m,\ r = 1,\dots,s. \end{aligned} \tag{8}$$

In this model, P means ‘‘probability’’ and α is a level of error between 0 and 1, which is a predetermined number. Now, we apply this model to define stochastic efficiency as follows.

Definition 5. (Stochastic Efficiency). DMU_o is stochastic efficient if and only if the following condition is satisfied:

$$\text{Max}\left(\sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+\right) = \sum_{i=1}^{m} s_i^{-*} + \sum_{r=1}^{s} s_r^{+*} = 0$$

The $\tilde{x}_{ij} = \tilde{x}_{io}$, $\tilde{y}_{ij} = \tilde{y}_{io}$ values for DMU_o emerge on the left as well as on the right inside the braces of (8). Therefore, we can always get a solution with $\lambda_o = 1$ and $\lambda_j = 0\,(\, j \neq o\,)$ and all slacks zero. Now, we apply chance-constrained problem and, propose the following stochastic model of the model (7) in which DMU_o is a stochastic efficient:

$$\begin{aligned} Min \quad & \sum_{i \in I} u_i^- + \sum_{r \in O} u_r^+ \\ s.t. \quad & P\left\{\sum_{j=1, j\neq o}^{n} \lambda_j \tilde{x}_{ij} \leq \tilde{x}_{io} + u_i^-\right\} \geq 1\text{-}\alpha \qquad i \in I \\ & P\left\{\sum_{j=1, j\neq o}^{n} \lambda_j \tilde{x}_{ij} \leq \tilde{x}_{io}\right\} \geq 1\text{-}\alpha \qquad i \notin I \\ & P\left\{\sum_{j=1, j\neq o}^{n} \lambda_j \tilde{y}_{rj} \geq \tilde{y}_{ro} - u_r^+\right\} \geq 1\text{-}\alpha \qquad r \in O \\ & P\left\{\sum_{j=1, j\neq o}^{n} \lambda_j \tilde{y}_{rj} \geq \tilde{y}_{ro}\right\} \geq 1\text{-}\alpha \qquad r \notin O \\ & \sum_{j=1, j\neq o}^{n} \lambda_j = 1 \\ & u_i^-, u_r^+, \lambda_j\,(j \neq o) \geq 0 \ , \qquad j = 1,\dots,n \end{aligned} \tag{9}$$

To analyse the sensitivity and determine the stability region of a stochastic DMU_o, model (9) can be converted into the deterministic model through the following procedures and using the notation conventions in Cooper et al. [14]. For this purpose, consider the first chance-constraint of the model (9). The input constraints can be transformed into equality form by adding $\varepsilon_i \geq 0$:

$$P\left\{\sum_{j=1, j\neq o}^{n} \lambda_j \tilde{x}_{ij} \leq \tilde{x}_{io} + u_i^-\right\} = 1\text{-}\alpha + \varepsilon_i \qquad i \in I \tag{10}$$

Remark 1. Let T be a random variable and a, b and c constant numbers, if $P\ (T \le a) = c$ and $b \le a$ then there exists $d \le c$ such that $P\ (T \le b) = d$.
By bringing the above remark into use; there exist $s_i^- \ge 0$ such that:

$$P\left\{ \sum_{j=1, j\ne o}^{n} \lambda_j \tilde{x}_{ij} \le \tilde{x}_{io} + u_i^- - s_i^- \right\} = 1 - \alpha \qquad i \in I \tag{11}$$

Using the above remark and defining new slack variables for the other three unequal constraints, we also convert them to equality. Applying these changes, we have the following model:

$$\begin{aligned}
Min \quad & \sum_{i\in I} u_i^- + \sum_{r\in O} u_r^+ \\
s.t. \quad & P\left\{ \sum_{j=1, j\ne o}^{n} \lambda_j \tilde{x}_{ij} - \tilde{x}_{io} - u_i^- \le -s_i^- \right\} = 1 - \alpha \qquad i \in I \\
& P\left\{ \sum_{j=1, j\ne o}^{n} \lambda_j \tilde{x}_{ij} - \tilde{x}_{io} \le -s_i^- \right\} = 1 - \alpha \qquad i \notin I \\
& P\left\{ - \sum_{j=1, j\ne o}^{n} \lambda_j \tilde{y}_{rj} + \tilde{y}_{ro} - u_r^+ \le -s_r^+ \right\} = 1 - \alpha \qquad r \in O \\
& P\left\{ - \sum_{j=1, j\ne o}^{n} \lambda_j \tilde{y}_{rj} + \tilde{y}_{ro} \le -s_r^+ \right\} = 1 - \alpha \qquad r \notin O \\
& \sum_{j=1, j\ne o}^{n} \lambda_j = 1 \\
& u_i^- \ (i \in I),\ u_r^+ \ (r \in O\ ),\ \lambda_j\ (j \ne o) \ge 0 \quad j = 1,\dots,n, \\
& s_i^- \ge 0 \ \ i = 1,\dots,m\ , \\
& s_r^+ \ge 0 \ \ r = 1,\dots,s.
\end{aligned} \tag{12}$$

Now we convert the stochastic sensitivity analysis model (12) with chance-constraints into a deterministic form. First, we obtain the deterministic form of the first constraint of the model (12) which is as follows:

$$P\left\{ \sum_{j=1, j\ne o}^{n} \lambda_j \tilde{x}_{ij} - \tilde{x}_{io} - u_i^- \le -s_i^- \right\} = 1 - \alpha \qquad i \in I \tag{13}$$

We define:

$$\tilde{h}_i = \sum_{j=1}^{n} \lambda_j \tilde{x}_{ij} - \tilde{x}_{io}, \qquad i \in I \tag{14}$$

Since each linear combination of normal random variables has a normal distribution, we have:

$$\begin{aligned}
& \tilde{h}_i \sim (h_i, (\sigma_i(\lambda))^2), && i \in I, \\
& h_i = E(\tilde{h}_i) = E(\sum_{j=1}^{n} \lambda_j \tilde{x}_{ij} - \tilde{x}_{io}) = \sum_{j=1}^{n} \lambda_j x_{ij} - x_{io}, && i \in I
\end{aligned} \tag{15}$$

$$\begin{aligned}(\sigma_i(\lambda))^2 &= Var(\tilde{h}_i) = Var(\sum_{j=1, j\neq o}^{n} \lambda_j \tilde{x}_{ij} - \tilde{x}_{io}), \qquad i \in I \\ &= Var(\sum_{j=1, j\neq o}^{n} \lambda_j \tilde{x}_{ij}) + Var(\tilde{x}_{io}) - 2\operatorname{cov}(\sum_{j=1, j\neq o}^{n} \lambda_j \tilde{x}_{ij}, \tilde{x}_{io}) \\ &= \sum_{j=1, j\neq o}^{n} \sum_{k=1, k\neq o}^{n} \lambda_j \lambda_k \operatorname{cov}(\tilde{x}_{ij}, \tilde{x}_{ik}) + Var(\tilde{x}_{io}) - 2 \sum_{j=1, j\neq o}^{n} \lambda_j \operatorname{cov}(\tilde{x}_{ij}, \tilde{x}_{io})\end{aligned} \qquad (16)$$

Taking into account the random variable $\tilde{h}_i$, Relation (13) is rewritten as follows:

$$P(\tilde{h}_i - u_i^- \le -s_i^-) = 1 - \alpha \qquad i \in I,$$

$$P\left(\frac{\tilde{h}_i - h_i}{\sigma_i(\lambda)} \le \frac{u_i^- - s_i^- - h_i}{\sigma_i(\lambda)}\right) = 1 - \alpha \qquad i \in I.$$

On the other hand, by placing $\tilde{Z}_i = \dfrac{\tilde{h}_i - h_i}{\sigma_i(\lambda)}$ and knowing that $\tilde{Z}_i$ has a standard normal distribution, we have:

$$P\left(Z_i \le \frac{u_i^- - s_i^- - h_i}{\sigma_i(\lambda)}\right) = 1 - \alpha \qquad i \in I$$

$$P\left(Z_i \le \frac{-u_i^- + s_i^- + h_i}{\sigma_i(\lambda)}\right) = \alpha \qquad i \in I$$

$$\Phi\left(\frac{-u_i^- + s_i^- + h_i}{\sigma_i(\lambda)}\right) = \alpha \qquad i \in I$$

In the above relation, Φ is a function of the standard normal cumulative distribution. Hence:

$$\frac{-u_i^- + s_i^- + h_i}{\sigma_i(\lambda)} = \Phi^{-1}(\alpha) \qquad i \in I$$

$$-u_i^- + s_i^- + h_i - \sigma_i(\lambda)\, \Phi^{-1}(\alpha) = 0 \qquad i \in I$$

where Φ^{-1} is the inverse of Φ and, is the so-called ‘‘fractile function’’. Therefore, the deterministic form of the chance-constraint will be as follows:

$$\sum_{j=1, j\neq o}^{n} \lambda_j x_{ij} - x_{io} - u_i^- + s_i^- - \Phi^{-1}(\alpha)\sigma_i(\lambda) = 0 \qquad i \in I \qquad (17)$$

In Relation (12), other chance-constraints, like the first constraint, become deterministic. But the objective function and the constraint $\sum_{j=1, j\neq o}^{n} \lambda_j = 1$ are not stochastic, so they remain unchanged. Therefore, the deterministic form of the model (12) which has derived from model (9) is as follows:

$$\begin{aligned}
Min \quad & \sum_{i\in I} u_i^- + \sum_{r\in O} u_r^+ \\
s.t. \quad & \sum_{j=1, j\neq o}^{n} \lambda_j x_{ij} - x_{io} - u_i^- + s_i^- - \Phi^{-1}(\alpha)\, \sigma_i(\lambda) = 0 \qquad i \in I \\
& \sum_{j=1, j\neq o}^{n} \lambda_j x_{ij} - x_{io} + s_i^- - \Phi^{-1}(\alpha)\, \sigma_i(\lambda) = 0 \qquad i \notin I \\
& -\sum_{j=1, j\neq o}^{n} \lambda_j y_{rj} + y_{ro} - u_r^+ + s_r^+ - \Phi^{-1}(\alpha)\, \sigma_r(\lambda) = 0 \qquad r \in O \\
& -\sum_{j=1, j\neq o}^{n} \lambda_j y_{rj} + y_{ro} + s_r^+ - \Phi^{-1}(\alpha)\, \sigma_r(\lambda) = 0 \qquad r \notin O \\
& \sum_{j=1, j\neq o}^{n} \lambda_j = 1 \\
& \mathrm{u}_i^- \ (i \in I),\ \mathrm{u}_r^+ \ (r \in O\ ), \lambda_j\, (j \neq o) \geq 0 \quad j = 1,\ldots,n, \\
& s_i^- \geq 0 \ \ i = 1,\ldots,m\ , \\
& s_r^+ \geq 0 \ \ r = 1,\ldots,s.
\end{aligned} \tag{18}$$

An optimal choice of the variables in Relation (18) will also be optimal for Relation (12) and, vice versa, an optimal solution of Relation (12) will also be optimal for Relation (18). See the chapter on chance-constrained programming in Ben Israel [6] which also develops a duality theory for these relations. Model (18) is a nonlinear programming problem because of the functional forms of $\sigma_i(\lambda)$ and $\sigma_r(\lambda)$. Let υ_i and ω_r are non-negative variables. Replace $\sigma_i(\lambda)$ by υ_i and $\sigma_r(\lambda)$ by ω_r in the model (18), and add two quadratic equality constraints, $\upsilon_i^2 = (\sigma_i(\lambda))^2$ and $\omega_r^2 = (\sigma_r(\lambda))^2$, to (18), then (18) is transformed to easily solvable quadratic programming problems.

$$\begin{aligned}
Min \quad & \sum_{i\in I} u_i^- + \sum_{r\in O} u_r^+ \\
s.t. \quad & \sum_{j=1, j\neq o}^{n} \lambda_j x_{ij} - x_{io} - u_i^- + s_i^- - \Phi^{-1}(\alpha)\, \upsilon_i = 0 \qquad i \in I \\
& \sum_{j=1, j\neq o}^{n} \lambda_j x_{ij} - x_{io} + s_i^- - \Phi^{-1}(\alpha)\, \upsilon_i = 0 \qquad i \notin I \\
& -\sum_{j=1, j\neq o}^{n} \lambda_j y_{rj} + y_{ro} - u_r^+ + s_r^+ - \Phi^{-1}(\alpha)\, \omega_r = 0 \qquad r \in O \\
& -\sum_{j=1, j\neq o}^{n} \lambda_j y_{rj} + y_{ro} + s_r^+ - \Phi^{-1}(\alpha)\, \omega_r = 0 \qquad r \notin O \\
& \sum_{j=1, j\neq o}^{n} \lambda_j = 1 \\
& \upsilon_i^2 = (\sigma_i(\lambda))^2 = \sum_{j=1, j\neq o}^{n} \sum_{k=1, k\neq o}^{n} \lambda_j \lambda_k \operatorname{cov}(\tilde{\mathrm{x}}_{\mathrm{ij}}, \tilde{\mathrm{x}}_{\mathrm{ik}}) + Var(\tilde{\mathrm{x}}_{\mathrm{io}}) - 2 \sum_{j=1, j\neq o}^{n} \lambda_j \operatorname{cov}(\tilde{\mathrm{x}}_{\mathrm{ij}}, \tilde{\mathrm{x}}_{\mathrm{io}}), \qquad i = 1,\ldots,m \\
& \omega_r^2 = (\sigma_r(\lambda))^2 = \sum_{j=1, j\neq o}^{n} \sum_{k=1, k\neq o}^{n} \lambda_j \lambda_k \operatorname{cov}(\tilde{\mathrm{y}}_{\mathrm{rj}}, \tilde{\mathrm{y}}_{\mathrm{rk}}) + Var(\tilde{\mathrm{y}}_{\mathrm{ro}}) - 2 \sum_{j=1, j\neq o}^{n} \lambda_j \operatorname{cov}(\tilde{\mathrm{y}}_{\mathrm{rj}}, \tilde{\mathrm{y}}_{\mathrm{ro}}), \qquad r = 1,\ldots,s \\
& \mathrm{u}_i^- \ (i \in I),\ \mathrm{u}_r^+ \ (r \in O\ ), \lambda_j\, (j \neq o) \geq 0 \quad j = 1,\ldots,n, \\
& s_i^- \geq 0 \ \ i = 1,\ldots,m\ , \qquad s_r^+ \geq 0 \ \ r = 1,\ldots,s.
\end{aligned} \tag{19}$$

Model (19) is a quadratic programming model whose optimal solution results in permissible input / output changes of $\mathrm{DMU_o}$. The formulated model detects the stability radius for all DMUs within which absolute data perturbations will not change the efficiency classification of test DMU. The approach considers not only the coinstantaneous absolute data perturbations of the inputs and outputs for all DMUs but also diverse input and output subsets.

4. Conclusions

Uncertainties always exist in practical management and engineering problems. In many applications, a model with deterministic inputs and outputs cannot reasonably encompass all the important features of the problem. To obtain reliable results, the uncertainties should be taken into account, and corresponding DEA methods that handle uncertainties should be developed. Stochastic DEA models may fit well in such applications. This study extends sensitivity analysis to find the largest region that preserves the classification of the DMU with stochastic data. This sensitivity analysis approach simultaneously considers data perturbation in all DMUs, that is, the change of the evaluating DMU_o and the changes of the remaining DMUs. The data perturbation in the evaluating DMU and the data perturbation in the remaining DMUs can be different when all remaining DMUs work in the direction of improving their efficiencies against the worsening of the efficiency of the evaluating efficient DMU_o.

One of the shortcomings of this study is the nonlinearity of the proposed model and also we are unable to discuss absolute changes directly through the modified CCR, because when absolute changes of data are considered, convexity condition is a necessary condition for performance sensitivity analysis. Future research will focus on improving the DEA models, theories, and applications for CCR models and the DEA models of fuzzy optimization and robust optimization to handle data uncertainty, for more details, see [18, 19, 40, 42-44, 47].

Conflict of Interest: The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper

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https://doi.org/10.30495/fomj.2021.1946159.1049

Received: 29 November 2021 Revised: 25 December 2021 Accepted: 26 December 2021