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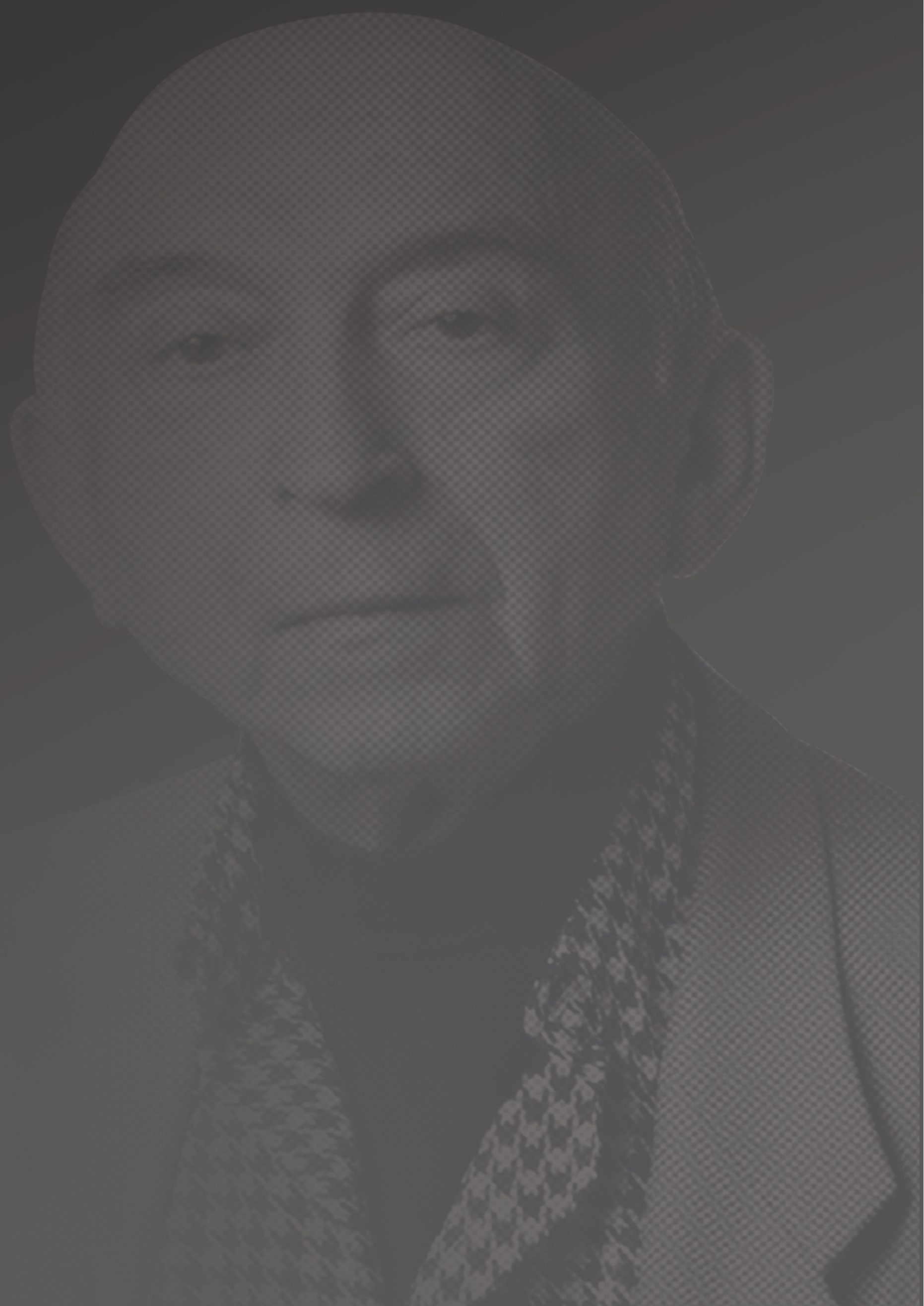

Transactions on
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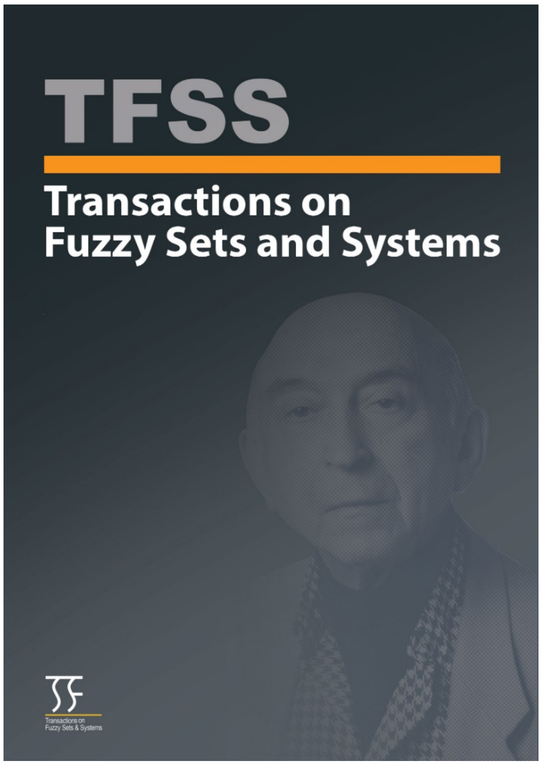

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A Novel Technique for Solving the Uncertainty under the Environment of Neutrosophic Theory of Choice

Tabasam Rashid*, **Aamir Mahboob**, **Ismat Beg**

Abstract. When it comes to solving dynamic programming challenges, it is essential to have a well-structured decision theory. As a result, the decision-makers must operate in a dynamically complicated environment where appropriate and rapid reaction in a cooperative way is the fundamental key to effectively completing the task. We express a theory of decision modeling and axiomatizing a decision-making process. The payoffs and probabilities are represented with simplified neutrosophic sets. We therefore, provide the theory of choice with the implementation of simplified neutrosophic sets. By exploiting the idea of pure strategy, we introduce two steps: in the first step, for each attractive point, some particular event is selected that can bring about a relatively neutrosophic upper payoff with a relatively neutrosophic upper probability or a relatively neutrosophic lower payoff with a relatively neutrosophic upper probability. A decision-maker selects the most favored attractive point in the second stage, based on the focus on all attractive points. Neutrosophic focus theory has been introduced to improve overall performance with more flexibility in complex decision-making. The approach suggested in this work has been implemented in a real-life example to determine its effectiveness. The proposed method is shown to be the most useful for ranking scenarios and addressing dynamic programming problems in decision-making.

AMS Subject Classification 2020: 03E72; 91A30; 91A86; 91B06
Keywords and Phrases: Fuzzy set, Neutrosophic set, Neutrosophic probability, Game theory, Focus theory.

1 Introduction

In various decision-oriented real-life problems, game theory plays a significant role. Nowadays, many such problems are generally described by different uncertainties. Uncertainties occur because of decision-makers the collection of information, perception, belief, opinion, actions, assessment, and finally, due to the problem itself. The definition of fuzzy set [1] with a membership degree initialized the treatment of ambiguity, but it was not sufficient. The concept of the intuitionistic fuzzy set was developed using membership and non-membership grades but struggled to convey truth more accurately. Then, with a new degree of uncertainty, say an indeterminacy degree, in addition to membership and non-membership degrees, neutrosophic logic was developed.

Ambiguities of fact exist everywhere. To explain the uncertainties, fuzzy logic [1] has emerged as one of the essential soft computing methods. From Zadeh to Atanassove [2], the fuzzy notion has been developed from its membership components to an intuitionistic fuzzy notion with non-membership components. For example; in the voting system, when we choose a candidate, one has the option to opt-out or remain independent, in addition to an election or a choice. Intuitionistic characters can not manage such circumstances. In these

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situations, Smarandache [3] introduced and was effectively applied to the neutrosophic set concept. The degree of indeterminacy still occurs in many cases, beyond the stages of acceptance and rejection. So many developments of neutrosophic sets have also been suggested, such as the single-value neutrosophic set [4], the neutrosophic interval set [5], the multi-value neutrosophic set [6], etc.

Game theory is a mathematical analysis in which there are the situation of two contrary ideas exist or strategic decision making. Game theory related to decision-making problems in a mathematical way invented by von Neumann [7]. The research on two-person zero-sum games quickly progressed following Neumann and Morgenstern's pioneering contributions, significantly influencing decision-making and strategic analysis in various disciplines. For example; on nash equilibrium [8, 9] various mathematical models of game theory [10] on decision making and so on. One game consists of multiple players, a set of tactics including a payoff that displays the overall results from every game's play in terms of the rewards won or lost by each based strategy player. According to the probability method, a player who selects a pure strategy randomly selects a row or a column that determines the opportunity for each pure strategy. For players, the probabilities are said to be a mixed strategy. In terms of probability, the measured payoffs represent the probability of each player to obtain and if the game is played a sufficiently large number of time, the player will eventually benefit on average. Due to the ambiguity and vagueness components included as well as what happened throughout the process, the strangeness of the prudence of gamers or decision-makers. We showed the characters of indeterminacy and falsity in matrix form. First of all, for solving fuzzy matrix games, Campos [11] used linear programming models. Later on, Li [12] used Attanasov's intuitionistic fuzzy sets to solve matrix games with different uncertainties. Nowadays, several writers [13, 14, 15, 16, 17] have examined some game models using payoff and probability using maximin, maximax and minmax rules.

After that, the matrix game solution was extended using intuitionistic fuzzy triangular payoff by Bandyopadhyay [18, 19]. He proposes the intuitionistic fuzzy numbers and arithmetic operation of score functions and introduces the matrix game using various strategies. Feng [12] gave the comprehensive idea of a matrix game with the help of intuitionistic payoffs. He also explained trapezoidal intuitionistic fuzzy numbers and interval-valued intuitionistic fuzzy sets and their properties.

Games with neutrosophy set the three contrasting collective grades to be compared: truth-membership, indeterminacy-membership and falsity-membership, whereas intuitionistic games have membership and non-membership degrees. Consequently, it is possible to apply the models and methods of intuitionistic fuzzy games to neutrosophic games. Some authors [20, 21, 22, 23, 24] applied the neutrosophic theory of games in our daily life.

Generally descriptive and normative theories, a decision-maker is believed to maintain a comprehensive understanding while analyzing s lottery game using an aggregated multiplicative model, like that of the SEU. Commonly, cumulative information found from studies using existing techniques makes clear that it is impossible that a risky decision based on weighing and summing procedures is unlikely [25, 26, 27]. Many research show that people assess a lottery by treating every result independently. Wedell [28] showed in his paper that justification for single play decisions is inclined to depend on a single feature of the gamble in which the amount that can be won or lost, the probability of doing so, or other variables are involved in a single attribute. These four characteristics are min payoff, probability of min payoff, max payoff and probability of max payoff suggested in [29]. Furthermore, numerous studies indicate that people judge a lottery based on a specific event associated with this lottery. i.e, they perceive a payoff and its probability [30]. In view of these theories, the Neutrosophic theory of choice claims that is rationally bounded and results with minimal attention, therefore, instead of selecting of all events of a lottery, decision-maker study the event according to the payoff and probability.

In section 2, on the game and neutrosophic set, we give some simple definitions and notations. In section 3, we explore how to pick positive attractive points and how to evaluate the optimal alternative using positive attractive points. An application of neutrosophic set in decision making is discussed and a comprehensive comparison analysis is shown in section 4 to explore the validity and effectiveness. The concluding remarks are given in section 5.

2 Preliminaries

In this section, we deliver concise analysis of neutrosophic set, simplified neutrosophic set, neutrosophic probability, accuracy function and score function. The neutrosophic set allows one to introduce indeterminacy, hesitant or ambiguity irrespective of the knowledge regarding membership and non-membership grades. Therefore, the notion of neutrosophic set is the generalization of fuzzy and intuitionistic fuzzy set. The following definition for a neutrosophic set was given by Smarandache [3].

Definition 2.1. [3] Let X be universe of discourse. A neutrosophic set NS is defined by $T_A(x)$, $I_A(x)$, $F_A(x)$, where $T_A(x)$ is the truth-membership function, $I_A(x)$ is the indeterminancey-membership function and $F_A(x)$ is the falsity-membership function, all of these functions are subset of $]0^-, 1^+[$ with condition $0^- \leq \sup T_A(x) + \sup I_A(x) + \sup F_A(x) \leq 3^+$ for all x belongs to X.

Definition 2.2. [31] A subclass of neutrosophic set is called simplified neutrosophic set (SNS) and it is defined as: $A = \{(x,\ T_A(x),\ I_A(x),\ F_A(x)) \mid x \in X\}$, where $T, I, F \in [0,1]$. For suitability, SNS can be written as: (a, b, c).

In general, if $I_A(x) = 0$, then the above set A can be reduced to intuitionistic fuzzy set, $IFSA = \{(x, T_A(x), F_A(x))|x \in X\}$ and if $I_A(x) = F_A(x) = 0$, then the set A ca be reduced to fuzzy set $FSA = \{(x, T_A(x)|x \in\}$. The relation between fuzzy set, intuitionistic fuzzy set and neutrosophic fuzzy set are shown in Figure 1.

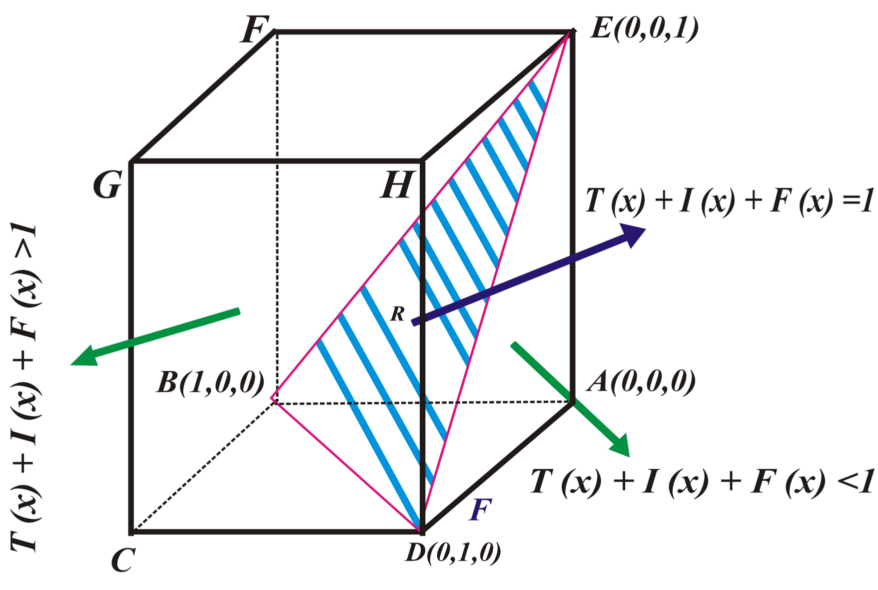

Figure 1: The environment of neutrosophic set

Definition 2.3. [31] Let A be a SNS, then the complement of SNS is denoted by A^c and defined as: $A^c = \{(x,\ F_A(x),\ 1 - I_A(x),\ T_A(x)) \mid x \in X\}$.

Definition 2.4. [31] Let $A = (a_1,\ b_1,\ c_1)$ and $B = (a_2,\ b_2,\ c_2)$ be the SNS, then A contained in B if and only if $a_1 \leq a_2$, $b_1 \geq b_2$ and $c_1 \geq c_2$ for every x in X.

Definition 2.5. [32] Let A be the SNS, then the score function S of a simplified neutrosophic value is defined as:

$$S(A) = \frac{1 + a - 2b - c}{2} \tag{1}$$

where $S(A) \in [-1, 1]$.

Definition 2.6. Let $A = (a_1,\ b_1,\ c_1)$ and $B = (a_2,\ b_2,\ c_2)$ be two simplified neutrosophic sets and $S(A)$ and $S(B)$ be their score functions, then
1) If $S(A) < S(B)$, then A is lesser than B;
2) If $S(A) > S(B)$, then A is greater than B;
3) If $S(A) = S(B)$, then A and B are equal.

Definition 2.7. A matrix game $S = \{Player_1,\ Player_2,\ M_1,\ M_2\}$ is denoted as two-player game;

1). Player_1 has countable game plan set M_1 accompanied by p elements.
2). Player_2 has countable game plan set M_2 accompanied by q elements.
3). The functions $v_1(m_1,\ m_2)$ and $v_2(m_1,\ m_2)$ are the payoff functions of the player_1 and player_2 respectively and $(m_1,\ m_2) \in M_1 \times M_2$.

The matrix game will be as: player_1 select $m_1 \in M_1$ at the certain time and player_2 select $m_2 \in M_2$ at the same time. When each player does this then he/she receives the payoff $v_i(m_1,\ m_2)$. If $M_1 = \{m_1^1, m_2^1, ..., m_p^1\}$, $M_2 = \{m_1^2, m_2^2, ..., m_q^2\}$ are the game plan of player_1 and player_2 respectively and we replace $a_{ij} = v_1\left(m_i^1, m_j^2\right)$ and $b_{ij} = v_2\left(m_i^1, m_j^2\right)$, then the payoffs can be organize in the form of $p \times q$ matrix.

Definition 2.8. Let S be the set of strategies of two player and M, N are the non-empty subset of set S. A triplet $(M,\ N,\ A)$ describes the strategies of two player for simplified neutrosophic set is defined as: $A = \{\langle (m, n),\ (T_A(m,n),\ I_A(m,n),\ F_A(m,n))\rangle \mid (m,n) \in M \times N\}$, where player_1 has M strategies, player_2 has N strategies and B be the simplifies neutrosophic set over $M \times N$.

The explanation is as: Player_1 choose the $m \in M$ and player_2 choose $n \in N$ at the same time and both of them don't know each other preference, at that point the payoff for player_1 is represented by $(T_A(m,n),\ I_A(m,n),\ F_A(m,n))$. Results of player_2 on the circumstance $(m,\ n)$ is negation of result of player_1. Therefore, the neutrosophic payoffs can be organized in matrix from shown in Table 1

Table 1: Gamble matrix

B	n_1	$\dots$	n_q
m_1	$(T_A(m_1,n_1),\ I_A(m_1,n_1),\ F_A(m_1,n_1))$	$\dots$	$(T_A(m_1,n_q),\ I_A(m_1,n_q),\ F_A(m_1,n_q))$
m_2	$(T_A(m_2,n_1),\ I_A(m_2,n_1),\ F_A(m_2,n_1))$		$(T_A(m_2,n_q),\ I_A(m_2,n_q),\ F_A(m_2,n_q))$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
m_p	$(T_A(m_p,n_1),\ I_A(m_p,n_1),\ F_A(m_p,n_1))$	$\dots$	$(T_A(m_p,n_q),\ I_A(m_p,n_q),\ F_A(m_p,n_q))$

For convenance, if we write $a_{ij} = (T_A(m_i,n_j),\ I_A(m_i,n_j),\ F_A(m_i,n_j))$ then the above matrix A can be written as:

$$A = \begin{pmatrix} a_{11} & a_{12} & ... & a_{1q} \\ a_{21} & a_{22} & ... & a_{2q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{p1} & a_{p2} & ... & a_{pq} \end{pmatrix}$$

Definition 2.9. Let $A = \{\langle (m,n)\,,\ (T_A(m,n),\ I_A(m,n),\ F_A(m,n))\rangle \mid (m,n) \in M \times N\}$ be the neutrosophic set of strategies of two person. It satisfied the following properties.

· $\max\Big\{T_A(m_i,n_j), I_A(m_i,n_j), F_A(m_i,n_j)\Big\} = \Big(T_A(m,n), I_A(m,n), F_A(m,n)\Big)$

· $\min\Big\{T_A(m_i,n_j), I_A(m_i,n_j), F_A(m_i,n_j)\Big\} = \Big(T_A(m,n), I_A(m,n), F_A(m,n)\Big)$

Example 2.10. Let $M = \{m_1,\ m_2,\ m_3\}$ and $N = \{n_1,\ n_2,\ n_3\}$ be the strategies for player$_1$ and player$_2$ respectively. The neutrosophic payoff is given as:

$$\begin{pmatrix} (0.95,\ 0.2,\ 0.1) & (0.86,\ 0.3,\ 0.2) & (0.76,\ 0.3,\ 0.3) & (1,\ 0,\ 0) & \\ (0.63,\ 0.3,\ 0.3) & (1,\ 0,\ 0) & (0.92,\ 0.2,\ 0.1) & (0.3,\ 0.4,\ 0.6) & \\ (0.43,\ 0.4,\ 0.6) & (0.38,\ 0.5,\ 0.6) & (1,\ 0,\ 0) & (0.98,\ 0.2,\ 0.2) & (0.85,\ 0.3,\ 0.3) \end{pmatrix}$$

let us consider the $a_{11} = (0.95,\ 0.2,\ 0.1)$ and $a_{12} = (0.86,\ 0.3,\ 0.2)$, according to definition (2.4),

$\max(a_{11},\ a_{12}) = \max\Big((0.95,\ 0.2,\ 0.1), (0.86,\ 0.3,\ 0.2)\Big) = (0.95,\ 0.2,\ 0.1) = a_{11}$

$\min(a_{12},\ a_{13}) = \min\Big((0.86,\ 0.3,\ 0.2),\ (0.76,\ 0.3,\ 0.3)\Big) = (0.76,\ 0.3,\ 0.3) = a_{13}$

3 Neutrosophic Evaluation System

3.1 Neutrosophic Attractive Point

Let E be the set of mutually exclusive events and $A = \{A_1,\ A_2, ..., A_p\}$ be the set of action. The neutrosophic probability is given as $\Big(P(T),\ P(I),\ \ P(F)\Big)$. An occurrence can be therefore be defined by $\Big(v(m_i,\ n_j),\ (P(T),\ \ P(I),\ \ P(F))\Big)$. An neutrosophic attractive point with events n_i is described to as a lottery $\{(v(m_1,\ n_1),\ (P(T_1),\ P(I_1),\ P(F_1))), ..., (v(m_{pi},\ n_{pi}),\ (P(T_{pi}),\ P(I_{pi}),\ P(F_{pi})))\}$.

Definition 3.1. Let $E_1, E_2 \in E$ if $p(SNS_1) \geq p(SNS_2)$ and $v(SNS_1) \geq v(SNS_2)$ and at least $p(SNS_1) > p(SNS_2)$ or $v(SNS_1) > v(SNS_2)$ at that point it is said to be E_1 is neutrosophic dominate E_2 for A_i.

Let us consider the following example to promote the comprehension of the above introduced definition and ideas.

Example 3.2. Let $A = \{A_1,\ A_2,\ A_3\}$ be the set of neutrosophic action, $N^1 = \{n_1^1,\ n_2^1,\ n_3^1\}$ and $N^2 = \{n_1^2,\ n_2^2,\ n_3^2\}$ be the strategies for player$_1$ and player$_2$ respectively. Then the neutrosophic payoff and their against neutrosophic probability is given as in Table 2 and 3:

Table 2: Neutrosophic payoff

(0.95, 0.2, 0.1)	(0.86, 0.3, 0.2)	(0.76, 0.3, 0.3)	(1, 0, 0)	(0, 0, 0)
(0.63, 0.3, 0.3)	(1, 0, 0)	(0.92, 0.2, 0.1)	(0.3, 0.4, 0.6)	(0, 0, 0)
(0.43, 0.4, 0.6)	(0.38, 0.5, 0.6)	(1, 0, 0)	(0.98, 0.2, 0.2)	(0.85, 0.3, 0.3)

Table 3: Neutrosophic probability

(0.1, 0.4, 0.8)	(0.4, 0.2, 0.6)	(0.3, 0.5, 0.5)	(0.2, 0.3, 0.5)	(0, 0, 0)
(0.15, 0.2, 0.8)	(0.15, 0.2, 0.8)	(0.3, 0.5, 0.5)	(0.4, 0.2, 0.6)	(0, 0, 0)
(0.15, 0.2, 0.8)	(0.24, 0.3, 0.7)	(0.35, 0.2, 0.6)	(0.13, 0.3, 0.8)	(0.13, 0.3, 0.8)

For A_1, according to definition (2.9), Clearly n_4^1 is neutrosophic dominate n_1^1, because using the definition (2.4) and equation (1), neutrosophic payoff of n_4^1 is $a_{14} = (1,\ 0,\ 0)$ greater than neutrosophic payoff n_1^1 is $a_{11} = (0.95,\ 0.2,\ 0.1)$ and at their corresponding neutrosophic probabilities $b_{14} = (0.2,\ 0.3,\ 0.5)$ is greater than $b_{11} = (0.1,\ 0.4,\ 0.8)$. Also we see that n_2^1 is neutrosophic dominates n_3^1, because neutrosophic payoff $a_{12} > a_{13}$ and neutrosophic probability $b_{12} > b_{13}$. Therefore, $\{n_2,\ n_4\}$ are the neutrosophic dominates for A_1.

For A_2, the analysis shows that n_2^2 is neutrosophic dominate n_1^2, the reason is that $a_{22} > a_{21}$ and $b_{22} = b_{21}$. Similarly, we can see that n_3^2 is neutrosophic dominate n_1^2, because neutrosophic payoff $a_{13} > a_{21}$ and neutrosophic probability $b_{23} > b_{21}$. Now it is clear that neutrosophic probability n_4^2 is higher than the other probabilities for A_2. So, with the help of definition (3.1), n_4^2 is neutrosophic dominated. Therefore, for A_2, the neutrosophic dominated vector is $\left\{n_2^2,\ n_3^2,\ n_4^2\right\}$. For A_3, it is clear that n_3^3 is neutrosophic dominated because neutrosophic payoff and neutrosophic probability are higher than all the other values. Therefore, $\left\{\ n_3^3\right\}$ is the only neutrosophic vector for A_3.

A decision-maker can select the most appealing event from E for each A_i. Obviously, a decision-maker choose the best attractive event from all the events against each activity. In the meantime, it means that then most attractive event is not necessarily extracted from a paired comparison.

Let us suppose that an event using upper value of neutrosophic probability and upper value of neutrosophic payoff would make the decision-maker more attractive. This is the naturally attractive way to characterize the selection process as it employs a relationship superiority that is know as the most generally accepted concept. This principle reflects an attitude of hope when analyzing events. These principles shows that the most desirable case of an alternative A_i is satisfied by overall state of nature E and denoted as $c_+^i(E)$. $c_+^i(E)$ is referred to describe the set of neutrosophic focus points of A_i over E in the event that there are several neutrosophic focus points A_i exist. Let's see how to recognize $c_+^i(E)$.

Definition 3.3. Let X be a space of points (objects) and $B = \Big(P(T),\ P(I),\ P(F)\Big)$ be the neutrosophic probability. A function $\pi : X \to [0,1]$ is called the neutrosophic relatively likelihood function and it is defined as: $\pi(x) = \left(\frac{P^i(T)}{\max\limits_{i\in X} P(T)}, \frac{P^i(I)}{\max\limits_{i\in X} P(I)}, \frac{P^i(F)}{\max\limits_{i\in X} P(F)}\right)$, where $0 \le \frac{P(T)}{\max\limits_{i\in X} P(T)} + \frac{P(I)}{\max\limits_{i\in X} P(I)} + \frac{P(F)}{\max\limits_{i\in X} P(F)} \le 3$ for all x belongs to X.
Suppose that x_1, x_2 belongs to X, then

$$\pi(x_1) > \pi(x_2) \iff s\left(\frac{P(T(x_1))}{\max\limits_{i\in X} P(T(x_1))}, \frac{P(I(x_1))}{\max\limits_{i\in X} P(I(x_1))}, \frac{P(F(x_1))}{\max\limits_{i\in X} P(F(x_1))}\right) > s\left(\frac{P(T(x_2))}{\max\limits_{i\in X} P(T(x_2))}, \frac{P(I(x_2))}{\max\limits_{i\in X} P(I(x_2))}, \frac{P(F(x_2))}{\max\limits_{i\in X} P(F(x_2))}\right).$$

Definition 3.4. Let a mapping η_i from payoff function to a closed interval zero and one for all A_i is called a satisfaction function and the satisfaction function is dependent to payoff function. i.e., $\eta_i : U_i \to [0,1]$ where, $\max \eta(u_i) = 1$ and if $u_1 > u_2$ then $\eta_i(u_1) > \eta_i(u_2)$.

The above definition is the general form of satisfaction function. The relative position of satisfaction function can be written as:

$$\eta_i(u_i) = \left(\frac{U(T)}{\max\limits_{i\in X} U(T)}, \frac{U(I)}{\max\limits_{i\in X} U(I)}, \frac{U(F)}{\max\limits_{i\in X} U(F)}\right), \text{ where } 0 \le \frac{U(T)}{\max\limits_{i\in X} U(T)} + \frac{U(I)}{\max\limits_{i\in X} U(I)} + \frac{U(F)}{\max\limits_{i\in X} U(F)} \le 3 \text{ for all } x \in X.$$

Let us consider the example 3.2, we rewrite the neutrosophic dominates for A_1 are $\left\{n_2^1,\ n_4^1\right\}$, the neutrosophic dominated vector for A_2 is $\left\{n_2^2,\ n_3^2,\ n_4^2\right\}$ and the neutrosophic dominated vector for A_3 is $\left\{\ n_3^3\right\}$. When considering the neutrosophic dominates for A_1 are $\left\{n_2^1,\ n_4^1\right\}$, then their neutrosophic payoff and neutrosophic probabilities are as: $\{[(0.86,\ 0.3,\ 0.2),\ (0.4,\ 0.2,\ 0.6)]\ [(1,\ 0,\ 0),\ (0.2,\ 0.3,\ 0.5)]\}$, using definition 3.3,

$\{[(0.86,\ 0.3,\ 0.2),\ (1.0,\ 0.67,\ 1.0)]\ [(1,\ 0,\ 0),\ (0.5,\ 1.0,\ 0.83)]\}$.

we calculate the attractive point between $\left\{n_2^1,\ n_4^1\right\}$ as: $\min(\pi(n_2), \eta(n_2)) = \min[(0.86,\ 0.3,\ 0.2),\ (1.0,\ 0.67,\ 1.0)]$. using equation 1, $\min[(0.86,\ 0.3,\ 0.2),\ (1.0,\ 0.67,\ 1.0)] = (1.0,\ 0.67,\ 1.0)$, similarly, $\min(\pi(n_4), \eta(n4)) = \min[(1,\ 0,\ 0),\ (0.5,\ 1.0,\ 0.83)] = (0.5,\ 1.0,\ 0.83)$. As the attractive point between $\left\{n_2^1,\ n_4^1\right\}$ is the upper value between n_2 and n_4. Therefore, $\max\left(n_2^1,\ n_4^1\right) = \max\Big((1.0,\ 0.67,\ 1.0),\ (0.5,\ 1.0,\ 0.83)\Big) = n_2^1$.

So, the neutrosophic attractive point is n_2^1. It shows that $c_+^1(A_1) = n_2^1$. Similarly, the neutrosophic payoff and neutrosophic probability for A_2 is: $\{[(1,\ 0,\ 0)\,,\ (0.15,\ 0.2,\ 0.8)]\ [(0.92,\ 0.2,\ 0.1)\,,\ (0.3,\ 0.5,\ 0.5)]\ [(0.3,\ 0.4,\ 0.6)\,,(0.4,\ 0.2,\ 0.6)]\}$.
Normalize the above vector using the definition 3.3, $\{[(1,\ 0,\ 0),\ (0.38,\ 0.4,\ 1.0)]\ [(0.92,\ 0.5,\ 0.1),\ (0.75,\ 1.0,\ 0.63)]\ [(0.3,\ 1.0,\ 1.0),(1.0,\ 0.4,\ 0.75)]\}$.

Likewise, we obtain $c_+^2(A_2) = n_2^2$. Because, for A_3, only the singleton set $\{\ n_3^3\}$ is the dominates vector, therefore, $\{\ n_3^3\}$ is the attractive point for A_3.

Therefore, the subsets of A_1, A_2 and A_3 are: $A_1 - \{\ n_2^1\} = \{n_1^1,\ n_3^1,\ n_4^1\}$, $A_2 - \{\ n_2^2\} = \{n_1^2,\ n_3^2,\ n_4^2\}$, $A_3 - \{\ n_3^3\} = \{n_1^3,\ n_2^3,\ n_4^3,\ n_5^3\}$.

Now the same process for the subsets have been done, and we have $c_+^1(A_1 - \{n_2^1\}) = \{n_4^1\}$, $c_+^2(A_2 - \{n_2^2\}) = \{n_1^2\}$, $c_+^3(A_3 - \{n_3^3\}) = \{n_4^3\}$.

3.2 Neutrosophic Ideal Alternatives

In the neutrosophic theory of choice, the first step is to calculate the neutrosophic attractive points and the second step is to calculate the neutrosophic ideal alternatives, these neutrosophic ideal alternatives are based on neutrosophic attractive points. In the problems of the neutrosophic theory of choice, a decision-maker believes that the NS attractive points are the most suitable points. Therefore, the alternatives are chosen that generate the ideal alternative after the selecting of NS attractive points. They are sum up of the following definitions.

Definition 3.5. Let $F \subseteq \bigcup_{j=1}^{n} E^j$, and Q_+ is the set of maximal elements of F, and $NF\,(F, Q_+) = \{t \in F, (t, e) \notin Q_+ \mid e \in F\}\,.$

$C_+ = \bigcup_{j=1}^{n} c_+^j\left(E^j\right)$ is the collection of attractive points with relatively high neutrosophic probabilities as well as relatively high neutrosophic payoffs. Suppose that $G = \bigcup_{j=1}^{n} G^j$, where $G^j \subseteq E^j$, then $D_+ = \left\{c_+^i \in NF(G, Q_+) \mid \forall A_i \in A\right\}$ is the neutrosophic set of action whose neutrosophic attractive points are in $NF\,(F, Q_+)$.

Let us turn Example 3.2. $c_+^1(A_1) = \left\{\ n_2^1\right\}$, $c_+^2(A_2) = n_2^2$, $c_+^3(A_3) = n_3^3$, according to definition 3.4, $C_+ = \left\{n_2^1,\ n_2^2, n_3^3\ \right\} = \{[(0.86,\ 0.3,\ 0.2),\ (0.4,\ 0.2,\ 0.6)]\ [(1.0,\ 0,\ 0)\,,\ (0.15,\ 0.2,\ 0.8)]\ [(1.0,\ 0,\ 0),(0.35,\ 0.2,\ 0.6)]\}$. It is clear that $n_3^3 > n_2^1$, and also $n_2^1 > n_2^2$. Hence $NF\,(F, Q_+) = \left\{n_2^1,\ n_3^3\right\}$. Corresponding to these actions, A_1 and A_3 are their respectively alternatives. Therefore, $D_+ = \{A_1,\ A_3\}$. Now, $\{n_2^1, n_3^3\} = \{[(0.86,\ 0.3,\ 0.2)\,,\ (0.4,\ 0.2,\ 0.6)]\ [(1,\ 0,\ 0)\,,\ (0.35,\ 0.2,\ 0.6)]\}$.

using definition 3.3 and 3.4, $\{n_2^1, n_3^3\} = \{[(0.86,\ 0.3,\ 0.2)\,,\ (1.0,\ 1.0,\ 1.0)]\ [(1,\ 0,\ 0)\,,\ (0.88,\ 1.0,\ 1.0)]\}$.

$\min\left(u\left(n_2^1\right),\ \pi\left(n_2^1\right)\right) = \min\left[(0.86,\ 0.3,\ 0.2)\,,\ (1.0,\ 1.0,\ 1.0)\right] = (1.0,\ 1.0,\ 1.0)$

$\min\left(u\left(n_3^3\right),\ \pi\left(n_3^3\right)\right) = \min\left[(1,\ 0,\ 0)\,,\ (0.88,\ 1.0,\ 1.0)\right] = (1,\ 0,\ 0)$

now the maximum value between the above is the optimal value, so the most attractive point is $\{n_3^3\}$ and hence, A_3 is the optimal alternative.

4 Application of Neutrosophic Set in Decision Making

4.1 Working Rule

In this section, a procedure for neutrosophic theory of choice is shown. The following steps show the algorithm of game problems.
Step 1: Calculate the score values of each neutrosophic payoff and neutrosophic probability.
Step 2: Calculate the neutrosophic dominate points according to definition 2.9 and deleting all other vectors.

Step 3: Apply the definition 3.1 and 3.3 for dominate vectors.
Step 4: Calculate all the neutrosophic attractive points using max(min $(\eta(u_i), \pi(x_i)))$.
Step 5: Collect the neutrosophic set of action for attractive points using definition 3.5.
Step 6: Obtained the optimal strategies from all the neutrosophic attractive points.
The conceptualization of the suggested strategy is shown in figure 2.

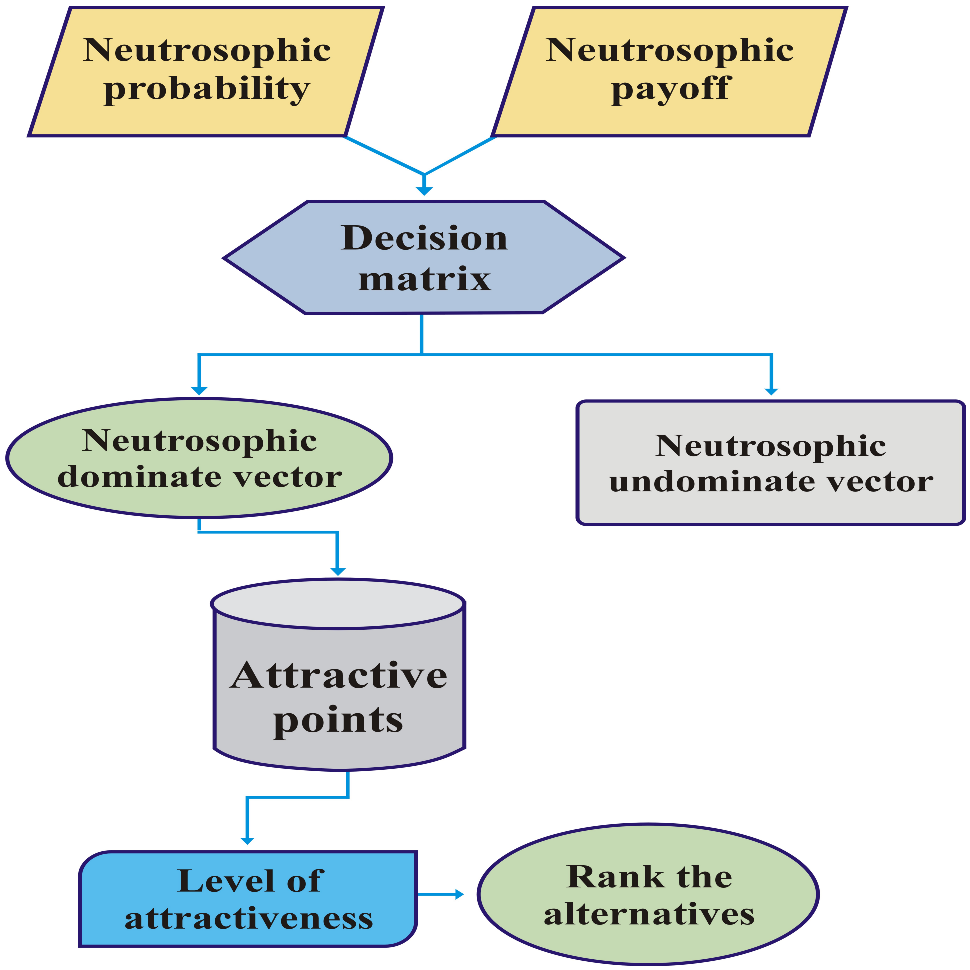

Figure 2: Algorithm of the proposed strategies under the environment of neutrosophic sets

4.2 Case Study

Let's take a real-life example to make the conceptual understanding easy, a person who wants to buy a new mobile phone. He, as decision-maker starts his research. Mainly he evaluates and analysis among the three most popular brands, i.e. Apple, Samsung and LG Mobiles. He compares the five major characteristics of a mobile phone which are the following: 1. camera pixels, 2. Battery power/ timing, 3. processor capacity, 4. mobile RAM & memory capacity, 5. screen resolution & size. The decision-maker collects the information given by the companies and online consumer views on these products. His satisfaction level about each of the characteristic is dependent on the customer's opinions on them. Suppose most of Apple customers are not satisfied with its battery timing, so Apple's probability in this regard is not good, leading to dissatisfaction. However, customers' views about Apple's screen resolution are exceptional, leading to high satisfaction to decision-makers. The neutrosophic satisfaction level for each alternative corresponding to their neutrosophic probabilities are as follows:

Suppose that $\{A_1, A_2, A_3\}$ be the set of neutrosophic action collaborated with the disjoint set of events N^i, where superscript i represents mathematical symbols for the action A_i. Suppose $N^1 = \{n_1^1, n_2^1, n_3^1, n_4^1, n_5^1\}$, $N^2 = \{n_1^2, n_2^2, n_3^2, n_4^2, n_5^2\}$, $N^3 = \{n_1^3, n_2^3, n_3^3, n_4^3, n_5^3\}$. The strategies of neutrosophic payoffs and their corresponding neutrosophic probability associated to each state shown in Table 4 and 5:

Table 4: Strategies of neutrosophic payoff for player.

$$\begin{pmatrix} (0.95,\ 0.2,\ 0.1) & (0.86,\ 0.3,\ 0.2) & (0.76,\ 0.3,\ 0.3) & (1,\ 0,\ 0) & (0,0,0) \\ (0.63,\ 0.3,\ 0.3) & (1,\ 0,\ 0) & (0.92,\ 0.2,\ 0.1) & (0.3,\ 0.4,\ 0.6) & (0,0,0) \\ (0.43,\ 0.4,\ 0.6) & (0.38,\ 0.5,\ 0.6) & (1,\ 0,\ 0) & (0.98,\ 0.2,\ 0.2) & (0.85,\ 0.3,\ 0.3) \end{pmatrix}$$

Table 5: Strategies of neutrosophic probability for player

$$\begin{pmatrix} (0.1,\ 0.4,\ 0.8) & (0.4,\ 0.2,\ 0.6) & (0.3,\ 0.5,\ 0.5) & (0.2,\ 0.3,\ 0.5) & (0,0,0) \\ (0.15,\ 0.2,\ 0.8) & (0.15,\ 0.2,\ 0.8) & (0.3,\ 0.5,\ 0.5) & (0.4,\ 0.2,\ 0.6) & (0,0,0) \\ (0.15,\ 0.2,\ 0.8) & (0.24,\ 0.3,\ 0.7) & (0.35,\ 0.2,\ 0.6) & (0.13,\ 0.3,\ 0.8) & (0.13,\ 0.3,\ 0.8) \end{pmatrix}$$

The neutrosophic dominates vectors for A_1, A_2, and A_3 are $\{n_1^1, n_2^1, n_3^1, n_4^1\}$, $\{n_2^2, n_5^2\}$ and $\{n_1^3, n_4^3\}$ respectively. For $\{n_1^1, n_2^1, n_3^1, n_4^1\}$, the payoff and their corresponding probabilities are:

$$\begin{pmatrix} (1,0,0) & (0.6,0.3,0.3) & (0.5,0.6,0.2) & (0.4,0.6,0.4) \\ (0.1,0.2,0.4) & (0.4,0.3,0.1) & (0.3,0.1,0.1) & (0.1,0.2,0.2) \end{pmatrix}$$

The neutrosophic satisfaction function and neutrosophic relatively likehood functions of above matrix can be written as:

$$\begin{pmatrix} (1,0,0) & (0.6,0.5,0.75) & (0.5,1.0,0.5) & (0.4,1.0,1.0) \\ (0.25,0.67,1.0) & (1,1,0.25) & (0.75,0.33,0.25) & (0.25,0.67,0.5) \end{pmatrix}$$

The optimal action point in neutrosophic theory of choice is; $\max(\min\left(v(n_i^1), \pi(n_i^1)\right)) = \max((0.25, 0.67, 1),$ $(1, 1, 0.25), (0.5, 1, 0.5), (0.4, 1, 1)) = n_2^1$. Therefore, $c_+^1(A_1) = n_2^1$. Similarly, $c_+^2(A_2) = n_2^2$ and $c_+^3(A_3) = n_4^3$. So, $C_+ = \bigcup_{j=1}^{n} c_+^j(A_j) = \{n_2^1, n_2^2, n_4^3\}$. Now the second step is to calculate the ideal alternative, for this, we write the neutrosophic payoff and their corresponding probabilities from Table (4) and (5) for the neutrosophic attractive points.

$$\underbrace{\begin{pmatrix} \overset{n_2^1}{(0.6,0.3,0.3)} & \overset{n_2^2}{(1,0,0)} & \overset{n_4^3}{(1,0,0)} \\ (0.4,0.3,0.1) & (0.2,0.4,0.7) & (0.35,0.4,0.7) \end{pmatrix}}_{\text{Neutrosophic attractive points}}$$

According to definition 3.4, $NF(F, Q_+) = \{n_2^1,\ n_4^3\}$ and $ND_+ = \{A_1,\ A_3\}$. Now we calculate the level of attractive for the alternatives A_1 and A_3 with the help of $\max(\min\left(v(n_2^1, A_1), \pi(n_2^1)\right), \left(v(n_4^3, A_3), \pi(n_4^3)\right))$. Therefore, the optimal action is A_3. Hence, $A_3 > A_1 > A_2$.

The graphical representation of the optimal point is shown in Figure 3. The points A, A', B, B' and C, C' shows the strategies of payoff and their corresponding strategies of probabilities, respectively. These points are the attractive points of the given decision matrix. Moreover, n_2^1,$n_2^1\prime$ and $n_4^3, n_4^3\prime$ represents the relative position of neutrosophic payoff and probabilities, respectively, of the points A, A' B, B' and C, C'.

4.3 Comparison Analysis

A comparative study between the proposed neutrosophic theory of choice and other methods like TOPSIS is discussed and analysis shows that the TOPSIS evaluates each alternative using the weighted of all the

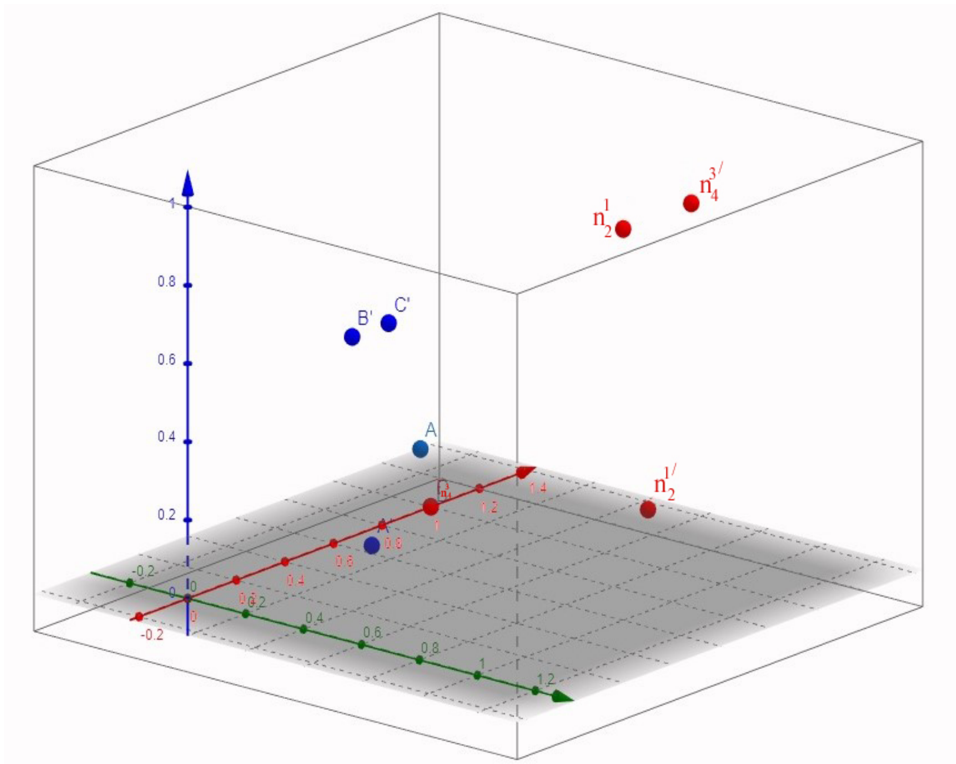

Figure 3: Neutrosophic optimal point

outcomes and then selecting the alternative with maximum relative closeness. Similarly, if we consider the subjective expected utility (SEU), this theory is also based on the weighted average. In, SEU, each alternative is selected by maximum average based on weight. These theories are related to the risk factor. A decision-maker avoids the risk, takes the risk or is neutral, the graph of utility is concave, convex and linear respectively. But the neutrosophic theory of choice, consists of two steps; the first step is to select the event of attractive points and 2nd step is the relationship trade of neutrosophic payoff and neutrosophic relative likelihood probability function. There is no risk factor for the decision-maker. Because in the proposed theory, the satisfaction function is the relative position of payoff, as well as the relative likelihood function, are used to make the decision, which shows the attitude of a decision-maker in uncertain situations. It means when a decision-maker chooses the attractive points. Decision-maker mark the same weight on probability and payoff, or when he tried to obtain an attractive point, the payoff and probability are equal degrees of importance. It shows that the neutrosophic theory of choice is most straight forward and more comfortable than other approaches. Neutrosophic theory of choice deals with totally different ways to select the scenario. SEU and other approaches use the weighted function to deal with the uncertainty, which is not the actual solution of the uncertainty problems. Suppose some alternative is repeated a large number of times. In that case, the obtained result is confidently reaching the maximum value. In contrast, the proposed theory shows a clear solution because of using neutrosophic payoff and relative likelihood function, and psychological evidence clearly supports them. Nowadays, many researchers [25, 26] give evidence gathered from studies using scanpath and different strategies suggesting that it is impossible that a risky decision would be made on a system of weighting and summing. Zhou et al.[27] proved that the proportion task of the information process sequence tends to be more compatible with the summing and weighting method. Therefore, we believe that our outcomes indicate the best results as compared to the weighting and summing process. So, the proposed technique would be an immense addition to decision-making problems.

5 Conclusion

The most generalization of intuitionistic fuzzy sets is the neutrosophic sets, in which ambiguity is introduced through an extra indeterminacy degree. In this research, we have implied neutrosophic frameworks. We chose game theories using neutrosophic logic. We have considered a neutrosophic payoff and probability approach to solving our constructed game phenomena. In this analysis, we have observed an indeterminacy function assuming neutrosophic sets with membership and falsity characteristics. Moreover, in focused recommendation systems, we have declared our proposed game model and have achieved some desirable outcomes. We have found that some have solved the problems in crisp data sets, while we have presented neutrosophic data sets that are more closely linked to the expressions of real-life problems. This can be seen as a limitation of our study's generalization. Some theoretical constructs can, however, be explored in various situations and other real-life issues with different levels of additional measures. In the future, research in multiple fields, such as medical diagnosis, business management optimization, aerospace engineering, space design management, manufacturing industry management, weapons, laboratory research management, wastewater management, optimization of renewable energy sources, supply chain management, can be carried out in game theory under different uncertainties. Game theories neutrosophic attributes can be comprehended utilizing different techniques from neuroscience, mechanical technology, artificial intelligence, humanitarian operations and so on. The neutrosophic focus theory of choice consists of two obligatory portions, from an external source, neutrosophic probability and payoff are given. For the selecting of the neutrosophic attractive points, we can't straightforward allocate the probabilities. Therefore, we use a two-level process to evaluate the probabilities. There are two types of theories for modeling rationality [33], which is substantively rational theory and the second is rational procedural theory. According to many researchers, the second model is more relaxed and latest logical approach. The fundamental principle of all these types of theories is to substitute or ease the portion of the expected utility theory or the expected subjective utility theory axioms. This paper contributes to a basic theory, including some logically satisfying axioms for the rationality procedural and deals with decision-making risk or uncertainty or ignorance. The key point of Neutrosophic theory of choice is that the most relevant occurrence leads to the most favoured attractive point. The Neutrosophic theory of choice dispose of two stages, one is to select an attractive event for each step and then the most occurrence event is chosen from all the attractive points. We have found many cognitive proofs in serval papers [31, 34] that all the evidence consists of the basic principles of the neutrosophic theory of choice. These theories used concave and convex functions to show the gain and loss; these functions are associated with risky situations. Whereas, in the proposed approach, the neutrosophic payoff function has no relation to risk situations. Decision-making models are categorized by Shafir et al.[35] into two groups; one is value-based and the second is reason-based. A value-based model is associated with a numerical value to every option and chooses the maximum value alternative on the other hand reason-based problems describes different goals and reasons that are expected to determine and affect and describes choices in terms of reasons for and against the various alternatives. But, there is no analysis of how these theories related to lottery base decisions. However, we imply the neutrosophic theory of choice that the reason for the alternative is the identify the attractive points of the optimal attractive. It is also possible that sometimes, a decision-maker often does not understand a particular factor when evaluating an ideal alternative [36]. Our proposed neutrosophic theory of choice can be implemented for complicated decisions and real-world problems where some existing approaches may be difficult to solve. In management fields, the proposed theory provides a comprehensive, structured framework for modeling rational thoughts. While it is well-known that behavioral variables are very significant in the study of research, however, it is complicated to integrate the characteristics and qualities of players into mathematical models because of the absence of proper theories. The proposed approach gives a conceptual framework for the development of behavioral models. This study has many limitations, while many recognized irregularities have been announced by this proposed theory, different axioms are proved using the logical procedure.

Conflict of Interest: "The authors declare no conflict of interest."

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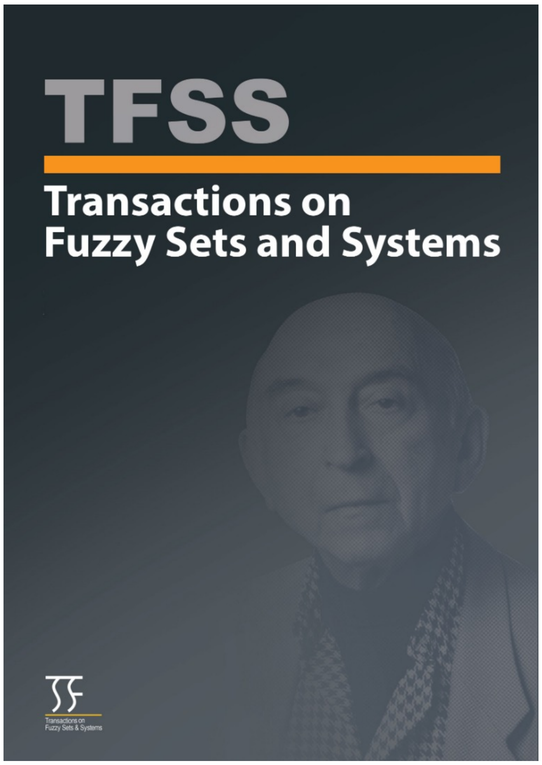

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Proposing a Conceptual Model of Critical Success Factors in Lean Production Using Interpretive Structural Modeling and Fuzzy MICMAC Analysis

Mazdak Khodadadi-Karimvand, **Hadi Shirouyehzad***, **Farhad Hosseinzadeh Lotfi**

Abstract. Since companies are inclined to implement lean production, researchers have proposed a number of fundamental success factors to facilitate the implementation of this production approach. This study analyzes the critical success factors (CSFs) in lean production extracted from 14 review studies. The interpretive structural modeling approach is utilized to analyze the impact of these critical success factors on one another. The aim is to enhance insights into lean production and facilitate informed decision-making. In this article, a seven-tiered model is presented. According to the conceptual model of success factors in lean production, leadership is positioned at the base of the model and serves as the origin for other factors. It should be regarded as the foremost critical success factor in lean production. When establishing lean production systems, organizations and senior managers should focus on higher levels and critical success factors that underlie the model. Subsequently, nonfuzzy and fuzzy driving and dependence power analyses were conducted that the fuzzy matrix cross-reference multiplication applied to a classification (MICMAC) analysis provides deeper insights into the analysis of driving and dependence power. The fuzzy matrix cross-reference multiplication applied to a classification analysis helped identify some key factors that are highly effective for successfully implementing lean manufacturing.

AMS Subject Classification 2020: 03B52, 03E72, 68T27
Keywords and Phrases: Critical success factors, Lean production, Interpretive structural modeling, Fuzzy MICMAC analysis.

1 Introduction

Lean thinking received considerable attention in the 1990s [1]. The notion was introduced to assist manufacturers in improving the performance of their manufacturing system by eliminating unnecessary activities [2]. Lean thinking helps organizations to identify types of waste, such as overproduction in mass production systems. When these wastes are reduced, and the production flow is streamlined, fewer resources would be required to perform operations. Consequently, waste reduction can result in improved performance, primarily characterized by lower costs, shorter lead times, and more stable quality. Additionally, it can lead to lesser work in progress, lower inventory levels, and higher product diversity. Subsequently, implementing lean concepts can lead to greater customer satisfaction and increased market competitiveness [3].

In recent years, there has been a renewed emphasis on lean production [4, 5]. The global economic recession has compelled companies operating in today's open global economy to raise productivity and lower

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costs. Accordingly, lean production has gained popularity as a strategy to enhance the competitiveness of industrial companies [6, 1].

Despite their best efforts, most companies are unable to successfully implement lean manufacturing programs [7, 4, 8]. Researchers and consultants have proposed a set of critical success factors (CSFs) to assist businesses in implementing lean manufacturing and avoiding costly failures. CSFs refer to factors that must go well to ensure the success of a manager or organization. These factors are directly associated with specific areas of management or the company that require consistent attention to achieve optimal performance [9].

There are several lists of CSFs for lean production implementation as well as improvement models, including total quality management, just-in-time production, Six Sigma, and total productive maintenance. On the whole, there is a strong theoretical consensus among studies as to what the CSFs are.

Interpretive structural modeling (ISM) is a useful approach for analyzing subjects that involve interrelated qualitative variables of varying importance [10]. ISM aids in identifying the internal relationships between variables and is an appropriate technique for analyzing the effect of one variable on others [11]. Additionally, ISM can rank and sort out system components, which greatly aids managers in implementing the intended model [12].

This study categorizes and evaluates 24 CFSs in the context of lean production. These CFSs are extracted by Netland from a review of 14 articles in this field and cited in "Critical Success Factors for Implementing Lean Production: The Effect of Contingencies" [13]. Here, in the current paper, they are partitioned into levels using the ISM method and categorized into four groups based on fuzzy driving and dependence power analyses.

2 Literature Review

CSFs are characteristics, conditions, or variables with a significant impact on the success of an organization in specific domains [14] if used and managed properly. Rungasamy, Antony, and Ghosh [15] argue that organizations can gain a competitive advantage by identifying and achieving favorable outcomes in CSFs. If an organization's objectives conflict with the CSFs in a particular domain, it may experience significant failure in that domain. Conversely, an organization's upper hand in one or more CSFs compared to competitors presents an exceptional opportunity for it to attain a competitive advantage.

Quality leaders such as Deming (1986), Crosby (1979), and Joran (1988), as well as advocates of lean manufacturing, such as Laker (2004) and Womack and Jones [2], have compiled comprehensive lists of CSFs (cited in Netland [13]).

The identification and proposal of CSFs for lean production, total quality management, just-in-time production, Six Sigma, and total productive maintenance, along with other methods, have consistently been the focus of scholarly articles and research in the field of operations management. Numerous explanations have been provided. Several researchers, including Achanga, Shehab, Roy, and Nelder [16], Cotte, Farber, Merchant, Paranikas, and Sirkin [17], Losonci, Demeter, and Jenei [18], and Vinodh and Joy [19], have produced lists of CSFs for lean manufacturing.

Many academic sources have synthesized a wealth of scientific literature on CSF for improvement programs. The articles concur that "Corporate management commitment", "Education", and "Employee involvement and support" are three of the key success factors. Refer, for example, to Sila and Ebrahimpour's review of 76 articles on total quality management [20], Nitin et al.'s review of success factors among 10 companies winning the National Quality Award [21], Brady and Allen's review of 201 published articles on Six Sigma [22], and Marodin and Saurin's review of 102 published studies on lean production [4]. The findings suggest that managers should play an active role in leading and supporting the implementation of lean production. This is important to ensure that all employees have a clear understanding of lean production and know how to effectively implement it. Organizations should educate employees and support them in implementing the designated changes. Scientific sources emphasize aligning improvement programs with business strategy, creating long-term plans, managing cultural changes, and involving supply chain partners as key factors.

Netland [13] extracted 22 CSFs from 14 structured review articles on total quality management, Six Sigma, total productive maintenance, just-in-time production, and lean production in his research work titled "Critical Success Factors for Implementing Lean Production: The Effect of Contingencies". These CSFs are summarized in Table 1.

Table 1: Critical success factors for implementing lean production programs [13]

Critical Success Factors					
1	Lead actively	9	Commit corporate management	17	Focus on areas and prioritize activities
2	Participate personally	10	Integrate lean in every day business	18	Invest time and money
3	Educate employees	11	Develop a vision and roadmap	19	Benchmark others
4	Educate managers	12	Use rewards and recognition	20	Emphasize team concept
5	Communicate, inform, and discuss	13	Monitor and audit implementation	21	Use external experts
6	Set and follow-up targets	14	Standardize and manage discipline	22	Hold regular implementation meetings
7	Involve and support employees	15	Find and share best practices	23	Emphasize safety and job attractiveness
8	Dedicate human resources	16	Stepwise approach	24	Use lean tools and methods

3 Methodology

The ISM is a systems design method initially proposed in 1973 by Warfield, a systems scientist at George Mason University in the United States. This approach was initially introduced and subsequently developed to facilitate the design of economic and social systems [23].

The ISM approach is an efficient methodology for addressing issues associated with interacting qualitative variables with varying degrees of importance [10]. It is a useful technique for analyzing the relationships between variables and assessing the influence of one variable on others [11]. ISM enables managers to prioritize and assess the importance of system elements, thereby facilitating the effective implementation of the model [12]. In order to apply the ISM technique and determine the internal relationships and priorities of system elements, the subsequent steps should be followed [24].

A- Identifying the variables to be used in the model

B- Developing the structural self-interaction matrix (SSIM) of the variables

C- Developing the reachability matrix

D- Checking the matrix for transitivity

E- Partitioning of the reachability matrix into different levels

F- Drawing the ISM

G- Analyzing the driving and dependence power using the MICMAC[2] diagram

[2]Matrix Cross-reference Multiplication Applied to a Classification (MICMAC)

4 The ISM of CSFs in Lean Production

4.1 Identifying the Variables to Be Used in the Model

The initial step in the ISM involves identifying variables relevant to the topic in question [25]. Here, the desired variables are the CSFs in lean production.

4.2 Developing the SSIM of the Variables

After identifying the variables, they need to be placed in the SSIM. The matrix in question has dimensions corresponding to the variables specified in its first row and column [26]. In order to determine the type of binary interaction between the variables, a questionnaire was developed, and experts were consulted. Ultimately, experts decided on the type of interaction. The experts in this study were selected from a pool of industry managers, professionals, and university professors.

4.3 Developing the Reachability Matrix

The reachability matrix is obtained by transforming the symbols of the SSIM relations into 0 and 1 [27].

Table 2: Aggregate initial binary reachability matrix

		Use lean tools and methods	Emphasize safety and job attractiveness	Hold regular implementation meetings	Use external experts	Emphasize team concept	Benchmark others	Invest time and money	Focus on areas and prioritize activities	Stepwise approach	Find and share best practices	Standardize and manage discipline	Monitor and audit implementation	Use rewards and recognition	Develop vision and roadmap	Integrate lean in everyday business	Commit corporate management	Dedicate human resources	Involve and support employees	Set and follow-up targets	Communicate. inform and discuss	Educate managers	Educate employees	Participate personally	Lead actively
#	**Critical Success Factors**	**24**	**23**	**22**	**21**	**20**	**19**	**18**	**17**	**16**	**15**	**14**	**13**	**12**	**11**	**10**	**9**	**8**	**7**	**6**	**5**	**4**	**3**	**2**	**1**
1	Lead actively	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**
2	Participate personally	**1**	**1**	**1**	**0**	**1**	**1**	**0**	**0**	**1**	**1**	**1**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**1**	**0**	**1**	**1**	**1**
3	Educate employees	**1**	**1**	**1**	**0**	**1**	**0**	**0**	**0**	**1**	**1**	**1**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**1**	**0**	**1**	**1**	**1**
4	Educate managers	**1**	**1**	**1**	**0**	**1**	**1**	**0**	**1**	**1**	**1**	**1**	**1**	**0**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**
5	Communicate. inform and discuss	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**0**	**0**	**1**	**0**	**0**	**0**	**1**	**1**	**0**	**0**	**1**	**1**	**1**	**1**	**1**	**1**	**1**
6	Set and follow-up targets	**1**	**0**	**0**	**0**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**0**	**0**	**1**	**1**	**1**	**1**	**1**	**1**	**0**	**1**	**1**	**0**	**0**
7	Involve and support employees	**0**	**1**	**0**	**0**	**1**	**0**	**0**	**0**	**0**	**1**	**0**	**0**	**1**	**1**	**1**	**0**	**1**	**1**	**0**	**1**	**0**	**0**	**1**	**0**
8	Dedicate human resources	**1**	**0**	**0**	**0**	**1**	**0**	**0**	**0**	**0**	**0**	**1**	**0**	**0**	**1**	**1**	**0**	**1**	**0**	**0**	**0**	**0**	**0**	**0**	**0**
9	Commit corporate management	**1**	**1**	**1**	**1**	**1**	**1**	**0**	**1**	**1**	**1**	**1**	**1**	**0**	**0**	**1**	**1**	**1**	**1**	**0**	**0**	**0**	**1**	**0**	**1**
10	Integrate lean in everyday business	**1**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**1**	**1**	**1**	**0**	**1**	**0**	**0**	**1**	**1**	**1**	**1**
11	Develop vision and roadmap	**1**	**1**	**0**	**0**	**0**	**1**	**1**	**1**	**0**	**0**	**1**	**1**	**1**	**1**	**1**	**1**	**0**	**1**	**0**	**1**	**1**	**1**	**1**	**1**
12	Use rewards and recognition	**0**	**1**	**1**	**0**	**1**	**0**	**0**	**1**	**1**	**1**	**1**	**1**	**1**	**0**	**1**	**1**	**0**	**1**	**0**	**0**	**0**	**0**	**1**	**1**
13	Monitor and audit implementation	**1**	**1**	**1**	**0**	**0**	**0**	**0**	**1**	**0**	**0**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**0**
14	Standardize and manage discipline	**1**	**1**	**1**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**1**	**0**	**0**	**1**	**1**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**
15	Find and share best practices	**1**	**1**	**0**	**1**	**1**	**1**	**0**	**1**	**0**	**1**	**1**	**0**	**1**	**1**	**1**	**0**	**0**	**1**	**0**	**1**	**0**	**0**	**0**	**1**
16	Stepwise approach	**0**	**0**	**1**	**0**	**1**	**0**	**0**	**1**	**1**	**0**	**1**	**0**	**0**	**1**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**
17	Focus on areas and prioritize activities	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**1**	**1**	**0**	**0**	**0**	**0**	**1**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**
18	Invest time and money	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**0**	**0**	**0**	**0**	**0**	**0**	**1**	**1**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**
19	Benchmark others	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**0**	**0**	**1**	**1**	**0**	**1**	**1**	**1**	**0**	**0**	**0**	**0**	**0**
20	Emphasize team concept	**0**	**1**	**0**	**0**	**1**	**0**	**0**	**0**	**1**	**0**	**0**	**0**	**0**	**0**	**1**	**1**	**0**	**1**	**0**	**0**	**0**	**0**	**0**	**0**
21	Use external experts	**1**	**1**	**0**	**1**	**0**	**1**	**0**	**1**	**0**	**1**	**1**	**0**	**0**	**1**	**1**	**0**	**0**	**1**	**1**	**1**	**1**	**1**	**0**	**0**
22	Hold regular implementation meetings	**0**	**0**	**1**	**0**	**1**	**0**	**0**	**1**	**0**	**1**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**1**	**0**	**0**	**0**	**0**
23	Emphasize safety and job attractiveness	**1**	**1**	**0**	**0**	**1**	**0**	**0**	**0**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**0**	**0**	**0**	**0**	**0**	**1**	**1**	**1**	**0**
24	Use lean tools and methods	**1**	**1**	**0**	**1**	**1**	**1**	**0**	**1**	**0**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**

4.4 Checking the Matrix for Transitivity

Once the initial reachability matrix is generated, its internal consistency must be established. For instance, if variable 1 leads to variable 2 and variable 2 leads to variable 3, variable 1 should also lead to variable 3. If the reachability matrix does not meet the condition, it should be revised by replacing the missing relationships. Multiple techniques exist for assessing the transitivity of a matrix. This article employs mathematical principles to establish consistency in the reachability matrix. Specifically, the reachability matrix is raised to the power of K+1, where K is a positive integer greater than or equal to 1. Notably, the exponentiation operation must adhere to the rules of Boolean algebra [27].

Table 3: Final (binary) reachability matrix (after transitivity check)

		Use lean tools and methods	Emphasize safety and job attractiveness	Hold regular implementation meetings	Use external experts	Emphasize team concept	Benchmark others	Invest time and money	Focus on areas and prioritize activities	Stepwise approach	Find and share best practices	Standardize and manage discipline	Monitor and audit implementation	Use rewards and recognition	Develop vision and roadmap	Integrate lean in everyday business	Commit corporate management	Dedicate human resources	Involve and support employees	Set and follow-up targets	Communicate. inform and discuss	Educate managers	Educate employees	Participate personally	Lead actively	**Driving Power**
#	**Critical Success Factors**	**24**	**23**	**22**	**21**	**20**	**19**	**18**	**17**	**16**	**15**	**14**	**13**	**12**	**11**	**10**	**9**	**8**	**7**	**6**	**5**	**4**	**3**	**2**	**1**	
1	Lead actively	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	24
2	Participate personally	1	1	1	0	1	1	0	0	1	1	1	0	0	0	0	0	0	0	0	1	0	1	1	1	12
3	Educate employees	1	1	1	0	1	0	0	0	1	1	1	0	0	0	0	0	0	0	0	1	0	1	1	1	11
4	Educate managers	1	1	1	0	1	1	0	1	1	1	1	1	0	1	1	1	1	1	1	1	1	1	1	1	21
5	Communicate. inform and discuss	1	1	1	1	1	1	1	0	0	1	0	0	0	1	1	0	0	1	1	1	1	1	1	1	17
6	Set and follow-up targets	1	0	0	0	1	1	1	1	1	1	1	0	0	1	1	1	1	1	1	0	1	1	0	0	16
7	Involve and support employees	0	1	0	0	1	0	0	0	0	1	0	0	1	1	1	0	1	1	0	1	0	0	1	0	10
8	Dedicate human resources	1	0	0	0	1	0	0	0	0	0	1	0	0	1	1	0	1	0	0	0	0	0	0	0	6
9	Commit corporate management	1	1	1	1	1	1	0	1	1	1	1	1	0	0	1	1	1	1	0	0	0	1	0	1	17
10	Integrate lean in everyday business	1	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	0	1	0	0	1	1	1	1	9
11	Develop vision and roadmap	1	1	0	0	0	1	1	1	0	0	1	1	1	1	1	1	0	1	0	1	1	1	1	1	17
12	Use rewards and recognition	0	1	1	0	1	0	0	1	1	1	1	1	1	0	1	1	0	1	0	0	0	0	1	1	14
13	Monitor and audit implementation	1	1	1	0	0	0	0	1	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	0	17
14	Standardize and manage discipline	1	1	1	0	0	0	0	0	0	0	1	0	0	1	1	0	0	0	0	0	0	0	0	0	6
15	Find and share best practices	1	1	0	1	1	1	0	1	0	1	1	0	1	1	1	0	0	1	0	1	0	0	0	1	14
16	Stepwise approach	0	0	1	0	1	0	0	1	1	0	1	0	0	1	0	0	0	0	0	0	0	0	0	0	6
17	Focus on areas and prioritize activities	0	0	0	0	0	0	0	1	1	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	3
18	Invest time and money	1	1	1	1	1	1	1	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	9
19	Benchmark others	1	1	1	1	1	1	1	1	1	1	1	0	0	1	1	0	1	1	1	0	0	0	0	0	16
20	Emphasize team concept	0	1	0	0	1	0	0	0	1	0	0	0	0	0	1	1	0	1	0	0	0	0	0	0	6
21	Use external experts	1	1	0	1	0	1	0	1	0	1	1	0	0	1	1	0	0	1	1	1	1	1	0	0	14
22	Hold regular implementation meetings	0	0	1	0	1	0	0	1	0	1	0	0	0	0	0	0	0	0	0	1	0	0	0	0	5
23	Emphasize safety and job attractiveness	1	1	0	0	1	0	0	0	1	1	1	1	1	1	1	0	0	0	0	0	1	1	1	0	13
24	Use lean tools and methods	1	1	0	1	1	1	0	1	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	21
Dependence Power		18	18	13	8	18	12	6	14	12	15	17	8	8	18	19	10	9	15	8	12	10	13	12	11	

4.5 Partitioning of the Reachability Matrix into Different Levels

In order to rank the variables, each variable's reachability set and antecedent set must be specified [20]. The reachability set for each variable includes the variables that can be reached through the variable of interest, and the antecedent set includes the variables through which the variable of interest can be reached. The level of each variable is determined after these sets and the shared elements are specified [25].

Table 4: Level partitioning

Levels	CSFs Row No.
1	10
2	15,18,19,24
3	13,14,16,17,21,22
4	7,12,20,23
5	2,3,4,8
6	5,6,9,115
7	1

4.6 Drawing the ISM

After determining the relationships and the level of the variables, they are translated into a model [11]. For this purpose, we first arrange the variables according to their level. In this research, the variables are placed in 7 levels, as illustrated in Figure 1.

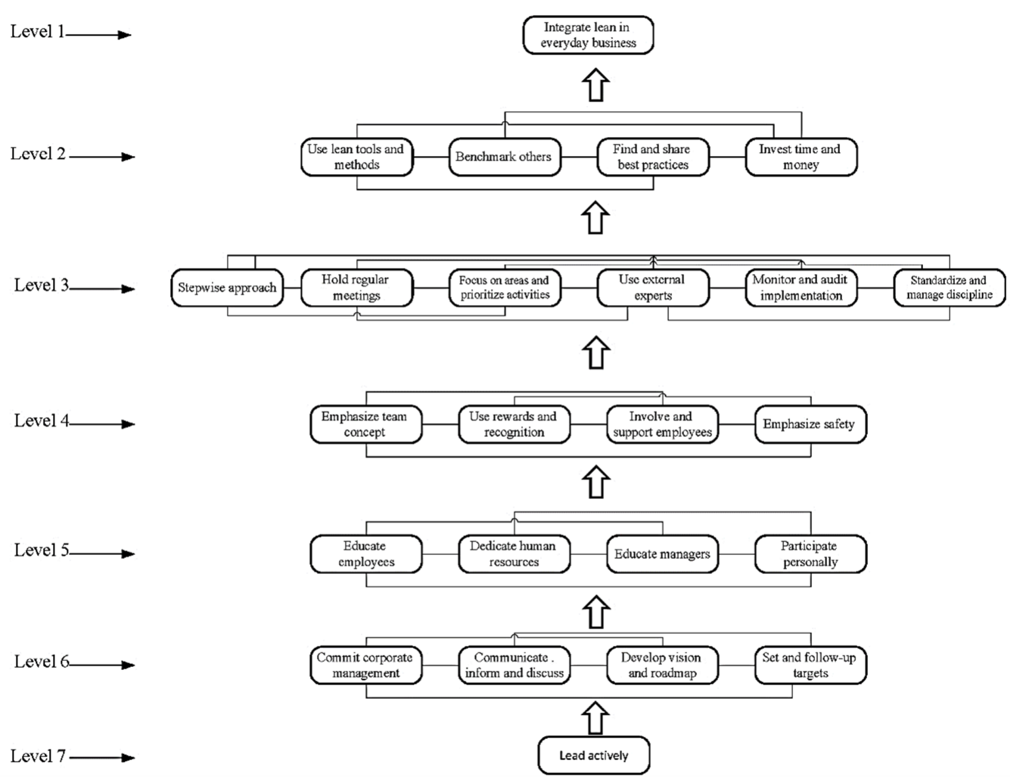

Figure 1: Interpretive structural model

4.7 Analyzing the Driving and Dependence Power Using the MICMAC Diagram

The MICMAC matrix is employed to analyze the reciprocal influence between variables and their categorization [28]. This analysis categorizes variables into four groups based on their driving and dependence power. The first category includes autonomous variables with weak driving and power dependence. These variables exhibit limited and weak connections to the system. No variables fell in this category in the current research, suggesting a substantial connection between variables in the developed model. Dependent variables are the second category, with weak driving but a strong dependence. These variables primarily consist of specific results that are the product of certain factors. Rarely can these variables serve as the basis for other variables. The third category, known as linkage, includes variables with strong driving and dependence. The variables are non-static, as changes to them can affect the whole system. The fourth category consists of variables with strong driving but weak dependence. This group serves as the fundamental basis of the model and should be given primary emphasis when initiating the system [11]. Figure 3 depicts the positions of each CSF.

Next, with the assistance of experts, these binary numbers from Table 3 are replaced with appropriate fuzzy values using Table 5, and a fuzzy direct reachability matrix (FDRM) is developed following Table 6. In fuzzy ISM, experts are free to consider the degree of relatedness when deciding whether to include or exclude an element as related or unrelated to another. In this study, experts could consider even the weakest degree of relation (0.1) as a relationship. In this FDRM, the sum of values between rows and columns indicates the driving and dependence power of each of the variables, respectively. The results are then used for fuzzy MICMAC analysis, as depicted in Figure 4 [29, 30, 31].

The selection of the fuzzy membership function for the seven linguistic variables is attributed to rank as follows.

The set of values related to the linguistic variable $= \{N, NL, L, M, H, VH, F\} = T(x)$ Variation range of the reference set $= [0, 1] = U$

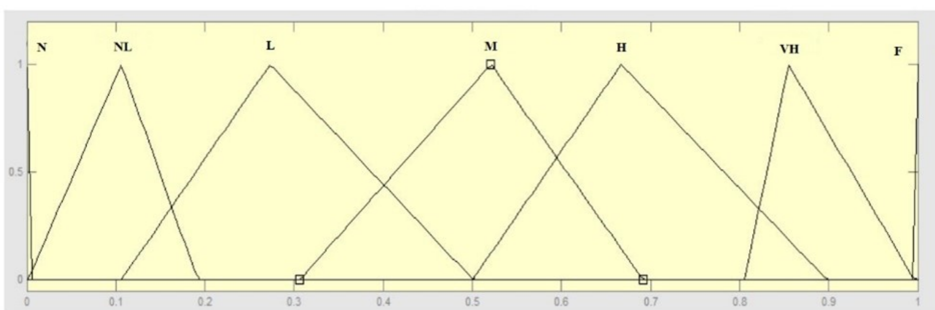

Figure 2: Membership function of linguistic variables (adopted from [32, 33])

The values presented in Table 3 are developed in the form of an FDRM using the values of Table 5, the results of which are reported in Table 6. Finally, the FDRM provides the possibility of MICMAC fuzzy analysis [30, 34].

Table 5: Scheme for the degree of perceived dominance factor (Adopted from [32, 33])

Value on the Scale	Fuzzy Triangular Numbers	Grade	Dominance of Interaction
0	(0,1,0)	N	No
0.1	(0,0.1,0.2)	NL	Very Low
0.3	(0.1,0.3,0.5)	L	Low
0.5	(0.3,0.5,0.7)	M	Medium
0.7	(0.5,0.7,0.9)	H	High
0.9	(0.8,0.9,1)	VH	Very high
1	(1,1,1)	F	Full

Table 6: Uzzy direct reachability matrix with driving and dependence power

		Use lean tools and methods	Emphasize safety and job attractivenes s	Hold regular implementation meetings	Use external experts	Emphasize team concept	Benchmark others	Invest time and money	Focus on areas and prioritize activities	Stepwise approach	Find and share best practices	Standardize and manage discipline	Monitor and audit implementation	Use rewards and recognition	Develop vision and roadmap	Integrate lean in everyday business	Commit corporate management	Dedicate human resources	Involve and support employees	Set and follow-up targets	Communicate. inform and discuss	Educate managers	Educate employees	Participate personally	Lead actively	**Driving Power**
#	**Critical Success Factors**	**24**	**23**	**22**	**21**	**20**	**19**	**18**	**17**	**16**	**15**	**14**	**13**	**12**	**11**	**10**	**9**	**8**	**7**	**6**	**5**	**4**	**3**	**2**	**1**	
1	Lead actively	**1**	**1**	**0.7**	**0.9**	**0.9**	**1**	**1**	**0.9**	**0.9**	**0.7**	**0.5**	**0.5**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**0.9**	**0.9**	**0.7**	**1**	21.5
2	Participate personally	**0.9**	**1**	**0.9**	**0**	**1**	**0.3**	**0**	**0**	**0.5**	**0.7**	**0.9**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0.7**	**0**	**1**	**1**	**1**	9.9
3	Educate employees	**1**	**1**	**0.7**	**0**	**0.9**	**0**	**0**	**0**	**1**	**0.9**	**0.9**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0.9**	**0**	**1**	**0.3**	**0.3**	8.9
4	Educate managers	**0.9**	**1**	**1**	**0**	**1**	**0.7**	**0**	**1**	**1**	**1**	**1**	**0.7**	**0**	**0.9**	**0.7**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**0.9**	**0.5**	19.3
5	Communicate. inform and discuss	**0.5**	**0.9**	**0.9**	**0.3**	**0.1**	**0.7**	**0.7**	**0**	**0**	**0.5**	**0**	**0**	**0**	**0.5**	**0.1**	**0**	**0**	**0.9**	**0.7**	**1**	**0.3**	**0.3**	**0.9**	**1**	10.3
6	Set and follow-up targets	**1**	**0**	**0**	**0**	**1**	**0.9**	**0.5**	**1**	**1**	**0.9**	**1**	**0**	**0**	**1**	**0.9**	**1**	**1**	**0.5**	**1**	**0**	**1**	**1**	**0**	**0**	14.7
7	Involve and support employees	**0**	**1**	**0**	**0**	**1**	**0**	**0**	**0**	**0**	**0.7**	**0**	**0**	**0.3**	**0.1**	**0.1**	**0**	**0.3**	**1**	**0**	**1**	**0**	**0**	**1**	**0**	6.5
8	Dedicate human resources	**0.7**	**0**	**0**	**0**	**0.9**	**0**	**0**	**0**	**0**	**0**	**0.7**	**0**	**0**	**0.3**	**0.3**	**0**	**1**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	3.9
9	Commit corporate management	**1**	**1**	**1**	**1**	**1**	**1**	**0**	**1**	**1**	**1**	**1**	**1**	**0**	**0**	**1**	**1**	**1**	**1**	**0**	**0**	**0**	**1**	**0**	**1**	17
10	Integrate lean in everyday business	**1**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0.9**	**1**	**0.7**	**0**	**1**	**0**	**0**	**0.7**	**0.7**	**1**	**1**	8
11	Develop vision and roadmap	**0.7**	**0.5**	**0**	**0**	**0**	**0.1**	**1**	**0.5**	**0**	**0**	**0.5**	**0.3**	**0.9**	**1**	**0.9**	**0.5**	**0**	**0.3**	**0**	**0.7**	**1**	**1**	**1**	**1**	11.9
12	Use rewards and recognition	**0**	**1**	**1**	**0**	**1**	**0**	**0**	**1**	**1**	**1**	**1**	**1**	**1**	**0**	**0.7**	**0.3**	**0**	**1**	**0**	**0**	**0**	**0**	**1**	**0.9**	12.9
13	Monitor and audit implementation	**1**	**1**	**0.7**	**0**	**0**	**0**	**0**	**0.3**	**0**	**0**	**0.5**	**1**	**0.7**	**0.9**	**0.9**	**0.7**	**0.3**	**0.1**	**0.9**	**0.9**	**1**	**1**	**1**	**0**	12.9
14	Standardize and manage discipline	**0.7**	**0.9**	**0.9**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**1**	**0**	**0**	**0.5**	**0.7**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	4.7
15	Find and share best practices	**1**	**0.5**	**0**	**1**	**1**	**1**	**0**	**0.1**	**0**	**1**	**1**	**0**	**1**	**1**	**0.9**	**0**	**0**	**0.9**	**0**	**0.7**	**0**	**0**	**0**	**1**	12.1
16	Stepwise approach	**0**	**0**	**0.7**	**0**	**0.5**	**0**	**0**	**0.7**	**1**	**0**	**1**	**0**	**0**	**0.9**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	4.8
17	Focus on areas and prioritize activities	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**1**	**1**	**0**	**0**	**0**	**0**	**0.9**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	2.9
18	Invest time and money	**0.3**	**0.1**	**0.1**	**0.3**	**0.1**	**0.5**	**1**	**0**	**0**	**0**	**0**	**0**	**0**	**1**	**0.1**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	3.5
19	Benchmark others	**0.9**	**0.9**	**0.5**	**1**	**0.7**	**1**	**0.1**	**0.3**	**0.9**	**1**	**1**	**0**	**0**	**0.9**	**1**	**0**	**1**	**0.9**	**0.9**	**0**	**0**	**0**	**0**	**0**	13
20	Emphasize team concept	**0**	**1**	**0**	**0**	**1**	**0**	**0**	**0**	**0.1**	**0**	**0**	**0**	**0**	**0**	**0.5**	**0.9**	**0**	**1**	**0**	**0**	**0**	**0**	**0**	**0**	4.5
21	Use external experts	**1**	**0.3**	**0**	**1**	**0**	**1**	**0**	**0.7**	**0**	**1**	**1**	**0**	**0**	**0.9**	**0.9**	**0**	**0**	**0.3**	**1**	**1**	**1**	**1**	**0**	**0**	12.1
22	Hold regular implementation meetings	**0**	**0**	**1**	**0**	**0.9**	**0**	**0**	**0.9**	**0**	**0.5**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**1**	**0**	**0**	**0**	**0**	4.3
23	Emphasize safety and job attractiveness	**0.7**	**1**	**0**	**0**	**0.7**	**0**	**0**	**0**	**0.5**	**0.5**	**1**	**1**	**1**	**0.7**	**0.3**	**0**	**0**	**0**	**0**	**0**	**1**	**1**	**1**	**0**	10.4
24	Use lean tools and methods	**1**	**0.9**	**0**	**1**	**0.9**	**1**	**0**	**0.9**	**0**	**0.7**	**0.9**	**1**	**1**	**0.5**	**1**	**0.5**	**0.9**	**0.9**	**1**	**0.9**	**1**	**1**	**0.7**	**1**	18.7
	Dependence Power	15	15	10	6.5	15	9.2	4.3	10	9.9	12	15	6.5	6.9	14	13	7.6	7.5	12	7.5	11	8.9	12	11	9.7	

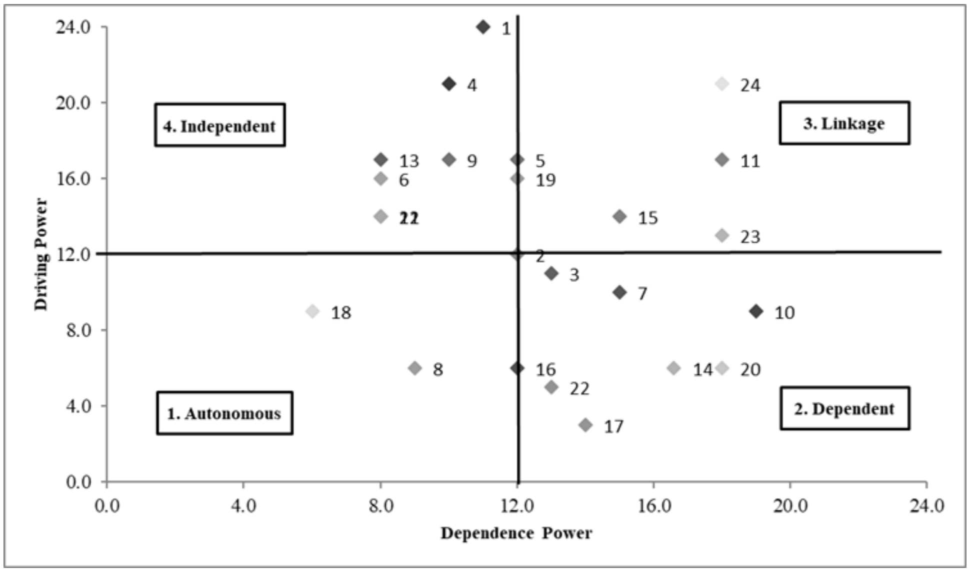

Figure 3: MICMAC analysis

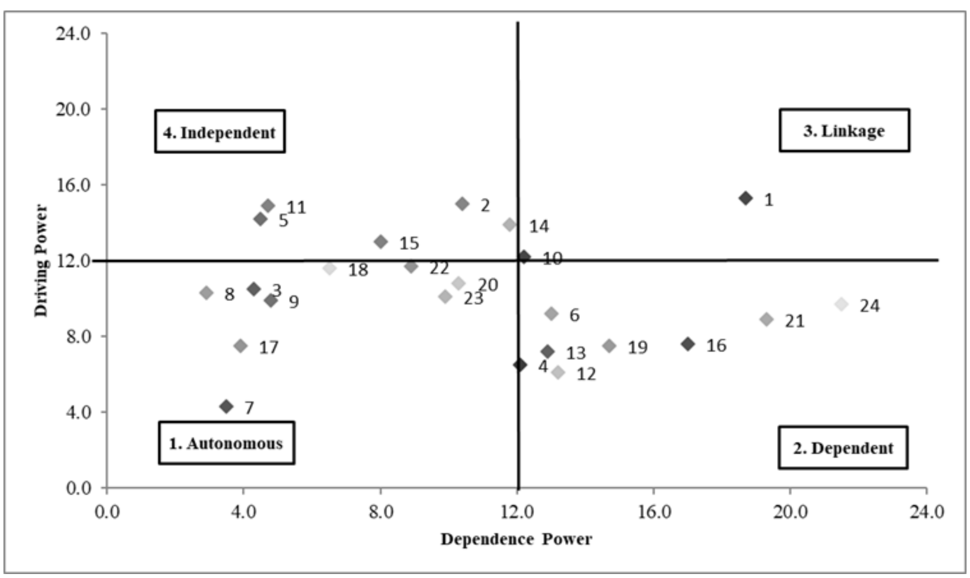

Figure 4: Fuzzy MICMAC analysis

In Table 7, the position of each CSF in fuzzy and nonfuzzy MICMAC analyses is specified.

Table 7: Driving and dependence power of critical success factors

Group	Critical Success Factors		
	Nonfuzzy		Fuzzy
1. Autonomous	18,8	2*16	3,7,8,9,17,18,20,22,23
2. Dependent	3,7,10,14,16,17,20,22		4,6,12,13,16,19,21,24
3. Linkage	11,15,23,24	2*5,19	1,10
4. Independent	1,4,6,9,13,22		2,5,11,14,15

5 Conclusion

As organizations are interested in implementing lean production systems, it would be of particular importance to identify CSFs to facilitate their implementation and provide a conceptual model in this area. ISM provides a proper framework and order for such systems. Thereby, decision-makers are given a clear picture of how various factors influence the system as a whole and how to best proceed to reach their objective. The current research utilized ISM to identify the type of relationship between factors and determine the levels of CSFs. Then, seven levels were assigned to the factors based on a summary of expert opinions and an ISM analysis. The conceptual model of CSFs in lean production indicates that leadership is the foundational element of the model and serves as the driver for other factors. Consequently, leadership is deemed the foremost CSF in lean production. When establishing lean production systems in organizations, senior managers should focus on higher levels and the CSFs underlying the model.

Based on the findings from Figures 3 and 4, as well as Table 7, it can be inferred that the fuzzy MICMAC analysis offers superior insights into the analysis of driving and dependence power. This eliminates the issue posed by the presence of border CSFs in the nonfuzzy MICMAC analysis. The fuzzy MICMAC analysis helped identify several key factors that are highly effective in achieving success in implementing lean production. These factors include “participate personally”, “communicate, inform, discuss”, “develop vision and roadmap”, “standardize and manage discipline”, and “find and share best practices”. These factors exhibit high driving and weak dependence power. The CSFs of leadership and lean business are linkage factors, which exhibit a significant correlation with the factors at the preceding levels of the model and a moderate driving on other factors.

The key factors of “educate managers”, “set and follow-up targets”, “use rewards and recognition”, “monitor and audit implementation”, “stepwise approach, “benchmark others”, “use external experts”, and “use lean tools and methods” are also dependent on the factors of the previous levels in the model. Other factors with low driving and dependence power include “educate employees”, “involve and support employees”, “dedicate human resources”, “commit corporate management”, “focus on areas and prioritizing activities”, “invest time and money”, “emphasize team concept”, “hold regular implementation meetings”, and “emphasize safety and job attractiveness” in the autonomous category.

In future studies, fuzzy-based conceptual models can be developed by FDRM or fuzzy ISM that builds on a questionnaire with linguistic variables and completely fuzzy calculations. Alternatively, ISM can be considered to develop models of quantitative approaches. In the current study, ISM was used as a tool whose performance is based on the judgment of experts for the development of the model.

Conflict of Interest: ”The authors declare no conflict of interest.”

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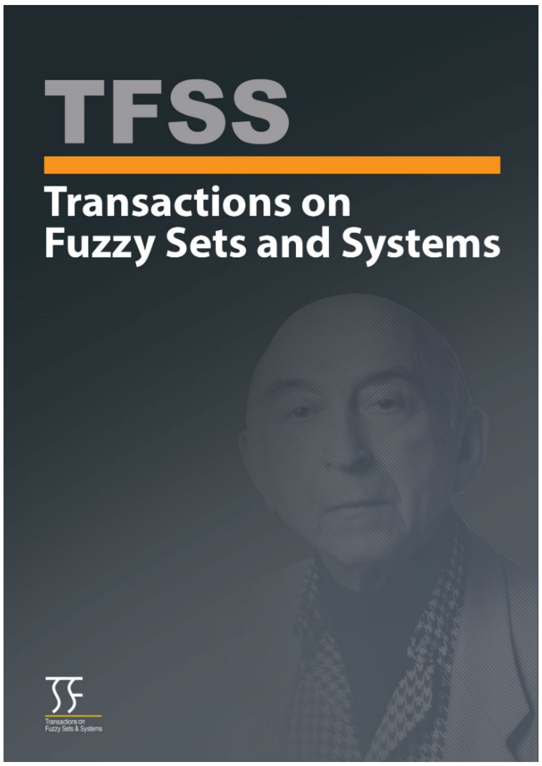

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Shannon Entropy Analysis of Serum C-Terminal Agrin Fragment as a Biomarker for Kidney Function: Reference Ranges, Healing Sequences and Insights

Mehmet Şengönül*

Abstract. This article focuses on evaluating the success or failure of kidney transplantation using Shannon entropy, fuzzy sets, and Scaf. The data for Scaf references used in this study for both healthy individuals and kidney transplant recipients have been collected from the relevant literature. For both groups, Scaf's Shannon entropy values have been calculated using an appropriate probability density function and formulation, and sequences have been generated for CAF and Scr biomarkers from entropy values, with findings interpreted. These sequences are called healing sequences. A case study demonstrating whether the transplant procedure was successful or unsuccessful was presented using sequences that we refer to as healing sequences. In this context, the utilization of mathematical tools such as fuzzy sets, Shannon entropy, and reference intervals becomes evident. These tools provide a systematic and quantitative approach to assessing the outcomes of kidney transplantation. By leveraging the principles of Shannon entropy, we gain insights into the degree of unpredictability and fuzziness associated with biomarker values, which can be indicative of the transplant's success. Furthermore, the concept of healing sequences provides a valuable framework for tracking the progression of patients post-transplantation. By monitoring changes in CAF and Scr biomarkers over time, healthcare professionals can make informed decisions and interventions to ensure the well-being of kidney transplant recipients.

AMS Subject Classification 2020: 03E72
Keywords and Phrases: Healing sequence, Shannon entropy, Fuzzy set, Renal transplant, Biomarker.

1 Introduction

In recent years, a rapid increase in the applications of fuzzy sets and fuzzy logic across various disciplines has been observed. One of these disciplines is medicine. For instance, in medical diagnostics [1], ECG interpretation [2] and [3], image processing [4], pacemaker control [5], anesthesia control [6], lung disease control [7] fuzzy sets or fuzzy logic have been widely used. Similarly, when we look at the literature, it will be seen that Ahmad et al., in [8], have used fuzzy logic-based systems to monitor chronic kidney diseases. Furthermore, Hamedan [9] and Norouzi [10] have used fuzzy expert systems to predict kidney diseases and predicting renal failure progression in chronic kidney disease, respectively. From another point of view, the ECG signal process was investigated by Czogala [11]. In [12] Rakkus, in [13] Tunç and Bloch[14] followed a different approach to medicine using fuzzy sets. Generally, when examining the previous studies related to this field, the severity of disease symptoms is transformed into fuzzy clusters, and with the assistance of

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expert systems, tasks such as sequencing during organ transplantation and determining the nature of the disease is addressed [15].

A literature review has revealed that there are either very few or no scientific studies specifically focusing on the compatibility or incompatibility of transplanted organs within the context of organ transplantation. This area has remained an open problem. As a starting point, Kılınç [16] has conducted research on patients with systemic lupus erythematosus, providing information about the condition of the tissue using entropy. However, the health status of an organ has not been investigated by Kılınç.

The scientific innovation, in this study, is that after kidney transplantation it is the application of Shannon entropy for the purpose of analyzing Serum C-Terminal Agrin Fragment (Scaf) as a biomarker to evaluate kidney function. While the use of biomarkers in clinical research and medical applications is increasing, the specific application of Shannon entropy to Scaf interval is a new approach in this study. Researchers can assess the degree of impairment in kidney function by calculating the Shannon entropy values of Scaf intervals for both healthy individuals and kidney transplant recipients. The findings indicate that the Shannon entropy of Scaf values is an indicator of the success or failure of kidney transplantation. This innovative use of Shannon entropy adds new insights to the assessment of kidney function using Scaf as a biomarker, potentially contributing to advanced diagnostic and treatment approaches in the field of nephrology. Thus, we believe that employing Shannon entropy in conjunction with a biomarker like Scaf can yield more insights into the health status of kidney transplant patients and enable more effective monitoring of kidney functions. It is important to note that the choice of using Shannon entropy is specific to this study, and while other types of entropy could also be used; the results should be evaluated accordingly.

2 Preliminaries

At the core of our research are reference intervals, fuzzy sets, and Shannon entropy. Therefore, in the following subsections, we will provide explanations of kidney reference intervals and subsequently, achieve the main goal of the study, we will define fuzzy sets and Shannon entropy.

2.1 References Intervals

Biomarkers are used to provide information about a biological condition, in clinical research and medical practice for diagnostic, prognostic, and therapeutic purposes. Biomarkers can include molecules, genes, cells, or physiological functions that are objectively measured and evaluated as indicators of normal biological processes, pathogenic processes, or pharmacologic responses to therapeutic interventions [17]. In addition to its wide use as a biomarker associated with neuromuscular junction (NMJ) dysfunction [18], C-terminal agrin fragment (CAF) has also been utilized by Yu et al. as a biomarker to evaluate kidney function after kidney transplantation [19]. Scaf consists of the C-terminal fragment of agrin, a protein secreted from kidney glomeruli. Agrin is an important protein for the formation and maintenance of the glomerular filtration barrier in the kidney. Scaf is a fragment of agrin that is released into circulation as a result of protein breakdown. As mentioned above, SCAF has been considered a promising biomarker for the evaluation of kidney function in recent years. Based on medical research, it has been demonstrated that when the kidney glomeruli are damaged, Scaf levels increase [20]. Therefore, biomarkers have great importance for the early diagnosis and treatment of significant health problems such as chronic kidney disease (CKD). Here are some scientific papers about Serum C-Terminal Agrin Fragment (Scaf) as a biomarker for evaluating kidney function [19, 18, 20]. These papers investigate the utility of Scaf as a biomarker for evaluating kidney function in various patient populations, highlighting its potential in clinical applications for the diagnosis, prognosis, and monitoring of kidney disease.

2.2 Fuzzy Sets

The following sentence emphasizes that medical data is a very rich source for fuzzy sets theory.

"Everything in medicine is fuzzy [21]."

Fuzzy sets are mathematical sets in which each element is defined by a certain degree of uncertainty. Considering an interval [a, b], where a and b are real numbers, it may initially seem that there is no specific uncertainty associated with the elements within this interval. However, this assumption may not reflect the reality. Therefore, a membership function can be used to provide a certain degree of uncertainty for each element in the interval [a, b]. This function, known as the membership function, expresses the degree of membership of a particular element in the interval. More formally, a fuzzy set is defined as follows:

Let $\mathscr{X}$ be a nonempty crisp set, $\mathbb{R}$ and $\mathbb{N}$ for the set of all real and natural numbers, respectively. According to Zadeh, a fuzzy subset of $\mathscr{X}$ is a nonempty subset $\{(x, u(x)) : x \in \mathscr{X}\}$ of $\mathscr{X} \times [0, 1]$ for some function $u : \mathscr{X} \to [0, 1]$, [22]. Consider a function $u : \mathbb{R} \to [0, 1]$ as a subset of a nonempty base space $\mathbb{R}$. If there exist reference functions L and R and scalars α and β, then a continuous fuzzy number (or fuzzy set) u can be represented in the $L - R$ form, where the membership function $u(x)$ of u, is defined as

$$u(x) = \begin{cases} L\left(\frac{\lambda - x}{\alpha}\right), & x \in [\lambda - \alpha, \lambda) \\ R\left(\frac{x - \lambda}{\beta}\right), & x \in [\lambda, \lambda + \beta] \end{cases} . \tag{1}$$

The notation $\lambda \in \mathbb{R}$ is called the mean value of u, α and β are called the left and right spreads, respectively. The support of u is stated as an interval $[\lambda - \alpha, \lambda + \beta]$. If take $L(x) = R(x) = 1 - x$ then the graphic shape of the membership of fuzzy set u will be triangular shape which is mostly used in the application of fuzzy sets [12]. In this study, we will also use a triangular membership functions.

2.3 Shannon Entropy

In the fuzzy set theory, measuring the degree of fuzziness of a fuzzy set is an important aspect, and various methods have been developed to determine it. Initially, it was believed that the degree of fuzziness could be quantified as the distance between a fuzzy set and its nearest non-fuzzy set. However, this method was later replaced by the use of entropy as a measure of fuzziness [23] and [24]. Thus, what exactly is entropy and how is it used to measure the fuzziness of a fuzzy set? The answer to this question is given as "Entropy is an information theory measure that quantifies the amount of uncertainty or unpredictability in a given set of data." In the context of fuzzy set theory, entropy is used to measure the degree of fuzziness of a fuzzy set by taking into account the membership values of the set's elements. Essentially, the higher the entropy value, the greater the degree of fuzziness of the set. Therefore, entropy provides a useful tool for evaluating the degree of fuzziness in fuzzy sets, which in turn can be used in a variety of applications, such as decision-making and pattern recognition. The entropy is defined as follows:

Let $u \in \mathscr{F}$ and $u(x)$ be the membership function of the fuzzy set u and consider the function $H : \mathscr{F} \to \mathbb{R}^+$, where $\mathscr{F}$ denotes all fuzzy sets on real numbers set.

If the function H satisfies the following conditions, then H is called an entropy function [25]:

(i) $H(u) = 0$ iff u is crisp set,

(ii) $H(u)$ has a unique maximum, if $u(x) = \frac{1}{2}$, for all $x \in \mathbb{R}$

(iii) For $u, v \in \mathscr{F}$, if $u(x) \leq v(x)$ for $u(x) \leq \frac{1}{2}$ and $u(x) \leq v(x)$ for $u(x) \geq \frac{1}{2}$ then $H(u) \geq H(v)$,

(iv) $H(u^c) = H(u)$, where u^c is the complement of the fuzzy set u.

Let's suppose that the function $h : [0, 1] \to [0, 1]$ satisfies the following properties:

1. Monotonically increasing at $[0, \frac{1}{2}]$ and decreasing $[\frac{1}{2}, 1]$,

2. $h(x) = 0$ if $x = 0$ and $h(x) = 1$ if $x = \frac{1}{2}$.

The function h is the called entropy function and the equality $H(u(x)) = h(u(x))$ holds for $x \in \mathbb{R}$. Some well-known entropy functions are given as follows:

$h_1(x) = 4x(1-x)$, $h_2(x) = -x\ln x - (1-x)\ln(1-x)$, $h_3(x) = \min\{2x, 2-2x\}$ and

$$h_4(x) = \begin{cases} 2x, & x \in [0, \frac{1}{2}] \\ 2(1-x), & x \in [\frac{1}{2}, 1] \end{cases}.$$

Note that the function h_1 is the logistic function, h_2 is the called Shannon function and h_3 is the tent function.

The probability density function (PDF) for a continuous random variable defined on the interval $[a, b]$ is a non-negative function $p(x)$ such that the integral of $p(x)$ over the entire interval equals 1. In other words, for any subset A of the interval $[a, b]$, the probability of the random variable taking a value in S is given by the integral of over A. Mathematically, this can be expressed as follows [24]:

$$0 \le \int_{x\in A} p(x)dx \le 1 \tag{2}$$

The PDF can be used to determine the probability of the random variable taking on a value in any subset of $[a, b]$, and can also be used to calculate expected values, variances, and other statistical properties of the random variable.

If $p(x)$ is any probability density function then the Shannon entropy of $p(x)$ is equal to [26]:

$$H = -\int_{x\in(-\infty,\infty)} p(x)\log_2 p(x)dx. \tag{3}$$

Let us suppose that $u(x)$ be any triangular membership function of fuzzy set u over a real numbers interval $[a, b]$ and

$$\Delta = \int_{x\in Supp\ u} u(x)dx, \tag{4}$$

where the notation $Supp\ u$ is $Supp\ u = \{x : u(x) \ge \lambda, \lambda \in [0, 1]\}$. Then the function

$$P(x) = \frac{1}{\Delta}u(x) \tag{5}$$

is satisfy the conditions of probability density functions. After that, in the following calculations, the function $P(x)$ will be considered the probability density function on the interval $[a, b]$ and the interval $[a, b]$ will be any interval of biomarkers for kidney diseases. Thus the Shannon entropy value on the interval $[a, b]$ will be equal to

$$H_{[a,b]} = -\int_{x\in Supp\ u} P(x)\log_2 P(x)dx. \tag{6}$$

Since Shannon's entropy can be thought of as the measure of "information content" in a variable, in the following section and next section, we will compare the Shannon entropy values of CAF(pM), Scr(μmol/l) and CysC (mg/l) obtained for healthy men and women in different age ranges.

Let u and v be two fuzzy set on any crisp set X. Then the Koczy similarity [27] of fuzzy sets u and v is defined as follows:

$$S(u, v) = \frac{1}{1 + d(u, v)} \tag{7}$$

where $d(u, v) = \max_{i=1,2,3}\{|u_i - v_i|\}$ and according to (1) the u_1, u_2, u_3, v_1, v_2 and v_3 are $u_1 = \lambda_1 - \alpha_1$, $u_2 = \lambda_1$ and $u_3 = \lambda_1 + \beta_1$, $v_1 = \lambda_2 - \alpha_2$, $v_2 = \lambda_2$ and $v_3 = \lambda_2 + \beta_2$. Let $(H_n^i) = \left(-\int_{x\in(-\infty,\infty)} P_n(x)\log_2 P_n(x)dx\right)$,

($n \in \mathbb{N}$) be the entropy sequence of the i^{th} biomarker of a patient who has undergone a kidney transplant and H^i_{both} be the entropy value calculated for healthy male or female individuals according to i^{th} biomarker. Let's modify the expression given in (7) to provide the following definition.

Definition 2.1. The sequence (H^i_n) is called healing sequence according to i^{th} biomarker that is $i \in \{CAF, Scr, Cys\}$, another word, if i is one of the CAF, Scr, or Cys. Let

$$\lim_n S(H^i_{both}, H^i_n) = \frac{1}{1+\lim_n d(H^i_{both}, H^i_n)} = \begin{cases} \text{Successful transplant}, & \text{if } \lim_n S(H^i_{both}, H^i_n) \in (\frac{1}{2}, 1] \\ \text{Unsuccessful transplant}, & \text{if } \lim_n S(H^i_{both}, H^i_n) \in [0, \frac{1}{2}] \end{cases}. \tag{8}$$

The limit value given in (8) is called the success of transplantation.

It may not always be possible to obtain a sequence using the entropy values obtained for a biomarker. In this case, using a sequence that has terms close in value to the terms of the obtained entropy sequence can be a solution. The (H^i_n) sequences used here will be taken as a sequence of real numbers that are approximately equal to the terms of the actual entropy sequence. As an example of the use of Definition 2.1, we will give the following case study:

3 A Case Study

In the [19], the intervals of serum levels of biomarkers among healthy subjects of different age and sex groups are given by Yu et al. in Table 1 and in Table 2. In this study the valuable data which obtained by Yu et al. will play a fundamental role.

Table 1: Serum levels of biomarkers among healthy subjects of different age and sex groups

Age of Groups	Gender	Numbers	CAF (pM)	Scr(μumol/l))	Sys C(mg/l)
18-34 Years	Male	25	131.2 ±71.6	110.6 ± 11.8	0.78 ±0.10
	Female	29	120.3 ± 56.1	89.3 ± 7.0	0.70 ± 0.09
	Both	54	125.3 ±63.3	99.2 ± 14.3	0.74 ± 0.10
35-49 Years	Male	25	138.6 ± 34.8	104.8 ± 13.7	0.77 ± 0.08
	Female	31	117.5 ±44.6	82.7 ±9.8	0.65 ± 0.10
	Both	56	126.9 ± 41.5	92.6± 16.0	0.70 ± 0.11
50-64 Years	Male	25	149.1±49.8	108.6 ±13.3	0.91± 0.22
	Female	19	202.6± 42.9	94.7 ± 11.3	0.82 ± 0.15
	Both	44	172.2 ± 53.6	102.6 ± 14.2	0.87 ± 0.20
≥65 Years	Male	22	154.1 ± 47.3	112.0 ± 12.6	1.09± 0.32
	Female	24	192.3 ± 53.3	91.8 ±13.7	0.91 ± 0.14
	Both	46	174.1 ± 53.6	101.5 ±16.5	1.00± 0.26

In this paper continuous data were expressed as mean ± standard deviation (SD) or median (minimum; maximum). That is, the means of the notation $a \pm b$ is the real number a is denotes arithmetic mean and the real number b is denotes standard deviation. In our calculations by using $a \pm b$ we will obtain support of fuzzy sets and these supports will denote $[a - b, a + b]$.

Again, in [19], Yu et al. have observed that time course changes in Serum CAF, Creatinine, eGFR (CKD-EPI), Cystatin C and NGAL in patients undergoing kidney transplantation and these changes are given in Table 2. If we compare the data in Table 1 and Table 2, we can observe that there is a certain variation

Table 2: Time course changes in serum CAF, creatinine, eGFR (CKD-EPI), cystatin C and NGAL in patients undergoing kidney transplantation.

	Before Tx	1 day after	2 days after	6 months after
Serum CAF (pM)	921.7 (618.1, 1508.8)	360.4 (85.9, 1291.3)*	164.1 (6.8, 977.3)*	164.8 (74.3, 338.0)*
Creatinine (lmol/l)	845.5 (476.0, 1856.0)	365.0 (115.0, 1254.0)*	204.5 (80.0, 1275.0)*	144.0 (67.0, 320.0)*
eGFR (ml/min/173 m2)	5.8 (2.6, 11.1)	17.4 (4.2, 58.2)*	35.2 (4.2,8 8.3)*	52.6 (20.1, 121.7)*
Cystatin C (mg/l)	5.49 (1.00, 12.00)	2.00 (1.00, 4.18)*	2.00 (1.00, 4.47)*	1.42 (0.77, 3.60)*
NGAL (ng/ml)	911.0 (305.3, 1783.2)	201.1 (71.0, 654.1)*	158.9 (52.5, 994.4)*	93.1 (11.9, 186.5)*

in the values. However, they do not provide us with information about the uncertainties contained in these variations. Therefore, it is necessary to calculate their entropies to determine these uncertainties. This will be performed in this study. If you need to more information about the notations in Table 1 and Table 2, you can see [19].

4 The Entropies of Serum CAF, Creatinine and Cystatin C in Patients Undergoing Kidney Transplantation

In this section, the biomarker intervals will convert into fuzzy sets according to age and sex using appropriate membership functions and determined their entropies values. Afterwards, obtained entropy values will be evaluated based on their magnitudes to determine the chaotic state contained in the biomarker intervals.

According to Table 1, CAF values for individuals who are male and aged between 18-34 are given in the interval $[59.6, 202.8]$. According to this, for males and aged between 18-34, the membership function of the CAF, $CAF_{(18-34)M}(x)$, is equal to

$$CAF_{(18-34)M}(x) = \begin{cases} \frac{x-59.6}{131.2-59.6}, & x \in [59.6, 131.2] \\ \frac{202.8-x}{202.2-131.2}, & x \in (131.2, 202.8] \\ 0, & \text{otherwise} \end{cases} . \tag{9}$$

If we consider equality (4) then we see that $\Delta = 71.6$. From equality (5) we obtain that the probability density function of CAF for individuals who are male and aged between 18-34 as follows:

$$P_{CAF(18-34)M}(x) = \frac{1}{\Delta} CAF_{(18-34)M}(x) = \begin{cases} \frac{x-59.6}{5126.56}, & x \in [59.6, 131.2] \\ \frac{202.8-x}{5126.56}, & x \in (131.2, 202.8] \\ 0, & \text{otherwise} \end{cases} . \tag{10}$$

The function $P_{(18-34)M}(x)$ given in (11) satisfies the conditions of the probability density function which it given in (2). Thus, the entropy of the CAF for individuals who are male and aged between 18-34 is computed as follows:

$$H_{CAF(18-34)M} = -\int_{x \in Supp\ CAF(18-34)M} P_{(18-34)M}(x) \log_2 P_{(18-34)M}(x) dx = 6.88324.$$

Similarly to above, since $\Delta = \frac{1}{56.1}$;

$$P_{CAF(18-34)F}(x) = \frac{1}{\Delta} CAF_{(18-34)F}(x) = \begin{cases} \frac{x-64.2}{3147.21}, & x \in [64.2, 120.3] \\ \frac{176.4-x}{3147.21}, & x \in (120.3, 176.4] \\ 0, & \text{otherwise} \end{cases} . \tag{11}$$

and the Shannon entropy of the CAF for individuals who are female and aged between 18-34 is

$$H_{CAF(18-34)F} = -\int_{x \in Supp\ CAF(18-34)F} P_{(18-34)F}(x) \log_2 P_{(18-34)F}(x) dx = 6.53128.$$

For individuals who are male and female and aged between 18-34 according to data of [19], similar calculations can made for Scr(μumol/l) and Sys C(mg/l), it can be seen that $H_{Scr(18-34)M} = 4.28206$, $H_{Scr(18-34)F} = 3.5287$. The $H_{Cys(18-34)M} = -2.60058$ and $H_{Cys(18-34)F} = -2.75258$. The Shannon entropies of Scr(μumol/l) and Sys C(mg/l) can be calculated for other age groups with similar calculations. These are given in Table 3 as a table. The term in the expression 6.46551 represents a very small imaginary component, which may

Table 3: Shannon entropy of biomarkers among healthy subjects of different age and sex groups

Age of Groups	Gender	Shannon Entropy of CAF (pM)	Shannon Entropy of Scr(μumol/l)	Shannon Entropy of Sys C(mg/l)
18-34 Years	Male	6.88324	4.28206	-2.60058
	Female	6.53128	3.5287	-2.75258
	Both	6.70548	4.55929	-2.60076
35-49 Years	Male	5.84236	4.49745	-2.92251
	Female	3.81713	4.01413	-2.70018
	Both	6.09639	4.72135	-2.46308
50-64 Years	Male	4.50536	4.4547	-1.46308
	Female	6.14425	3.63403	-2.01562
	Both	6.46551	4.54917	-1.60058
≥65 Years	Male	6.28512	4.3767	-0.922509
	Female	6.45741	4.49745	-2.11515
	Both	6.46551	4.76574	-1.22207

arise due to rounding errors or other computational reasons. Therefore, the real part of the result should be considered as 6.46 to evaluate the Shannon entropy. This is very important: Entropy is a concept that measures the uncertainty of a probability distribution. If a negative Shannon entropy is obtained, it is indicated that this may be due to the characteristics of the probability distribution. For example, if the sum of probabilities is not equal to 1 or if there is an inverse relationship between the probabilities, a negative entropy can be obtained. This indicates that the distribution is regular and predictable, and the information content is low. If the entropy value is 0, the distribution becomes completely predictable, and there is no uncertainty.

In some cases, a negative entropy result can be a realistic outcome. For example, if a group of items in a dataset exhibits a more distinct characteristic than all other items, then the probability distribution for that group may have a lower entropy and this entropy could be negative. Therefore, negative entropy in the table may not be perceived as a problem.

5 The Shannon Entropy of Reference Ranges and Healing Sequences for Serum c-terminal Agrin Fragment Used as a Biomarker for Kidney Function in Kidney Recipients

In the [19] (see, Table 3), the values of CAF, Scr and $Cys\ C$ have given as a table for the baseline characteristics of kidney transplants. According to Table 3. of Yu et al., Serum CAF (pM) takes the value in the interval $[618.1, 1508.8]$ before Tx, takes the value of $[85.9, 1291.3]$ after one days, takes the value of $[6.8, 977.3]$ after two days and the takes value of [74.3, 338.0] after six months. Similarly, Creatinine (μmol/l) takes the value of in the interval [476.0, 1856.0] before Tx, the value of $[115.0, 1254.0]$ after one days, the value of

$[80.0, 1275.0]$ after two days and the value of $[67.0, 320.0]$ after six months and Cystatin C (mg/l) takes the value of in the interval $[1.00, 12.00]$ before Tx, the value of $[1.00, 4.18]$ after one days, the value of $[1.00, 4.47]$ after two days and the value of $[0.77, 3.60]$ after six months, where Tx denotes transplantation.

Similarly to Section 4, we can construct membership functions of the Serum CAF (pM), Creatinine, Cystatin C after we can obtain probability density functions and using these functions we can calculate Shannon entropy of kidney recipients.

According to data of [19], Serum CAF (pM) takes the value of in the interval [618.1, 1508.8] before Tx. According to this, the membership function Serum CAF (pM) before Tx, $CAF_{Tx}(x)$, is equal to

$$CAF_{Tx}(x) = \begin{cases} \frac{x-618.1}{1063.45-618.1}, & x \in [618.1, 1063.45] \\ \frac{1508.8-x}{1508.8-1063{,}45}, & x \in (1063, 45, 1508.8] \\ 0, & \text{otherwise} \end{cases} . \tag{12}$$

If we consider equality (4) then we see that $\Delta = 445.35$. From equality (5) we obtain that the probability density function of CAF for individuals who are male and aged between 18-34 as follows:

$$P_{CAF_{Tx}}(x) = \frac{1}{\Delta} CAF_{Tx}(x) = \begin{cases} \frac{x-618.1}{198336{,}6225}, & x \in [618.1, 1063.45] \\ \frac{1508.8-x}{198336{,}6225}, & x \in (1063.45, 1508.8] \\ 0, & \text{otherwise} \end{cases} . \tag{13}$$

The function $P_{CAF_{Tx}}(x)$ with given in (13) satisfies the conditions of the probability density function which is given in (2). Thus, the entropy of the CAF_{Tx} is computed as follows:

$$H_{CAF_{Tx}} = -\int_{x \in Supp\ P_{CAF_{Tx}}} P_{CAF_{Tx}}(x) \log_2 P_{CAF_{Tx}}(x) dx = 9.52014.$$

Similarly to above, the entropy values of the CAF, Scr and Sys are given in Table 4 .

Table 4: Healing entropy values for various biomarkers

	The Entropy Values according to Days after Transplantation				
Biomarkers	1.Day	2. Day	...	180.Day	...
CAF	9.95664	9.94393	...	7.7641	...
Scr	9.8749	9.9441	...	7.70434	...
Cys	1.39037	9.94414	...	1.22215	...

Now, we can determine the success of the transplantation of the kidney using the healing sequence which it given in Definition (2.1).

Taking into account the data in Table 4, we can write the healing sequences of CAF as $(H_n^{CAF}) = (6.68 + 3.27e^{(-0.2116(n-1))})$ for kidney transplantation. The value of H_{both}^{CAF} is 6.4332 by taking the arithmetic average of the sum of the "both" values of CAF in Table 3. In this case,

$$\frac{1}{2} < \lim_n S(H_{both}^{CAF}, H_n^{CAF}) = \frac{1}{1 + \lim_n d(6.4332, H_n^{CAF})} \leq 1, \tag{14}$$

where d denotes the natural metric on real numbers set.

Again, taking into account the data in Table 4, we can write the healing sequences of Scr as $(H_n^{Scr}) = (4.55 + (9.94 - 4.55)/(1 + (n-1)/59)$ for kidney transplantation. The value of H_{both}^{Scr} is 4.642 by taking the arithmetic average of the sum of the "both" values of Scr in Table 3. In this case,

$$\frac{1}{2} < \lim_n S(H_{both}^{Scr}, H_n^{Scr}) = \frac{1}{1 + \lim_n d(4.642, H_n^{Scr})} \leq 1. \tag{15}$$

Due to Definition 2.1, the meaning of (14) and (15) are that the kidney transplant is successful according to healing sequences $(H_n^{CAF}) = (6.68 + 3.27e^{(-0.2116(n-1))})$ and $(H_n^{Scr}) = (4.55 + (9.94 - 4.55)/(1 + (n-1)/59)$. Furthermore, the (14) and (15) give us the result that the transplants of kidney transplant recipients, who were the subject of the research of Yu et al. in [19], were successful.

It is note that the healing sequence is obtained differently for each biomarker. These sequences will also vary from patient to patient. Therefore, a healing sequence should be obtained according to the patient under observation. Sometimes it may not be possible to obtain a healing sequence.

A similar calculation can also be made for Sys. However, I could not obtain an improvement sequence for Sys. I will address this as a separate problem in another study.

6 Discussions

We will consider full part and three decimals of the Shannon entropy values of the biomarkers CAF, Scr and Cys C which they have given in Table 3 and are discussed as follows:

6.1 Discussions for Table 3

The discussions can be summarized as follows:

1. For the age group of 18-34 years, the Shannon entropy value provides valuable insights into the uncertainty associated with the CAF (pM) interval, as indicated in Table 1. The entropy value, being significantly greater than 1 for all sex types, suggests that there is a substantial amount of variability in CAF levels within this age range. This variability could be attributed to various factors such as individual differences, lifestyle choices, and underlying health conditions.

   A Shannon entropy value greater than 1 implies that the distribution of CAF levels within the specified age group is widely spread, leading to a higher degree of uncertainty. In practical terms, this means that within the 18-34 age range, kidney transplant patients may exhibit diverse CAF concentrations, making it challenging to draw definitive conclusions solely based on these values. The observation of high entropy underscores the importance of further investigation to understand the underlying factors contributing to this variability in CAF levels. It also emphasizes the need for additional studies involving larger and more diverse patient populations to validate these findings and establish more robust reference ranges for CAF in kidney transplant recipients. Moreover, healthcare professionals should be cautious while interpreting CAF levels in young adult patients, considering the significant uncertainty associated with the biomarker's values in this specific age group. The use of complementary diagnostic tools and the integration of patient-specific data may be essential in making accurate clinical decisions and evaluating kidney function effectively in young kidney transplant recipients.

   In conclusion, the Shannon entropy analysis highlights the considerable uncertainty in CAF levels among kidney transplant patients aged 18-34 years. This finding encourages further research to enhance our understanding of this phenomenon and underscores the importance of personalized and comprehensive approaches when evaluating kidney function in this particular age group.

2. For the age group of 18-34 years, the Shannon entropy values for the intervals of Scr (μmol/l) and Sys C (mg/l) in kidney transplant patients (Table 1) reveals a substantial degree of uncertainty across all sex types. These high entropy values suggest considerable variability in serum creatinine and systemic C levels within this specific age range. Such pronounced uncertainty underscores the complexity of kidney function in young transplant recipients, emphasizing the importance of careful monitoring and tailored medical interventions to address the diverse needs and responses observed in this demographic. Further research and analysis of these biomarkers' fluctuations can potentially lead to enhanced strategies

for managing kidney transplant patients within this age category and improving their overall health outcomes.

3. For the 35-49 years, the Shannon entropy value indicates that the interval of CAF(PM), which is given Table 1, contains a large amount of uncertainty as it is much greater than 1 for all sex types.

4. In the 35-49 years age group, the Shannon entropy value reveals a notable level of uncertainty within the intervals of Scr (μumol/l) and Sys C (mg/l), as presented in Table 1, across all sex types. The entropy value, significantly greater than 1, suggests substantial variability in the levels of serum creatinine (Scr) and serum C (Sys C) biomarkers among kidney transplant patients within this specific age range. This finding indicates that kidney function and other related physiological processes represented by these biomarkers exhibit diverse and complex patterns in individuals aged 35-49 years.

   The observed high entropy underscores the importance of carefully monitoring kidney function and related health parameters in this particular age group of kidney transplant recipients. The variability in Scr and Sys C levels may be influenced by various factors, such as lifestyle, comorbidities, and response to immunosuppressive medications. Therefore, healthcare professionals must take these fluctuations into account when designing personalized treatment plans and assessing the overall health status of patients within this age category.

5. In the 50-64 years age group, the Shannon entropy value highlights significant uncertainty within the intervals of CAF (pM), as reported in Table 1, across all sex types. The entropy value, being much greater than 1, suggests substantial variability in the levels of CAF biomarkers in kidney transplant patients within this specific age range. This finding implies that kidney function and neuromuscular junction (NMJ) dysfunction, which the CAF biomarker represents, may exhibit diverse patterns and responses in this demographic.

   The observed high entropy underscores the complexity and heterogeneity of kidney-related health conditions in individuals aged 50-64 who have undergone kidney transplantation. It also emphasizes the need for precise and individualized monitoring and treatment strategies to manage the varying health challenges that may arise in this age category.

6. For the 50-64 years, the Shannon entropy value indicates that the intervals of Scr(μumol/l) and Sys C(mg/l), which are given in Table 1, contains a large amount of uncertainty as it is much greater than 1 for all sex type.

Shannon entropy is generally used as a measure of uncertainty in an information source. If the entropy of CAF or other biomarkers is greater than 1; it usually indicates uncertainty in that source. This may also suggest that the information source is less predictable or more complex. However, this is only a general interpretation and more context are needed for a more accurate interpretation.

6.2 Discussions for Results of Section 5

Considering the insights gained from this section, the initial measurements taken during the early days of kidney transplantation reveal a highly intricate state of kidney function, as indicated by the remarkably high values of entropy. This finding suggests that the biomarkers used to assess kidney function, as reported by Yu and other researchers in [19], exhibit a significant degree of chaos and variability within their respective ranges. The heightened entropy values imply that kidney transplant recipients experience considerable fluctuations and unpredictability in these biomarkers during the immediate post-transplant period.

As patients progress through the post-transplant period and undergo proper medical management, a notable transformation occurs. The once chaotic state of kidney function gradually stabilizes, as reflected by a decrease in the entropy values over time. This reduction in entropy signifies a trend toward greater regularity and predictability in the levels of biomarkers associated with kidney function.

The decreasing entropy values can be attributed to various factors, such as the healing and recovery process of the transplanted kidney, the adjustment of immunosuppressive medications, and the body's adaptation to the new organ. As the transplanted kidney becomes integrated into the recipient's body and begins to function optimally, the overall dynamics of kidney-related biomarkers tend to stabilize, leading to a less chaotic state.

The observed trend of decreasing entropy over time is promising and reinforces the significance of continuous monitoring and medical intervention during the early phases of kidney transplantation. By closely observing the changes in entropy values and biomarker levels, healthcare professionals can better understand the trajectory of kidney function recovery and identify potential complications or abnormalities that may require timely intervention.

Furthermore, this knowledge could pave the way for refining post-transplant care protocols and developing personalized treatment strategies tailored to individual patients. By promoting the transition from a chaotic to a more stable state of kidney function, healthcare providers can enhance the long-term success of kidney transplantations and improve the overall quality of life for transplant recipients.

In conclusion, the fluctuations in entropy values during the early post-transplant period highlight the complexity and dynamic nature of kidney function. The subsequent decrease in entropy underscores the positive evolution of kidney function over time, offering hope for improved patient outcomes and reinforcing the importance of meticulous monitoring and care throughout the kidney transplantation journey.

6.3 The Disadvantages of This Study

The entropy values, in this calculation, maybe depend

1. to measure devices and individuals,
2. to conditions of the environment,
3. to alimentation of people,
4. results may change from one region to an other region
5. entropy values may depend on the species of the person.

Converting reference ranges to fuzzy sets and calculating Shannon entropy can be used to confirm a diagnostic method or identify a disease. However, these data are only a part of the picture. These data should be considered in conjunction with many other factors such as disease symptoms, medical history, medication use, age, gender and genetic factors.

Therefore, a more comprehensive data analysis is necessary to collect, model, and interpret data more accurately. This analysis may include data mining techniques, artificial intelligence methods, and other mathematical and statistical tools to obtain more comprehensive results. In general, it has been concluded that combining biomarkers such as Scaf with mathematical techniques such as fuzzy sets and Shannon entropy can provide a valuable understanding of the diagnosis and treatment of kidney diseases; advanced research can lead to the development of more effective diagnostic and treatment approaches for kidney diseases.

7 Conclusion

The Shannon entropy, discussed in this article, can be used to measure the disorder in the tissue; high entropy values indicate the disorder is high, while low entropy values indicate that more order. This information can help machine learning algorithms be more successful in recognition or classification tasks by better understanding of the structure of the tissue.

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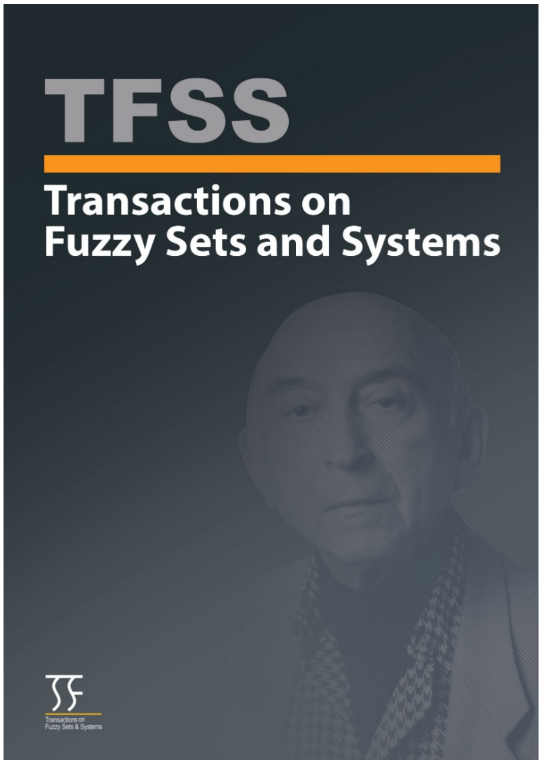

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Triangle Algebras and Relative Co-annihilators

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Triangle Algebras and Relative Co-annihilators

Emile Djomgoue Nana , **Ariane Gabriel Tallee Kakeu*** , **Blaise Bleriot Koguep Njionou** , **Lele Celestin**

Abstract. Triangle algebras are an important variety of residuated lattices enriched with two approximation operators as well as a third angular point (different from 0 and 1). They provide a well-defined mathematical framework for formalizing the use of closed intervals derived from a bounded lattice as truth values, with a set of structured axioms. This paper introduces the concept of relative co-annihilator of a subset within the framework of triangle algebras. As filters of triangle algebras, these relative co-annihilators are explored and some of their properties and characterizations are given. A meaningful contribution of this work lies in its proof that the relative co-annihilator of a subset T with respect to another subset Y in a triangle algebra $\mathcal{L}$ inherits specific filter's characteristics of Y. More precisely, if Y is a Boolean filter of the second kind, then the co-annihilator of T with respect to Y is also a Boolean filter of the second kind. The same statement applies when we replace the Boolean filter of the second kind with an implicative filter, pseudo complementation filter, Boolean filter, prime filter, prime filter of the third kind, pseudo-prime filter, or involution filter, respectively. Finally, we establish some conditions under which the co-annihilator of T relative to Y is a prime filter of the second kind.

AMS Subject Classification 2020: MSC 03G10; MSC 03G25; MSC 08A72
Keywords and Phrases: Triangularization, Triangle algebra, Filter, Co-annihilator, Relative co-annihilator.

1 Introduction

George Boole's endeavor to formalize propositional logic led to the concept of Boolean algebra ([1]). Unfortunately, the discrete nature of the truth values fails to handle situations in which the accuracy of statements is not precisely known. In his attempt to solve this problem, Zadeh ([2]) proposed the idea of working with the unit interval $[0, 1]$ equipped with the usual order, giving rise to fuzzy logic. Considering the potential non-comparability of elements within the set of truth values, a substantial advancement occurred in 1967 when Goguen [3] brought in a novel approach: replacing the unit interval with a bounded lattice, and using triangular norms and co-norms to extend the concepts of logical conjunction and disjunction. Among the significant features of triangular norms and co-norms, their compatibility with the principle of residuation stands out, resulting in the algebraic structure called *residuated lattice* (see [4]). In 2008, Van Gasse et al. ([5]) established residuated lattices based on lattices of closed intervals, also known as triangular lattices, thereby introducing the concept of *Interval-valued residuated lattices (IVRLs)*. Subsequently, they equipped the latter with two approximation operators and with a third angular point, leading to the so called *extended Interval-valued residuated lattices*, which are equationally represented by *triangle algebras* [6].

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A crucial concept in algebraic structures used for formal fuzzy logic, is that of a filter, since filters have a natural interpretation as sets of provable formulas, and therefore are important in the proof of the completeness of these logics. Indeed, the theory of triangle algebras has been endowed with the filter theory (see [7, 8, 9]). In 2017, Zahiri et al. [8] conducted an investigation into a particular class of filters in triangle algebras, namely, co-annihilators. Our main purpose is to introduce and thoroughly explore relative co-annihilators in triangle algebras, as a generalization of co-annihilators.

In the literature, the concept of co-annihilator of an element a relative to a filter F was introduced in BL-algebra by Meng and Xin [10]. Following this, Maroof et al. ([11]) and Rasouli ([12]) extended this notion to residuated lattices. In [11], they examinined the co-annihilator of an arbitrary subset T with respect to another subset Y within a residuated lattice. Nevertheless, the concept of relative co-annihilator remains unexplored in triangle algebras.

This paper is organized as follows: In **Section 2**, we recall some preliminary notions in order to make the document self-contained. **Section 3** is devoted to the notion of relative co-annihilator in triangle algebras, with some of its properties. In **Section 4**, we provide more properties of relative co-annihilators through filters of triangle algebras. We prove that for any two nonempty subsets T and Y of a triangle algebras $\mathcal{L}$, if Y is a Boolean filter of the second kind (respectively, pseudo-complementation filter, implicative filter, Boolean filter, prime filter, prime filter of the third kind, pseudo-prime filter, involution filter), then, so is the co-annihilator of T relative to Y. Finally, we highlight some conditions under which the co-annihilator of T relative to Y is a prime filter of the second kind.

2 Preliminaries

In this section, we recall some notions that will be useful in this paper.

Definition 2.1. [13, 4] A *residuated lattice* is an algebra $\mathcal{L} = (L, \vee, \wedge, \odot, \rightarrow, 0, 1)$ with four binary operations and two constants such that:

(R1) $(L, \vee, \wedge, 0, 1)$ is a bounded lattice;

(R2) $(L, \odot, 1)$ is a commutative monoid;

(R3) $x \odot y \leq z$ iff $x \leq y \rightarrow z$, for all x, y and z in L.

Unless otherwise specified, by $\mathcal{L}$ we will denote the residuated lattice $(L, \vee, \wedge, \odot, \rightarrow, 0, 1)$. The negation $\neg$ in $\mathcal{L}$ is defined by $\neg x = x \rightarrow 0$, for all x in L.

Theorem 2.2. *[14, 11, 6, 4] Let $\mathcal{L}$ be a residuated lattice. Then, the following properties are valid, for all $x, x_1, x_2, y, y_1, y_2, z \in L$:*

(RL1) $1 \rightarrow x = x, x \rightarrow x = 1$, $\neg 1 = 0$, *and* $\neg 0 = 1$;

(RL2) $x \odot y \leq x, y$ *hence* $x \odot y \leq x \wedge y$, $y \leq x \rightarrow y$ *and* $x \odot 0 = 0$;

(RL3) $x \odot y \leq x \rightarrow y$, *and* $x \odot y = 0$ *iff* $x \leq \neg y$;

(RL4) $x \leq y$ *iff* $x \rightarrow y = 1$;

(RL5) $x \odot (x \rightarrow y) \leq y$, $x \leq (x \rightarrow y) \rightarrow y$, $((x \rightarrow y) \rightarrow y) \rightarrow y = x \rightarrow y$;

(RL6) $x \rightarrow y \leq (x \odot z) \rightarrow (y \odot z)$;

(RL7) $x \leq y$ *implies* $(x \odot z) \leq (y \odot z)$, $z \rightarrow x \leq z \rightarrow y$, $y \rightarrow z \leq x \rightarrow z$, *and* $\neg y \leq \neg x$;

(RL8) $x \to y \leq (z \to x) \to (z \to y)$, $x \to y \leq (y \to z) \to (x \to z)$;

(RL9) $x \odot (y \to z) \leq y \to (x \odot z) \leq (x \odot y) \to (z \odot z)$;

(RL10) $x \to (y \to z) = (x \odot y) \to z = y \to (x \to z)$;

(RL11) $x_1 \to y_1 \leq (y_2 \to x_2) \to [(y_1 \to y_2) \to (x_1 \to x_2)]$;

(RL12) $(x \to z) \vee (y \to z) \leq x \wedge y \to z$;

(RL13) $x \odot (y \vee z) = (x \odot y) \vee (x \odot z)$, $z \vee (x \odot y) \geq (z \vee x) \odot (z \vee y)$;

(RL14) $x \leq \neg\neg x \leq \neg x \to x$, $\neg\neg\neg x = \neg x$;

(RL15) $\neg(x \odot y) = x \to \neg y, y \to \neg x = \neg\neg x \to \neg y$, *and* $x \to y \leq \neg y \to \neg x$.

Recall from [7] that a *filter* of a residuated lattice $\mathcal{L}$ is a nonempty subset F of L such that for all $x, y \in L$:

(F1) if $x \in F$ and $x \leq y$, then $y \in F$;

(F2) if $x, y \in F$, then $x \odot y \in F$.

We now recall the notion of interval-valued residuated lattices, which are residuated lattices on triangularizations. This has led to the development of triangle algebras through the use of approximation operators, describing the aspect of incompleteness inherent in interval-valued residuated lattices.

Definition 2.3. [5, 6] Let $\mathcal{L} = (L, \vee, \wedge, 0, 1)$ be a bounded lattice. We call *triangularization or triangular lattice* of $\mathcal{L}$ the bounded lattice, $\mathbb{T}(\mathcal{L})$ of the closed intervals of L defined by

$$\mathbb{T}(\mathcal{L}) = (Int(\mathcal{L}), \vee_{Int(\mathcal{L})}, \wedge_{Int(\mathcal{L})}, [0,0], [1,1])$$

such that $Int(\mathcal{L}) = \{[x_1, x_2] : x_1, x_2 \in L \text{ and } x_1 \leq x_2\}$, and for all $x_1, x_2, y_1, y_2 \in L$,

- $[x_1, x_2] \vee_{Int(\mathcal{L})} [y_1, y_2] = [x_1 \vee y_1, x_2 \vee y_2]$;
- $[x_1, x_2] \wedge_{Int(\mathcal{L})} [y_1, y_2] = [x_1 \wedge y_1, x_2 \wedge y_2]$;
- $[x_1, x_2] \leq_{Int(\mathcal{L})} [y_1, y_2]$ iff $x_1 \leq y_1$ and $x_2 \leq y_2$.

The set $D(\mathcal{L}) = \{[x, x] : x \in L\}$ is called *diagonal* of $\mathbb{T}(\mathcal{L})$.

From [5, 6], an *interval-valued residuated lattice (IVRL)* is a residuated lattice $(\text{Int}(\mathcal{L}), \vee, \wedge, \odot, \to_{\odot}, [0,0], [1,1])$ on the triangularization $\mathbb{T}(\mathcal{L})$ of a bounded lattice $\mathcal{L}$, in which the diagonal $D(\mathcal{L})$ is closed under $\odot$ and $\to_{\odot}$, i.e., $[x,x] \odot [y,y] \in D(\mathcal{L})$ and $[x,x] \to_{\odot} [y,y] \in D(\mathcal{L})$, for all x, y in L.

Definition 2.4. [6, 15] An *extended IVRL* is a structure $(Int(\mathcal{L}), \vee, \wedge, \odot, \to, pr_v, pr_h, [0,0], [0,1], [1,1])$ where $u = [0,1]$ is a constant interval, pr_v and pr_h are maps from $Int(\mathcal{L})$ to $Int(\mathcal{L})$, respectively called vertical and horizontal projections defined by $pr_v([x_1, x_2]) = [x_1, x_1]$ and $pr_h([x_1, x_2]) = [x_2, x_2]$, for all $[x_1, x_2] \in Int(\mathcal{L})$.

The following definition presents the concept of triangle algebra, which serves as an equational representation of interval-valued residuated lattices.

Definition 2.5. [5, 16] A *triangle algebra* is a structure $\mathcal{L} = (L, \vee, \wedge, \odot, \to, \nu, \mu, 0, u, 1)$ in which $(L, \vee, \wedge, \odot, \to, 0, 1)$ is a residuated lattice, ν and μ are unary operations on L, u $(0 \neq u \neq 1)$ a constant, all satisfying the following conditions:

$(T.1)\nu x \leq x$; $(T.1')\, x \leq \mu x$;
$(T.2)\nu x \leq \nu\nu x$; $(T.2')\, \mu\mu x \leq \mu x$;
$(T.3)\nu(x \wedge y) = \nu x \wedge \nu y$; $(T.3')\, \mu(x \wedge y) = \mu x \wedge \mu y$;
$(T.4)\nu(x \vee y) = \nu x \vee \nu y$; $(T.4')\, \mu(x \vee y) = \mu x \vee \mu y$;
$(T.5)\nu u = 0$; $(T.5')\, \mu u = 1$;
$(T.6)\nu\mu x = \mu x$; $(T.6')\, \mu\nu x = \nu x$;
$(T.7)\nu(x \to y) \leq \nu x \to \nu y$;
$(T.8)(\nu x \leftrightarrow \nu y) \odot (\mu x \leftrightarrow \mu y) \leq (x \leftrightarrow y)$;
$(T.9)\nu x \to \nu y \leq \nu(\nu x \to \nu y)$.

Note that the statement $x \leftrightarrow y$ stands for $(x \to y) \wedge (y \to x)$.

Remark 2.6. $\nu 0 = \mu 0 = 0$ and $\nu 1 = \mu 1 = 1$.

Unless otherwise specified, the triangle algebra $(L, \vee, \wedge, \odot, \to, \nu, \mu, 0, u, 1)$ will be denoted by $\mathcal{L}$.

Proposition 2.7. *[17] Let $\mathcal{L}$ be a triangle algebra. Then, for all $x, y \in L$ we have:*

1. $\nu(x \odot y) = \nu x \odot \nu y$;
2. $\mu(x \odot y) \leq \mu x \odot \mu y$.

Lemma 2.8. *Let $\mathcal{L}$ be a triangle algebra. For all $x, y \in L$, if $\nu x \vee y = 1$, then $x \odot y = x \wedge y$.*

Proof.

Let $x, y \in L$. We already know from (*RL2*) of Theorem 2.2 that $x \odot y \leq x \wedge y$. All we need to prove is $x \wedge y \leq x \odot y$. We have:

$$\begin{aligned} x \wedge y &= 1 \odot (x \wedge y) \\ &= (\nu x \vee y) \odot (x \wedge y), && \text{as } \nu x \vee y = 1 \\ &= [\nu x \odot (x \wedge y)] \vee [y \odot (x \wedge y)], && \text{from } (RL13) \\ &\leq (\nu x \odot y) \vee (x \odot y), && \text{as } x \wedge y \leq x, y \\ &\leq (x \odot y) \vee (x \odot y), && \text{as } \nu x \leq x \\ &= x \odot y. \end{aligned}$$

□

Definition 2.9. [7, 16] A *filter* (or *IVRL-filter*) of a triangle algebra $\mathcal{L}$ is a nonempty subset F of $\mathcal{L}$ satisfying:

(F1) if $x \in F$, $y \in L$ and $x \leq y$, then $y \in F$;

(F2) if $x, y \in F$, then $x \odot y \in F$;

(F3) if $x \in F$, then $\nu x \in F$.

It is worth noticing that, for every filter F of a triangle algebra $\mathcal{L}$, $1 \in F$, and [$x \in F$ if and only if $\nu x \in F$], see [18].

Definition 2.10. [9, 8] Let F be a filter of a triangle algebra $\mathcal{L}$. Then, F is said to be:

1. a *Boolean filter* (BF) if for all $x \in L$, $\nu(x \vee \neg x) \in F$.
2. a *Boolean filter of the second kind* (BF2) if for all $x \in L, \nu x \in F$ or $\nu(\neg x) \in F$.
3. a *prime filter* (PF) if for all $x, y \in L$, $\nu(x \to y) \in F$ or $\nu(y \to x) \in F$ (or both).
4. a *prime filter of the second kind* (PF2) if for all $x, y \in L$, $\nu(x \vee y) \in F$ implies $\nu x \in F$ or $\nu y \in F$ (or both).
5. a *prime filter of the third kind* (PF3) if for all $x, y \in L$, $\nu[(x \to y) \vee (y \to x)] \in F$.
6. a *pseudo-prime filter* (PPF) if for all $x, y \in L$, $\nu x \to \nu y \in F$ or $\nu y \to \nu x \in F$ (or both).
7. an *implicative filter* (IF) if for all $x, y, z \in L$, $\nu[x \to (y \to z)] \in F$ and $\nu(x \to y) \in F$ imply $\nu(x \to z) \in F$ (first form) or equivalently, $\nu[x \to (x \to z)] \in F$ implies that $\nu(x \to z) \in F$ (second form).
8. a *pseudocomplementation filter* (PSF) if for all $x \in L$, $\nu[\neg(x \wedge \neg x)] \in F$.
9. an *involution filter* (VF) iff for all $x \in L, \nu(\neg\neg x \to x) \in F$.

Proposition 2.11. *[8] Let F be a filter of $\mathcal{L}$. Then, F is an implicative filter iff $\nu(x \to x^2) \in F$, for all $x \in L$.*

Definition 2.12. [16]
Let A be a nonempty subset of a triangle algebra $\mathcal{L}$. Then, the *co-annihilator* of A, denoted by $A^\top$ is the filter defined by $A^\top = \{x \in L \mid \nu x \vee a = 1, \text{ for all } a \in A\}$.

3 Relative Co-annihilators in Triangle Algebras

In this section, we introduce the notion of relative co-annihilator in a triangle algebra $\mathcal{L}$ and investigate some of its properties.

Definition 3.1. Let $\mathcal{L}$ be a triangle algebra, A and B be subsets of L. The *co-annihilator of A relative to B* is the set $(A^\top, B) = \{a \in L \mid (\forall b \in A), \nu a \vee b \in B\}$.

If $B = \{x\}$, then we will denote $(A^\top, \{x\})$ by $(A^\top, x)$.
In a similar way, If $A = \{a\}$, then we will denote $(\{a\}^\top, B)$ by $(a^\top, B)$.

Remark 3.2. For any subset A of L, $(A^\top, 1) = A^\top$.

Example 3.3.
Let $L = \{[0,0], [0,a], [0,b], [a,a], [b,b], [0,1], [a,1], [b,1], [1,1]\}$ be the lattice whose associated Hasse diagram is depicted in Figure 1. Define $\odot$ and $\Rightarrow$ as presented in Table 1.

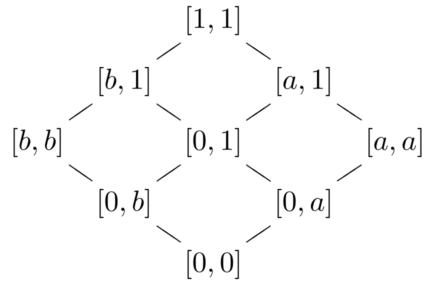

Figure 1: Hasse diagram of $\mathcal{L}$ in Example 3.3

Table 1: Operation tables of $\odot$ and $\Rightarrow$ for $\mathcal{L}$ in Example 3.3

$\odot$	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

$\Rightarrow$	0	a	b	1
0	1	1	1	1
a	b	1	b	1
b	a	a	1	1
1	0	a	b	1

Consider the actions on L of ν, μ, $\odot$ and $\to$ defined as follows: for all $[x_1, x_2]$, $[y_1, y_2] \in L$, $\nu\,[x_1, x_2] = [x_1, x_1]$; $\mu\,[x_1, x_2] = [x_2, x_2]$; $[x_1, x_2] \odot_L [y_1, y_2] = [x_1 \odot y_1, x_2 \odot y_2]$;
$[x_1, x_2] \to [y_1, y_2] = [(x_1 \Rightarrow y_1) \wedge (x_2 \Rightarrow y_2), x_2 \Rightarrow y_2]$.

Then, $\mathcal{L} = (L, \vee, \wedge, \odot_L, \to, \nu, \mu, [0, 0], [0, 1], [1, 1])$ is a triangle algebra [16].

Set $A = \{[b, b], [b, 1], [1, 1]\}$ and $B = \{[0, 0], [a, a], [1, 1]\}$. One easily verifies that $(A^\top, B) = \{[a, a], [a, 1], [1, 1]\}$. In a similar manner, $(B^\top, A) = \{[b, b], [b, 1], [1, 1]\}$.

Some of the following properties of relative co-annihilators have been established within the framework of residuated lattices (see [11]). Nevertheless, the formulations presented here are specific to triangle algebras, since the approximation operator ν is involved.

Proposition 3.4. *Let $\mathcal{L}$ be a triangle algebra. Let A and B be subsets of L. Then,*

(1) $A = \emptyset$, *implies* $(A^\top, B) = L$;

(2) With $A \neq \emptyset$:

(i) if $B = \emptyset$, *then* $(A^\top, B) = \emptyset$;

(ii) for $A \neq \{0\}$, $(A^\top, 0) = \emptyset$;

(iii) $(A^\top, 1) \subseteq \{x \in L \mid (\forall y \in A), x \odot y = x \wedge y\}$;

(iv) $(0^\top, 1) = \{1\}$,$(1^\top, 0) = \emptyset$, $(L^\top, 1) = \{1\}$, $(1^\top, 1) = L$, $(0^\top, 0) = \{x \in L \mid \nu x = 0\}$;

(v) $(L^\top, A) = \emptyset$ *iff* $1 \notin A$;

(vi) $(L^\top, A) \subseteq A$.

Proof.

(1) We write $(A^\top, B)$ in a more logical form as $\{x \in L \mid (\forall y)(y \in A \text{ implies } \nu x \vee y \in B)\}$. Thus, $(\emptyset^\top, B) = \{x \in L \mid (\forall y)(y \in \emptyset \text{ implies } \nu x \vee y \in B)\} = L$, since the statement "$(\forall y)(y \in \emptyset \text{ implies } \nu x \vee y \in B)$" is always true, for all $x \in L$.

2 Consider $A \neq \emptyset$:

(i) if $B = \emptyset$, then :

$$\begin{aligned}(A^\top, B) &= (A^\top, \emptyset)\\ &= \{x \in L \mid (\forall y \in A), \nu x \vee y \in \emptyset\}\\ &= \emptyset.\end{aligned}$$

(ii) If $A \neq \{0\}$, then,

$$\begin{aligned}(A^\top, 0) &= \{x \in L \mid (\forall y \in A), \nu x \vee y = 0\}\\ &= \emptyset\end{aligned}$$

(iii) For all $z \in L$,

$$\begin{aligned}z \in (A^\top, 1) &\Rightarrow \forall y \in A, \nu z \vee y = 1\\ &\Rightarrow \forall y \in A, z \odot y = z \wedge y, \quad \textit{(by Lemma 2.8)}\\ &\Rightarrow z \in \{x \in L \mid (\forall y \in A), x \odot y = x \wedge y\}\end{aligned}$$

Thus, $(A^\top, 1) \subseteq \{x \in L \mid (\forall y \in A), x \odot y = x \wedge y\}$.

(iv) We have:

$$\begin{aligned}(0^\top, 1) &= \{x \in L \mid \nu x \vee 0 = 1\}\\ &= \{1\};\\ (1^\top, 0) &= \{x \in L \mid \nu x \vee 1 = 0\}\\ &= \emptyset;\\ (L^\top, 1) &= L^\top, \quad \textit{(by Remark 3.2)}\\ &= \{x \in L \mid (\forall y \in L), \nu x \vee y = 1\}\\ &= \{1\};\\ (0^\top, 0) &= \{x \in L \mid \nu x \vee 0 = 0\}\\ &= \{x \in L \mid \nu x = 0\};\\ (1^\top, 1) &= \{x \in L \mid \nu x \vee 1 = 1\}\\ &= L.\end{aligned}$$

(v) Let $(L^\top, A) = \emptyset$. Then $1 \notin A$, otherwise we would have $(L^\top, 1) = \emptyset$, which implies from *(iv)* that $\{1\} = \emptyset$, a contradiction.

Conversely,

suppose by contrary that $(L^\top, A) \neq \emptyset$ and let $x \in (L^\top, A)$. Then, for all $y \in L, \nu x \vee y \in A$. Since $1 \in L$, then $1 = \nu x \vee 1 \in A$, which contradicts the fact that $1 \notin A$.

It follows that $(L^\top, A) = \emptyset$ iff $1 \notin A$.

(vi) Suppose by contrary that $(L^{\top}, A) \nsubseteq A$. Then, there is $x \in (L^{\top}, A)$ such that $x \notin A$, i.e., for all $y \in L$, $\nu x \vee y \in A$ and $x \notin A$. In particular, for $y = x$, we have $\nu x \vee x \in A$ and $x \notin A$, i.e., $x \in A$ (since $\nu x \leq x$) and $x \notin A$, which is absurd. Thus, $(L^{\top}, A) \subseteq A$.

□

The reverse inclusion in Proposition 3.4 (*vi*) is not always true, as it is deduced from Proposition 3.4 (*v*) that $A \nsubseteq \emptyset = (L^{\top}, A)$ whenever $1 \notin A$.

Proposition 3.5. *Let $\mathcal{L}$ be a triangle algebra. Let T, T_1, T_2, Y_1, Y_2, Y and Z be nonempty subsets of L. Then,*

(i) $T_1 \subseteq T_2$ *implies* $\left(T_1^{\top}, Y\right) \subseteq \left(T_2^{\top}, Y\right)$;

(ii) $Y_1 \subseteq Y_2$ *implies* $\left(T^{\top}, Y_1\right) \subseteq \left(T^{\top}, Y_2\right)$;

(iii) $\left(T_1^{\top}, Y\right) \cap \left(T_2^{\top}, Z\right) \subseteq \left((T_1 \cap T_2)^{\top}, Y \cap Z\right)$;

(iv) $(T^{\top}, (T^{\top}, Y \cap Z)) \subseteq (T^{\top}, (T^{\top}, Y)) \cap (T^{\top}, (T^{\top}, Z))$;

(v) $\left(T^{\top}, \bigcap_{i\in I} Y_i\right) \subseteq \bigcap_{i\in I} (T^{\top}, Y_i) \subseteq \left(T^{\top}, \bigcup_{i\in I} Y_i\right) \subseteq \bigcup_{i\in I} (T^{\top}, Y_i)$;

(vi) $\left(\bigcap_{i\in I} T_i^{\top}, Y\right) \subseteq \bigcap_{i\in I} (T_i^{\top}, Y) \subseteq \left(\bigcup_{i\in I} T_i^{\top}, Y\right) \subseteq \bigcup_{i\in I} (T_i^{\top}, Y)$;

(vii) $T \cap (T^{\top}, Y) \subseteq Y$;

(viii) $(T^{\top}, Y) = \bigcap_{t\in T} (t^{\top}, Y)$.

Proof.

(i) Suppose that $T_1 \subseteq T_2$ and let $x \in \left(T_1^{\top}, Y\right)$. Then, for all $t_1 \in T_1 \subseteq T_2$, $\nu x \vee y \in Y$. Thus, $x \in \left(T_2^{\top}, Y\right)$ and consequently, $\left(T_1^{\top}, Y\right) \subseteq \left(T_2^{\top}, Y\right)$.

(ii) Let $x \in \left(T^{\top}, Y_1\right)$. Then, for all $z \in T$, $\nu x \vee z \in Y_1 \subseteq Y_2$, that is, $x \in \left(T^{\top}, Y_2\right)$. Therefore, $\left(T^{\top}, Y_1\right) \subseteq \left(T^{\top}, Y_2\right)$.

(iii) Let $x \in L$. Then, $x \in \left(T_1^{\top}, Y\right) \cap \left(T_2^{\top}, Z\right)$ implies that for all $y \in T_1$ and $z \in T_2$, $\nu x \vee y \in Y$ and $\nu x \vee z \in Z$. Given that $T_1 \cap T_2 \subseteq T_1, T_2$, we deduce that for all $y \in T_1 \cap T_2$, $\nu x \vee t \in Y \cap Z$, i.e., $x \in \left((T_1 \cap T_2)^{\top}, Y \cap Z\right)$.

(iv) We have $Y \cap Z \subseteq Y, Z$. Then by *(ii)*, $(T^{\top}, Y \cap Z) \subseteq (T^{\top}, Y), (T^{\top}, Z)$. Applying *(ii)* again, $(T^{\top}, (T^{\top}, Y \cap Z)) \subseteq (T^{\top}, (T^{\top}, Y)), (T^{\top}, (T^{\top}, Z))$. Therefore, $(T^{\top}, (T^{\top}, Y \cap Z)) \subseteq (T^{\top}, (T^{\top}, Y)) \cap (T^{\top}, (T^{\top}, Z))$.

(v) *(*)* Let us prove that $\left(T^{\top}, \bigcap_{i\in I} Y_i\right) \subseteq \bigcap_{i\in I} (T^{\top}, Y_i)$.

Since $\bigcap_{i\in I} Y_i \subseteq Y_i$ for all $i \in I$, by *(ii)*, we have $\left(T^{\top}, \bigcap_{i\in I} Y_i\right) \subseteq (T^{\top}, Y_i)$, for all $i \in I$.

Thus, $\left(T^{\top}, \bigcap_{i\in I} Y_i\right) \subseteq \bigcap_{i\in I} (T^{\top}, Y_i)$.

(**) To show that $\bigcap_{i\in I} (T^{\top}, Y_i) \subseteq \left(T^{\top}, \bigcup_{i\in I} Y_i\right)$, for all $i \in I$, we have $Y_i \subseteq \bigcup_{i\in I} Y_i$. Thus, by *(ii)*, we

obtain that for all $i \in I$, $(T^\top, Y_i) \subseteq \left(T^\top, \bigcup_{i\in I} Y_i\right)$, that is, $\bigcap_{i\in I} (T^\top, Y_i) \subseteq \left(T^\top, \bigcup_{i\in I} Y_i\right)$.

(***) Now we prove that $\left(T^\top, \bigcup_{i\in I} Y_i\right) \subseteq \bigcup_{i\in I} (T^\top, Y_i)$. Let $x \in \left(T^\top, \bigcup_{i\in I} Y_i\right)$. Then, for all $y \in T$, there is $i \in I$ such that $\nu x \vee y \in Y_i$. Thus, there is $i \in I$ such that $x \in (T^\top, Y_i)$, that is $x \in \bigcup_{i\in I} (T^\top, Y_i)$. Therefore, $\left(T^\top, \bigcup_{i\in I} Y_i\right) \subseteq \bigcup_{i\in I} (T^\top, Y_i)$.

(vi) *(*)* We have $\bigcap_{i\in I} T_i^\top \subseteq T_i^\top$, for all $i \in I$. Then by *(i)*, we obtain that $\left(\bigcap_{i\in I} T_i^\top, Y\right) \subseteq (T_i^\top, Y)$, for all $i \in I$. Thus, $\left(\bigcap_{i\in I} T_i^\top, Y\right) \subseteq \bigcap_{i\in I} (T_i^\top, Y)$.

(**) For all $i \in I$, $T_i^\top \subseteq \bigcup_{i\in I} T_i^\top$. By applying *(i)*, we have $(T_i^\top, Y) \subseteq \left(\bigcup_{i\in I} T_i^\top, Y\right)$, for all $i \in I$. Therefore, $\bigcap_{i\in I} (T_i^\top, Y) \subseteq \left(\bigcup_{i\in I} T^\top, Y\right)$.

(***) Let $x \in \left(\bigcup_{i\in I} T_i^\top, Y\right)$. Then, there exists $i \in I$ such that $\nu x \vee y \in Y$, for all $y \in T_i$. Thus, there exist $i \in I$ such that $x \in (T_i^\top, Y)$, that is, $x \in \bigcup_{i\in I} (T_i^\top, Y)$. Therefore, $\left(\bigcup_{i\in I} T_i^\top, Y\right) \subseteq \bigcup_{i\in I} (T_i^\top, Y)$.

(vii) If $x \in T \cap (T^\top, Y)$, then $x \in T$ and $\nu x \vee y \in Y$, for all $y \in T$. In particular, $\nu x \vee x \in Y$, which implies that $x \in Y$, as $\nu x \leq x$. Thus, $T \cap (T^\top, Y) \subseteq Y$.

(viii) Let $x \in L$. Then, $x \in (T^\top, Y)$ iff for all $t \in T, \nu x \vee y \in Y$ iff for all $t \in T, x \in (t^\top, Y)$ iff $x \in \bigcap_{t\in T} (t^\top, Y)$. Therefore, $(T^\top, Y) = \bigcap_{t\in T} (t^\top, Y)$.

□

4 Relative Co-annihilators as Filters of Triangle Algebras.

Exploring the relative co-annihilator $(A^\top, B)$, where A and B are arbitrary subsets of L, prompts a natural query: what happens when B is a filter of $\mathcal{L}$? This section examines relative co-annihilators with respect to filters of triangle algebras, providing additional properties.

Proposition 4.1. *Let A and B be two nonempty subsets of a triangle algebra $\mathcal{L}$. If B is a filter of $\mathcal{L}$, then $(A^\top, B)$ is a filter of $\mathcal{L}$.*

Proof. Since B is a filter of L, then $1 \in B$. Also, for all $a \in A$, $\nu 1 \vee a = 1 \in B$ (by Remark 2.6). Thus, $1 \in (A^\top, B)$, and therefore $(A^\top, B)$ is nonempty.
Let $x \in (A^\top, B)$ and $y \in L$ such that $x \leq y$. Then, $\nu x \vee a \in B$. But $x \leq y$ implies $x = x \wedge y$. By $(T.3)$, we have $\nu x = \nu(x \wedge y) = \nu x \wedge \nu y$, that is, $\nu x \leq \nu y$, which implies that $\nu x \vee a \leq \nu y \vee a$, for all $a \in A$. But since B is a filter of $\mathcal{L}$, we deduce that $\nu y \vee a \in B$. Thus, $y \in (A^\top, B)$.
Now, let $x, y \in (A^\top, B)$. Then, for all $a \in A$, we have $\nu x \vee a \in B$ and $\nu y \vee a \in B$. Since B is a filter of $\mathcal{L}$, then $(\nu x \vee a) \odot (\nu y \vee a) \in B$. But by $(RL13)$ and Proposition 2.7 (1), $(\nu x \vee a) \odot (\nu y \vee a) \leq a \vee \nu(x \odot y)$ and since B is a filter of $\mathcal{L}$, we have $a \vee \nu(x \odot y) \in B$, for all $a \in A$. Hence, $x \odot y \in (A^\top, B)$.
Moreover, let $x \in (A^\top, B)$. This implies that $\nu x \vee a \in B$, that for all $a \in A$. But $\nu x \leq \nu\nu x$, which implies

that $\nu x \vee a \leq \nu\nu x \vee a$, for all $a \in A$. It follows that $\nu\nu x \vee a \in B$, since B is a filter of $\mathcal{L}$. Hence, $\nu x \in (A^\top, B)$. □

The converse of Proposition 4.1 is not necessarily true. Indeed, consider the triangle algebra $\mathcal{L}$ from Example 3.3. For $X = \{[a, 1]\}$ and $Y = \{[a, a], [1, 1]\}$, we observe that $(X^\top, Y) = \{[b, b], [b, 1], [1, 1]\}$ which is a filter of $\mathcal{L}$. However, Y is not a filter, as $[a, a] \leq [a, 1] \notin Y$.

Proposition 4.2. *Let T be a filter of a triangle algebra $\mathcal{L}$, and Y a nonempty subset of L. Then,*

(i) $T \subseteq (Y^\top, T)$;

(ii) $(Y^\top, T) = L$ *iff* $Y \subseteq T$ *(specifically,* $(Y^\top, L) = L$, *and* $(T^\top, T) = L$*);*

(iii) $(L^\top, T) = T$;

(iv) $((T^\top, T)^\top, T)) = T$ *and* $((T^\top, (T^\top, T)) = L$;

(v) $Y \cap (Y^\top, T) = Y \cap T$;

(vi) $T \subseteq Y$ *implies* $(Y^\top, T) \cap Y = T$;

(vii) *If* $Y \subseteq T$, *then* $((Y^\top, T)^\top, T) = T$;

(viii) $(Y^\top, T)^\top, T) \cap (Y^\top, T) = T$.

Proof.

(i) Let $x \in T$. Then, $\nu x \in T$ since T is a filter. We have $\nu x \leq \nu\nu x \leq \nu\nu x \vee y$, for all $y \in Y$. Therefore, $\nu\nu x \vee y \in T$, as T is a filter. Thus, $\nu x \in (Y^\top, T)$ and consequently, $T \subseteq (Y^\top, T)$.

(ii) Suppose that $(Y^\top, T) = L$ and $y \in Y$. Since $0 \in L = (Y^\top, T)$, then $y = \nu 0 \vee y \in T$. Therefore, $Y \subseteq T$. Reciprocally, for any $y \in Y \subseteq T$, $\nu 0 \vee y = y \in T$, i.e., $0 \in (Y^\top, T)$. Hence, $L = (Y^\top, T)$.

(iii) By *(i)*, we have $T \subseteq (L^\top, T)$.
Also, from Proposition *3.4 (vi)*, we have $(L^\top, T) \subseteq T$. Thus, $(L^\top, T) = T$.

(iv) $(T^\top, T) = L$ (by *(ii)*). This implies that $((T^\top, T)^\top, T) = (L^\top, T) = T$, by *(iii)*.
Also, $(T^\top, (T^\top, T)) = (T^\top, T) = L$, by *(ii)*.

(v) Clearly, $T \subseteq (Y^\top, T)$ by *(i)* , which implies that $Y \cap T \subseteq Y \cap (Y^\top, T)$.
In addition, $Y \cap (Y^\top, T) \subseteq T$ by Proposition *3.5 (vii)*. Thus, $Y \cap (Y^\top, T) = Y \cap [Y \cap (Y^\top, T)] \subseteq Y \cap T$, i.e., $Y \cap (Y^\top, T) \subseteq Y \cap T$. Therefore, $Y \cap (Y^\top, T) = Y \cap T$.

(vi) Assume that $T \subseteq Y$. Then, $Y \cap T = T$. Thus, *(v)* becomes $Y \cap (Y^\top, T) = T$.

(vii) Since $Y \subseteq T$, then by *(ii)*, $(Y^\top, T) = L$. We obtain from *(iii)* that $((Y^\top, T)^\top, T) = (L^\top, T) = T$.

(viii) From *(i)*, we have $T \subseteq (Y^\top, T)$. Then, from *(vi)*, we deduce that $((Y^\top, T)^\top, T) \cap (Y^\top, T) = T$.

□

Lemma 4.3. *Let $\mathcal{L}$ be a triangle algebra, T, Y and Z, nonempty subsets of L. If Z is a filter, then* $(T^\top, (Y^\top, Z)) \subseteq \bigcap_{t \in T, y \in Y} ((\nu t \vee y)^\top, Z)$.

Proof. Let $x \in L$, then,

$$\begin{aligned} x \in (T^\top, (Y^\top, Z)) &\Rightarrow \forall t \in T, \nu x \vee t \in (Y^\top, Z) \\ &\Rightarrow \forall t \in T, \forall y \in Y, \nu(\nu x \vee t) \vee y \in Z \\ &\Rightarrow \forall t \in T, \forall y \in Y, (\nu\nu x \vee \nu t) \vee y \in Z \quad \text{(by (T.4))} \\ &\Rightarrow \forall t \in T, \forall y \in Y, \nu\nu x \vee (\nu t \vee y) \in Z \quad \text{(by associativity)} \\ &\Rightarrow \nu x \in \bigcap_{t \in T, y \in Y} ((\nu t \vee y)^\top, Z). \end{aligned}$$

But since $\nu x \leq x$ and Z is a filter, then by Proposition 4.1, $\bigcap_{t \in T, y \in Y} ((\nu t \vee y)^\top, Z)$ is also a filter and we have $x \in \bigcap_{t \in T, y \in Y} ((\nu t \vee y)^\top, Z)$.
Consequently, $(T^\top, (Y^\top, Z)) \subseteq \bigcap_{t \in T, y \in Y} ((\nu t \vee y)^\top, Z)$. □

Theorem 4.4. *Let T and Y be two nonempty subsets of a triangle algebra $\mathcal{L}$. If Y is a BF2 (respectively PSF, IF, BF, PF, PF3, PPF, VF), then so is $(T^\top, Y)$.*

Proof. *We establish the first three properties, and the remaining ones are demonstrated in a similar manner.*

Let T and Y be two nonempty subsets of a triangle algebra $\mathcal{L}$:

(i) Suppose that Y is a (BF2) and that for all $x \in L$, $\nu(\neg x) \notin (T^\top, Y)$. Let us show that $\nu x \in (T^\top, Y)$. Since Y is a (BF2), we have $\nu x \in Y$ or $\nu(\neg x) \in Y$. But since Y is a filter of triangle algebra, and that $\nu x \leq \nu x \vee a$ and $\nu(\neg x) \leq \nu(\neg x) \vee a$ for all $a \in T \subseteq L$, then, we have $\nu x \vee a \in Y$ or $\nu(\neg x) \vee a \in Y$, for all $a \in T$. That is, $x \in (T^\top, Y)$ or $\neg x \in (T^\top, Y)$. But, $(T^\top, Y)$ is a filter and $\nu(\neg x) \notin (T^\top, Y)$ by assumption, therefore $x \in (T^\top, Y)$, and hence, $\nu x \in (T^\top, Y)$.

(ii) Suppose that Y is a (PSF). For all $x \in L$, let us show that $\nu[\neg(x \wedge \neg x)] \in (T^\top, Y)$. Now, since Y is a (PSF), then $\nu[\neg(x \wedge \neg x)] \in Y$. But $\nu[\neg(x \wedge \neg x)] \leq \nu[\neg(x \wedge \neg x)] \vee a$ for all $a \in T \subseteq L$. And since Y is a filter of $\mathcal{L}$, we have $\nu[\neg(x \wedge \neg x)] \vee a \in Y$, for all $a \in T$. It yields that, $\neg(x \wedge \neg x) \in (T^\top, Y)$. Hence, since $(T^\top, Y)$ is a filter, we have $\nu[\neg(x \wedge \neg x)] \in (T^\top, Y)$.

(iii) Suppose that Y is an (IF). let us prove that $(T^\top, Y)$ is also an (IF).
Let x be an arbitrary element of L. By Proposition 2.11, it is sufficient to show that $\nu(x \to x^2) \in (T^\top, Y)$. We have $\nu(x \to x^2) \in Y$ and $\nu(x \to x^2) \leq \nu\nu(x \to x^2) \leq \nu\nu(x \to x^2) \vee a$, for all $a \in T$. Since Y is a filter, we have $\nu\nu(x \to x^2) \vee a \in Y$ for all $a \in T$. Hence, $\nu(x \to x^2) \in (T^\top, Y)$.

□

The following property is specific to PF2 (Prime filter of second kind).

Proposition 4.5. *Let $\mathcal{L}$ be a triangle algebra, T be PF2 of $\mathcal{L}$, and Y a subset of L such that $Y \nsubseteq T$. Then, $(Y^\top, T) = T$ (and hence $(Y^\top, T)$ is PF2).*

Proof.
Since T is a filter of $\mathcal{L}$, then $T \subseteq (Y^\top, T)$, by Proposition 4.2 *(i)*.
For the converse, let us suppose by contrary that $(Y^\top, T) \nsubseteq T$. Then, there is $x \in L$ such that $x \in (Y^\top, T)$ and $x \notin T$. This means that for all $a \in Y, \nu x \vee a \in T$ and $\nu x \notin T$ (as T is a filter), which implies that for all $a \in Y, \nu(\nu x \vee a) \in T$ and $\nu x \notin T$ (from (F3)).
Since T is a PF2, it follows that for all $a \in Y, [\nu\nu x \in T$ or $\nu a \in T]$ and $\nu x \notin T$.

This implies that for all $a \in Y, [\nu x \in T$ or $a \in T$ (as T is a filter)] and $\nu x \notin T$. Which is absurd, since $a \in Y \nsubseteq T$ from hypothesis. Therefore, $(Y^\top, T) \subseteq T$. □

It is evident that the converse of Proposition 4.5 may not always hold. Specifically, consider the triangle algebra $\mathcal{L}$ from Example 3.3:

- Let $T = [1,1]$. We have $(L^\top, T) = T$, but T does not satisfy the PF2 property.
- For $X = \{[a,1]\}$ and $Y = \{[a,a],[1,1]\}$, we obtain $(X^\top, Y) = \{[b,b],[b,1],[1,1]\}$ which is $PF2$. But Y is not even a filter.

Proposition 4.6. *Let $\mathcal{L}$ be a triangle algebra, T a filter of $\mathcal{L}$ and Y a nonempty subset of L. If $\mathcal{L}$ is linear, then $(Y^\top, T) = T$ or $(Y^\top, T) = L$.*

Proof. Let us suppose that $(Y^\top, T) \neq L$ and prove that $(Y^\top, T) = T$. Since T is a filter of $\mathcal{L}$, then $T \subseteq (Y^\top, T)$, from Proposition 4.2 *(i)*.
Since $(Y^\top, T) \neq L$ then by *(ii)* of Proposition 4.2, we have $Y \nsubseteq T$. Thus, there exists $b \in L$ such that $b \in Y$ and $b \notin T$. Let $a \in (Y^\top, T)$; then for all $y \in Y$, $\nu a \vee y \in T$, which implies that, $\nu a \vee b \in T$ as, $b \in Y$. Also, since $\nu a \leq a$, and that $\nu a \vee b \leq a \vee b$, then $a \vee b \in T$ due to the fact that T is a filter. Now, since $\mathcal{L}$ is linear, then either $a \leq b$ or $b \leq a$. We claim that $a \nleq b$ otherwise, we would have $b = a \vee b \in T$ which is a contradiction. So, $b \leq a$. Consequently, $a = a \vee b \in T$. This shows that $(Y^\top, T) \subseteq T$. It yields that $(Y^\top, T) = T$. □

The converse of Proposition 4.6 is not always guaranteed. Revisiting Example 3.3, if we set ($Y = L$ and $T = \{[1,1]\}$) or ($Y = \{[1,1]\}$ and $T = \{[1,1]\}$), in both cases, we find that $(Y^\top, T) = T$ or $(Y^\top, T) = L$, whereas $\mathcal{L}$ is not linear.

5 Conclusion and Future Work

This article is in the general framework of the study of filters in triangle algebras. We have introduced the notion of relative co-annihilator, established some of its properties, and built the link with filters in triangle algebras. In addition, we proved that the co-annihilator of a nonempty subset T of a triangle algebra $\mathcal{L}$ relative to a filter Y of $\mathcal{L}$ preserves certain characteristics of the filter Y. In particular, if Y is a Boolean filter of the second kind, then the co-annihilator of T with respect to Y is also a Boolean filter of the second kind; the same applies to an implicative filter, a pseudo complementation filter, a Boolean filter, a prime filter, a prime filter of the third kind, a pseudo-prime filter or an involution filter, respectively. Moreover, we have presented certain conditions under which the co-annihilator of T with respect to Y is a prime filter of the second type.

Filters are particularly interesting as they are closely related to congruence relations, which are used to construct quotient algebras: from each filter, a congruence relation can be defined (see [7]). However, we have identified some inaccuracies in [16]: the relation θ given in [16, Example 4.2] is not a congruence relation since it is not reflexive. Also, contrary to what they claimed in [16, Example 4.3], the congruence relation θ does preserve co-annihilators. In the process of asserting that every congruence relation on an MTL-triangle algebra preserves co-annihilators, they claimed that the relation $\theta(Y) = \{(a,b) \in A \times A; \varphi(a) \cap Y = \varphi(b) \cap Y\}$ is a congruence relation, which is not always true. Hence, this remains an open problem in the framework of triangle algebras for further examination in future works.

In our forthcoming work, we will extend our exploration of algebraic structures, with a specific focus on triangle algebras. More precisely, since *ideals* also represent sets of provable formulas within algebraic structures, and knowing that *ideals* and *filters* are not dual notions in residuated lattices, it follows that they will not be dual notions in triangle algebras either, given that triangle algebras are enriched residuated lattices. A subsequent paper introducing *ideals in Triangle algebras* and proving soundness and completeness with respect to triangle algebras is in preparation.

Another challenge for the future is the investigation of the concepts of *annihilator and relative annihilator* in triangle algebras, viewed as special types of ideals.

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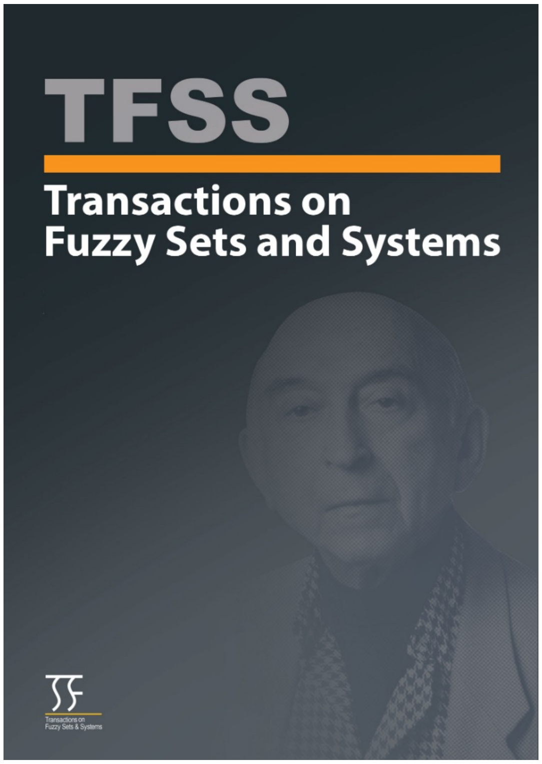

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Fuzzy Implication Operators Applied to Country Health Preparation

John N. Mordeson, **Sunil Mathew***, **Aswathi Prabhath**

Abstract. We use a new method to determine a fuzzy similarity measure using fuzzy implication operators. We use this method to determine the fuzzy similarity between the two rankings of countries involving health security and health care. We then find a fuzzy similarity of countries involving the two rankings of countries with respect to national disaster and political disaster.

AMS Subject Classification 2020: 94D05; 03E72
Keywords and Phrases: Fuzzy implication operators, Fuzzy similarity measures, Global health security index, Health care, National disaster, Political stability, Country rankings.

1 Introduction

The Global Health Security Index states that all countries remain dangerously unprepared for future epidemic and pandemic threats, including threats potentially more devastating than COVID-19, [1]. In [2], we ranked the Organization for Economic Cooperation and Development (OECD) countries with respect to their preparation. In [3], countries are ranked with respect to their health care. We find the fuzzy similarity measure between these two rankings. We use implication operators to define a new fuzzy similarity measure to find the fuzzy similarity of these rankings. We also consider the natural disaster risk, the political stability of OECD countries. We provide the rankings as given in [4, 5, 6]. The report in [4] considers a country's vulnerability and exposure to natural hazards to determine a ranking of countries around the world based on their natural disaster risk. The index of Political Stability and Absence of Violence/Terrorism measures perceptions of the likelihood that the government will be destabilized or overthrown by unconstitutional or violent means, including politically motivated violence and terrorism. We used five different fuzzy similarity measures. In three cases, we found the similarities to be medium and in two, we found the similarity to be low.

Let X be a set. Then the **fuzzy power set** of X, denoted $\mathcal{FP}(X)$, is the set of all fuzzy subsets of X. Define the relations $\vee, \wedge$ on the closed interval $[0,1]$ by for all $a, b \in [0,1], a \vee b$ is the maximum of a and b and $a \wedge b$ is the minimum of a and b.

Define $\overline{\wedge} : [0,1] \times [0,1] \to [0,1]$ by $\overline{\wedge}(a,b) = 1$ if $a = b$ and $a \wedge b$ if $a \neq b$. Define $\varnothing : [0,1] \times [0,1] \to [0,1]$ by $\varnothing(a,b) = 1$ if $a = b$ and $\frac{a \wedge b}{a \vee b}$ if $a \neq b$. Note that for all $a, b \in [0,1]$, $\varnothing(a,b) = \frac{a \wedge b}{a \vee b}$.

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2 Preliminary Results

Definition 2.1. *Let S be a function of $\mathcal{FP}(X) \times \mathcal{FP}(X)$ into $[0,1]$. Then S is called a **fuzzy similarity measure** on $\mathcal{FP}(X)$ if the following properties hold: $\forall \mu, \nu, \rho \in \mathcal{FP}(X)$:*
(1)$S(\mu, \nu) = S(\nu, \mu)$;
(2)$S(\mu, \nu) = 1$ *if and only if* $\mu = \nu$;
(3) *If* $\mu \subseteq \nu \subseteq \rho$, *then* $S(\mu, \rho) \leq S(\mu, \nu) \wedge S(\nu, \rho)$;
(4) *If* $S(\mu, \nu) = 0$, *then* $\forall x \in X, \mu(x) \wedge \nu(x) = 0$.

Example 2.2. Let μ, ν be fuzzy subsets of a set X. Then M and S are fuzzy similarity measures on $\mathcal{FP}(X)$, where

$$\begin{aligned} M(\mu, \nu) &= \frac{\sum_{x \in X} \mu(x) \wedge \nu(x)}{\sum_{x \in X} \mu(x) \vee \nu(x)}, \\ S(\mu, \nu) &= 1 - \frac{\sum_{x \in X} |\mu(x) - \nu(x)|}{\sum_{x \in X} (\mu(x) + \nu(x))}. \end{aligned}$$

Results concerning fuzzy similarity measures can be found in [7, 8].

Definition 2.3. *([9], p. 14) Let I be a function of $[0,1] \times [0,1]$ into $[0,1]$ such that $I(0,0) = I(0,1) = I(1,1) = 1$ and $I(1,0) = 0$. Then I is called an **implication operator**.*

An implication operator I is said to satisfy the **identity principle** if $I(x, x) = 1$ for all $x \in [0,1]$. An implication operator is said to satisfy the **ordering principle** if $x \leq y \Leftrightarrow I(x, y) = 1$, [10]. Clearly, the ordering principle implies the identity principle.

I_1, I_2, and L defined below are implication operators that satisfy the ordering principle.

Example 2.4. Let $x, y \in [0,1]$.
(1) Godel implication operator:$I_1(x, y) = 1$ if $x \leq y, I_1(x, y) = y$ otherwise.
(2) Goguen implication operator: $I_2(x, y) = 1$ if $x \leq y$ and $I_2(x, y) = y/x$ otherwise
(3) Luckasiewicz implication operator: $L(x, y) = (1 - x + y) \wedge 1$.

By ([2], Theorem 3.1), S_L is a fuzzy similarity, where $S_L(\mu_A, \mu_B) = \frac{1}{n} \sum_{x \in X} (1 - \mu_A(x)) \wedge (1 - \mu_B(x)) \wedge (\mu_A(x)) \wedge \mu_B(x)$.

Definition 2.5. *([9], p. 15) Let I be an implication operator. Define the fuzzy subset E_I of $\mathcal{FP}(X) \times \mathcal{FP}(X)$ by for all $\mu, \nu \in \mathcal{FP}(X)$,*

$$E_I(\mu, \nu) = \wedge\{\wedge\{I(\mu(x), \nu(x)) | x \in X\}, \wedge\{I(\nu(x), \mu(x)) | x \in X\}\}.$$

*Then $E_I(\mu, \nu)$ is called the **degree of sameness** of μ and ν.*

In [2], it was decided that the following definition would be more suitable than the previous definition for defining fuzzy similarity measures from implication operators.

Definition 2.6. *Let I be an implication operator. Define $S : \mathcal{FP}(X) \times \mathcal{FP}(X) \to [0,1]$ by for all $(\mu, \nu) \in \mathcal{FP}(X) \times \mathcal{FP}(X), S(\mu, \nu) = \frac{1}{n} \sum_{x \in X} I((\mu(x), \nu(x)) \wedge I((\nu(x), \mu(x)))$. Then S is called a **degree of likeness**.*

In ([2], Theorem 2.7), it was shown that the function S of Definition 2.6 is a fuzzy similarity measure.

An implication operator I is called a **hybrid monotonous implication operator if** $I(x, \_)$ is non decreasing for all $x \in [0,1]$ and $i(\_, y)$ is nonincreasing for all $y \in [0,1]$.

Other implication operators can be found in [9].

Let X be a set with n elements, $n > 1$, say $X = \{x_1, ..., x_n)$. Let A be one-to-one function of X onto $\{1, ..., n\}$. Then A is called a **ranking** of X. Define the fuzzy subset μ_A of X by for all $x \in X, \mu_A(x) = \frac{A(x)}{n}$. Then μ_A is called the **fuzzy subset associated with** A.

For two rankings A and B of $X, \sum_{x\in X}(A(x) + B(x)) = n(n+1)$ and so $\sum_{x\in X}(\mu_A(x) + \mu_B(x)) = n+1$. Thus for S of Example 2.2,

$$S(\mu_A, \mu_B) = 1 - \frac{\sum_{x\in X} |\mu_A(x) - \nu_B(x)|}{n+1}.$$

3 Main Results

Let S_1 and S_2 be the fuzzy similarity measures defined by I_1 and I_2 under Definition 2.6, respectively. Then

$$\begin{aligned} S_1(\mu, \nu) &= \frac{1}{n} \sum_{x\in X} \mu(x) \bar{\wedge} \nu(x), \\ S_2(\mu, \nu) &= \frac{1}{n} \sum_{x\in X} \mu(x) \varnothing \nu(x). \end{aligned}$$

We next consider how small S_1 can be with respect to rankings A and B.

Suppose n is even. Let A be the ranking: $1, 2, ...\frac{n}{2}, \frac{n+2}{2}..., n-1, n$ and let B be the ranking $n, n-1, ..., \frac{n+2}{2}, \frac{n}{2}, , , , , 2, 1$. Then

$$\begin{aligned} S_1(\mu_A, \mu_B) &= \frac{1}{n} \sum_{x\in X} \mu_A(x) \bar{\wedge} \mu_B(x) = \frac{1}{n}(2(1+2+...+\frac{n}{2}))\frac{1}{n} \\ &= \frac{1}{n}(2([\frac{n}{2}(\frac{n}{2}+1)])/2)\frac{1}{n} = \frac{1}{n^2}(\frac{n^2}{4} + \frac{n}{2}) = \frac{1}{4} + \frac{1}{2n}. \end{aligned}$$

Suppose that n is odd. Let A be the ranking $1, 2, ..., \frac{n+1}{2}, ..., n-1, n$ and B be the ranking $n, n-1, ..., \frac{n+1}{2}, ..., 2, 1$. Then

$$\begin{aligned} S_1(\mu_A, \mu_B) &= \frac{1}{n} \sum_{x\in X} \mu_A(x) \bar{\wedge} \mu_B(x) = \frac{1}{n}(1 + 2(1+2+...+\frac{n-1}{2}))\frac{1}{n} \\ &= \frac{1}{n^2}(1 + 2(\frac{n-1}{2})(\frac{n-1}{2}+1)/2) = \frac{1}{n^2}(1 + 2(\frac{n-1}{2}\frac{n+1}{2}\frac{1}{2})) \\ &= \frac{1}{n^2}(1 + \frac{n^2-1}{4}) = \frac{1}{n^2} + \frac{1}{4} - \frac{1}{4n^2} = \frac{1}{4} + \frac{3}{4n^2}. \end{aligned}$$

Example 3.1. Let $n = 6$. Let A be the ranking $1, 2, ..., 5, 6$ and B the ranking $6, ..., 2, 1$. Then $\mu_A(x_i) = \frac{i}{6}$ and $B(x_i) = \frac{6-i+1}{6}, i = 1, 2, ..., 6$. Hence

$$\begin{aligned} \frac{\mu_A(x_1) \wedge \mu_B(x_1)}{\mu_A(x_1) \vee \mu_B(x_1)} &= \frac{\frac{1}{6}}{\frac{6}{6}} = \frac{\mu_A(x_6) \wedge \mu_B(x_6)}{\mu_A(x_6) \vee \mu_B(x_6)}, \\ \frac{\mu_A(x_2) \wedge \mu_B(x_2)}{\mu_A(x_2) \vee \mu_B(x_2)} &= \frac{\frac{2}{6}}{\frac{5}{6}} = \frac{\mu_A(x_5) \wedge \mu_B(x_5)}{\mu_A(x_5) \vee \mu_B(x_5)}, \\ \frac{\mu_A(x_3) \wedge \mu_B(x_3)}{\mu_A(x_3) \vee \mu_B(x_3)} &= \frac{\frac{3}{6}}{\frac{4}{6}} = \frac{\mu_A(x_4) \wedge \mu_B(x_4)}{\mu_A(x_4) \vee \mu_B(x_4)}. \end{aligned}$$

Let $n = 5$. Let A be the ranking $1, 2, ..., 5$,and B the ranking $5, ..., 2, 1$. Then $\mu_A(x_i) = \frac{i}{5}$ and $B(x_i) = \frac{5-i+1}{5}, i = 1, 2, ..., 5$. Hence

$$\begin{aligned}
\frac{\mu_A(x_1) \wedge \mu_B(x_1)}{\mu_A(x_1) \vee \mu_B(x_1)} &= \frac{\frac{1}{5}}{\frac{5}{5}} = \frac{\mu_A(x_5) \wedge \mu_B(x_5)}{\mu_A(x_5) \vee \mu_B(x_5)}, \\
\frac{\mu_A(x_2) \wedge \mu_B(x_2)}{\mu_A(x_2) \vee \mu_B(x_2)} &= \frac{\frac{2}{5}}{\frac{4}{5}} = \frac{\mu_A(x_4) \wedge \mu_B(x_4)}{\mu_A(x_4) \vee \mu_B(x_4)}, \\
\frac{\mu_A(x_3) \wedge \mu_B(x_3)}{\mu_A(x_3) \vee \mu_B(x_3)} &= \frac{\frac{3}{5}}{\frac{3}{5}} = \frac{\mu_A(x_3) \wedge \mu_B(x_3)}{\mu_A(x_3) \vee \mu_B(x_3)}.
\end{aligned}$$

We see that for n odd, the middle term will yield the value 1.

The following discussion is to determine the smallest value a fuzzy similarity measure can be with respect to rankings. Let S be any fuzzy similarity measure with respect to some rankings A and B. We determine the smallest value S can be for the following reason: Say, the smallest value S can be is S^*. Then the ratio $\frac{S-S^*}{1-S^*}$ ranges from 0 to 1. A clearer picture of the similarity is thus provided.

Lemma 3.2. *(1) Suppose n is even. Let A be the ranking: $1, 2, ...\frac{n}{2}, \frac{n+2}{2}..., n-1, n$ and let B be the ranking $n, n-1, ..., \frac{n+2}{2}, \frac{n}{2}, , , , , 2, 1$. Then $\frac{1}{n}(\frac{\frac{n}{2}}{\frac{n}{2}+1} + ... + \frac{2}{n-1} + \frac{1}{n}) = (n+1)(\sum_{j=\frac{n}{2}+1}^{n} \frac{1}{j}) - \frac{n}{2}$.*

(2) Suppose n is odd. Let A be the ranking $1, 2, ..., \frac{n+1}{2}, ..., n-1, n$ and B be the ranking $n, n-1, ..., \frac{n+1}{2}, ..., 2, 1$. Then $\frac{\frac{n-1}{2}}{\frac{n+3}{2}} + ... + \frac{2}{n-1} + \frac{1}{n} = (n+1)\sum_{j=\frac{n+1}{2}}^{n} \frac{1}{j}) - \frac{n-1}{2}$.

Proof. (1) $\frac{\frac{n}{2}}{\frac{n}{2}+1} + ... + \frac{2}{n-1} + \frac{1}{n} =$

$$\begin{aligned}
\sum_{i=1}^{\frac{n}{2}} \frac{i}{n-i+1} &= \sum_{j=\frac{n}{2}+1}^{n} \frac{n-j+1}{j} = \sum_{j=\frac{n}{2}+1}^{n} (\frac{n}{j} - 1 + \frac{1}{j}) \\
&= (n+1)(\sum_{j=\frac{n}{2}+1}^{n} \frac{1}{j}) - \frac{n}{2}.
\end{aligned}$$

(2) $\frac{\frac{n-1}{2}}{\frac{n+3}{2}} + ... + \frac{2}{n-1} + \frac{1}{n} = \sum_{i=1}^{\frac{n-1}{2}} \frac{i}{n-i+1}$.

Let $j = n - i + 1$. Then $i = n - j + 1$ and $j = n, n-1, ..., \frac{n}{2} + \frac{3}{2}$. Now

$$\begin{aligned}
\sum_{i=1}^{\frac{n-1}{2}} \frac{i}{n-i+1} &= \sum_{j=\frac{n+3}{2}}^{n} \frac{n-j+1}{j} = \sum_{j=\frac{n+3}{2}}^{n} (\frac{n}{j} - 1 + \frac{1}{j}) \\
&= (n+1)(\sum_{j=\frac{n+3}{2}}^{n} \frac{1}{j}) - \frac{n-1}{2}.
\end{aligned}$$

□

Theorem 3.3. *(1) Suppose n is even. Let A be the ranking: $1, 2, ...\frac{n}{2}, \frac{n+2}{2}..., n-1, n$ and let B be the ranking $n, n-1, ..., \frac{n+2}{2}, \frac{n}{2}, , , , , 2, 1$. Then $S_2(\mu_A, \mu_B) = \frac{2}{n}[(n+1)(\sum_{j=\frac{n}{2}+1}^{n} \frac{1}{j}) - \frac{n}{2}]$.*

(2) Suppose n is odd. Let A be the ranking $1, 2, ..., \frac{n+1}{2}, ..., n-1, n$ and B be the ranking $n, n-1, ..., \frac{n+1}{2}, ..., 2, 1$. Then $S_2(\mu_A, \mu_B) = \frac{1}{n}[[(n+1)(\sum_{j=\frac{n+3}{2}}^{n} \frac{1}{j}) - \frac{n-1}{2}]2 + 1]$.

Proof. (1) $S_2(\mu_A, \mu_B) = \frac{1}{n}(\frac{\frac{n}{2}}{\frac{n+2}{2}} + ... + \frac{2}{n-1} + \frac{1}{n})2 = \frac{1}{n}(\frac{\frac{n}{2}}{\frac{n}{2}+1} + ... + \frac{2}{n-1} + \frac{1}{n})2 = \frac{2}{n}[(n+1)(\sum_{j=\frac{n}{2}+1}^{n} \frac{1}{j}) - \frac{n}{2}]$ by Lemma 3.2 (1).

(2) $S_2(\mu_A, \mu_B) = \frac{1}{n}((\frac{\frac{n-1}{2}}{\frac{n+3}{2}} + ... + \frac{2}{n-1} + \frac{1}{n})2 + 1) = \frac{1}{n}[[(n+1)(\sum_{j=\frac{n+3}{2}}^{n} \frac{1}{j}) - \frac{n-1}{2}]2 + 1]$ by Lemma 3.2 (2). □

We next determine approximate values for $\sum_{j=\frac{n}{2}+1}^{n} \frac{1}{j}$ when n is even and $\sum_{j=\frac{n+3}{2}}^{n} \frac{1}{j}$ when n is odd. Recall that $H_n = \sum_{j=1}^{n} \frac{1}{j}$ is a harmonic sum which sums approximately to $\gamma + \ln 2$, where γ is the Euler-Mascheroni constant, $\gamma \approx 0.5772$ and where $\approx$ denotes approximately equal to.

Let n be even. Consider $\sum_{j=\frac{n}{2}+1}^{n} \frac{1}{j}$. We have $\sum_{j=\frac{n}{2}+1}^{n} \frac{1}{j} = \sum_{j=1}^{n} \frac{1}{j} - \sum_{j=1}^{\frac{n}{2}} \frac{1}{j} \approx \gamma + \ln n - (\gamma + \ln \frac{n}{2}) = \ln n - \ln \frac{n}{2} = \ln 2$.

Let n be odd. Consider $\sum_{j=\frac{n+3}{2}}^{n} \frac{1}{j}$. We have $\sum_{j=\frac{n+3}{2}}^{n} \frac{1}{j} = \sum_{j=1}^{n} \frac{1}{j} - \sum_{j=1}^{\frac{n+1}{2}} \frac{1}{j} \approx \gamma + \ln n - (\gamma + \ln \frac{n+1}{2}) = \ln n - \ln \frac{n+1}{2} = \ln \frac{2n}{n+1}$.

Theorem 3.4. *(1) Suppose n is even. Let A be the ranking: $1, 2, ... \frac{n}{2}, \frac{n+2}{2} ..., n-1, n$ and let B be the ranking $n, n-1, ..., \frac{n+2}{2}, \frac{n}{2}, , , , , 2, 1$. Then $S_2(\mu_A, \mu_B) \approx 0.386 + \frac{2}{n} \ln 2$.*

(2) Suppose n is odd. Let A be the ranking $1, 2, ..., \frac{n+1}{2}, ..., n-1, n$ and B be the ranking $n, n-1, ..., \frac{n+1}{2}, ..., 2, 1$. Then $S_2(\mu_A, \mu_B) \approx 2 \ln \frac{2n}{n+1} + \frac{2}{n} \ln \frac{2n}{n+1} - 1 + \frac{2}{n}$.

Proof. Theorem 3.3 is used in the following arguments.

(1) We have

$$\begin{aligned} S_2(\mu_A, \mu_B) &= \frac{1}{n}(\sum_{j=1}^{\frac{n}{2}} \frac{j}{n-j+1})2 \\ &= \frac{2}{n}[(n+1)(\sum_{j=\frac{n}{2}+1}^{n} \frac{1}{j}) - \frac{n}{2}] \\ &\approx \frac{2}{n}[(n+1)\ln 2 - \frac{n}{2}] \\ &= (2 + \frac{2}{n})\ln 2 - 1 \\ &= 2\ln 2 + \frac{2}{n}\ln 2 - 1 \\ &\approx 0.386 + \frac{2}{n}\ln 2. \end{aligned}$$

(2) We have

$$\begin{aligned} S_2(\mu_A, \mu_B) &= \frac{1}{n}[[(n+1)(\sum_{j=\frac{n+3}{2}}^{n} \frac{1}{j}) - \frac{n-1}{2}]2 + 1] \\ &= \frac{2}{n}[(n+1)(\sum_{j=\frac{n+3}{2}}^{n} \frac{1}{j}) - \frac{n-1}{2}] + \frac{1}{n} \\ &\approx \frac{2}{n}[(n+1)\ln \frac{2n}{n+1} - \frac{n-1}{2}] + \frac{1}{n} \\ &= (2 + \frac{2}{n})\ln \frac{2n}{n+1} - (1 - \frac{1}{n}) + \frac{1}{n} \\ &= 2\ln \frac{2n}{n+1} + \frac{2}{n}\ln \frac{2n}{n+1} - 1 + \frac{2}{n}. \end{aligned}$$

□

Proposition 3.5. *Let $S_1, ..., S_n$ be fuzzy similarity measures on $\mathcal{FP}(X)$. Let $w_i \in [0,1]$ be such that $\sum_{i=1}^n w_i = 1$. Then $\sum_{i=1}^n w_i S_i$ is a fuzzy similarity measure on $\mathcal{FP}(X)$.*

Proof. Let $S = \sum_{i=1}^n w_i S_i$ and $\mu, \nu, \rho \in \mathcal{FP}(X)$. Then $S(\mu,\nu) = \sum_{i=1}^n w_i S_i(\mu,\nu) = \sum_{i=1}^n w_i S_i(\nu,\mu) = S(\nu,\mu)$. Now $S(\mu,\nu) = 1 \Leftrightarrow \sum_{i=1}^n w_i S_i(\mu,\nu) = 1 \Leftrightarrow S_i(\mu,\nu) = 1$ for $i = 1, ..., n \Leftrightarrow \mu = \nu$. Suppose that $\mu \subseteq \nu \subseteq \rho$. Then $S_i(\mu,\rho) \le S_i(\mu,\nu) \wedge S_i(\nu,\rho), i = 1, ..., n$. Hence

$$\begin{aligned}\sum_{i=1}^n w_i S_i(\mu,\rho) &\le \sum_{i=1}^n w_i[S_i(\mu,\nu) \wedge S_i(\nu,\rho)] = \sum_{i=1}^n w_i S_i(\mu,\nu) \wedge w_i S_i(\nu,\rho) \\ &\le \sum_{i=1}^n w_i S_i(\mu,\nu) \wedge \sum_{i=1}^n w_i S_i(\nu,\rho) = S(\mu,\nu) \wedge S(\nu,\rho).\end{aligned}$$

Suppose $S(\mu,\nu) = 0$. Then $\sum_{i=1}^n w_i S_i(\mu,\nu) = 0$. Thus $S_i(\mu,\nu) = 0$ for all i such that $w_i > 0$. Thus for all $x \in X, \mu(x) \wedge \nu(x) = 0$. □

Proposition 3.6. *Let $S_1, ..., S_n$ be fuzzy similarity measures on $\mathcal{FP}(X)$. Let $w_i \in [0,1]$ be such that $\sum_{i=1}^n w_i = 1$. Let a_i be the smallest value S_i can be, $i = 1, ..., n$. Then $\sum_{i=1}^n w_i a_i$ is the smallest value $\sum_{i=1}^n w_i S_i$ can be.*

Proof. Suppose $(\sum_{i=1}^n w_i S_i)(\mu,\nu) = b$. Then $\sum_{i=1}^n (w_i S_i)(\mu,\nu) = b$. Let $S_i(\mu,\nu) = b_i, i = 1, ..., n$. Then $b_i \ge a_i, i = 1, ..., n$. Now $b = \sum_{i=1}^n w_i b_i$ and so $b \ge \sum_{i=1}^n w_i a_i$. □

Converting a fuzzy similarity measures to a measure using the smallest value it can be, converts the measure to the interval $[0,1]$. We can say if this converted value lies between 0 and 0.2, the similarity is very low, from 0.2 to 0.4 the similarity is low, from 0.4 to 0.6 the similarity is medium, from 0.6 to 0.8 high, and from 0.8 to 1 very high.

4 Country Health

The 2021 Global Health Security Index measures the capacities of 195 countries to prepare for epidemics and pandemics. All countries remain dangerously unprepared for future epidemics and pandemic threats, including threats potentially more devastating than Covid-19, [3]. In [1], a ranking of countries with respect to health care is provided. We provide the ranking with respect to OECD countries.

Table 1: OECD health security and health care rankings

Country	Health Security	Health Care	Country	Health Security	Health Care
Australia	2	9	Korea, Rep.	8	1
Austria	22	5	Latvia	14	
Belgium	19	4	Lithuania	18	26
Canada	4	19	Luxembourg	35	
Chile	23	30	Mexico	21	23
Czech Rep.	30	12	Netherlands	10	3
Denmark	11	8	New Zealand	12	16
Estonia	25	18	Norway	17	13
Finland	3	11	Poland	24	29
France	13	6	Portugal	27	22
Germany	7	10	Slovak Rep.	29	28
Greece	32	31	Slovenia	5	27
Hungary	28	33	Spain	15	7
Iceland	34		Sweden	9	20
Ireland	26	32	Switzerland	20	17
Israel	36	15	Turkey	33	21
Italy	31	25	United Kingdom	6	14
Japan	16	2	United States	1	24

Let M and S be the fuzzy similarity measures of Example 2.2. We deleted the countries in the Health Security ranking that were not in the Health Care ranking and then reranked the Health Security countries. We found that $S(\mu_A,\mu_B) = 1 - \frac{223}{1122} = 1 - 0.199 = 0.801$. By ([10], Theorem 2.10), $S(\mu_A,\mu_B) = \frac{2M(\mu_A,\mu_B)}{1+M(\mu_A,\mu_B)}$. Hence $M(\mu_A,\mu_B) = \frac{S(\mu_A,\mu_B)}{2-S(\mu_A,\mu_B)} = \frac{0.801}{1.199} = 0.668$. With the countries deleted, $n = 33$. Thus the smallest M can be is $\frac{n+1}{3n-1} = \frac{34}{98} = 0.347$. The smallest S can be is $\frac{1}{2} + \frac{1}{2n} = \frac{1}{2} + \frac{1}{66} = 0.515$. Therefore,

$$\frac{0.668 - 0.347}{1 - 0.347} = \frac{0.321}{0.653} = 0.492$$

and

$$\frac{0.801 - 0.515}{1 - 0.515} = \frac{0.286}{0.485} = 0.590.$$

We see that in both cases the similarity is medium.

A fuzzy similarity measure using implication operators was defined in [2]: $S_L(\mu_A,\mu_B) = \frac{1}{n}\sum_{x\in X}[(1-\mu_A(x)) \wedge (1-\mu_B(x)) + \mu_A(x) \wedge \mu_B(x)]$. We have by ([2], Proposition 3.5) that $S_L = S + \frac{1}{n}(S-1)$. Thus $S_L(\mu_A,\mu_B) = 0.801 + \frac{1}{33}(0.801-1) = 0.801 - 0.006 = 0.795$. The smallest $S_L(\mu_A,\mu_B)$ is $\frac{1}{2} + \frac{1}{2n^2} = \frac{1}{2} + \frac{1}{2178} = 0.5 + 0.000459$ which we round off to 0.5. Thus$\frac{0.795-0.5}{1-0.5} = 0.59$. The similarity is thus medium.

We have that

$$\begin{aligned} S_1(\mu_A,\mu_B) &= \frac{1}{n}\sum_{x\in X} \mu_A \overline{\wedge} \mu_B(x) \\ &= \frac{1}{33}\left(\frac{428}{33}\right) = 0.393. \end{aligned}$$

The smallest S_1 can be is $\approx \frac{1}{4} + \frac{3}{4n^2} = 0.25 + 0.003 = 0.253$. Thus $\frac{0.393-0.253}{1-0.253} = \frac{0.140}{0.747} = 0.187$ and so the similarity is very low.

We find that $S_2(\mu_A, \mu_B) = \frac{17.921}{33} = 0.543$. The smallest S_2 can be is $\approx 2\ln\frac{66}{34} + \frac{2}{34} - 1 + \frac{3}{33} = 0.426$. Thus we have $\frac{0.543-0.426}{1-0.426} = \frac{0.117}{0.514} = 0.228$.

Hence the similarity is low.

We have that $\frac{1}{3}S_1 + \frac{1}{3}S_2 + \frac{1}{3}S_L \approx \frac{1}{3}(0.393+0.543+0.795) = \frac{1}{3}(1.649) = 0.577$. The smallest $\frac{1}{3}S_1 + \frac{1}{3}S_2 + \frac{1}{3}S_L$ can be is $\approx \frac{1}{3}(0.276 + 0.426 + 0.500) = \frac{1}{3}(1.202) = 0.401$.

Now $\frac{0.577-0.401}{1-0.401} = \frac{0.176}{0.499} = 0.353$. Here the similarity is low.

5 Natural Disaster, Political Stability, and Political Risk

We next consider the natural disaster risk, [4], the political stability, [6], and the political risk, [5], of OECD countries. We provide the rankings as given in [4, 5, 6]. The report in [4] systematically considers a country's vulnerability and exposure to natural hazards to determine a ranking of countries around the world based on their natural disaster risk. The index of Political Stability and Absence of Violence/Terrorism measures perceptions of the likelihood that the government will be destabilized or overthrown by unconstitutional or violent means, including politically motivated violence and terrorism. The index is an average of several other indexes from the Economist Intelligence Unit, the Economic Forum, and the Political Risk Services, among others, [6]. The Political Risk Index is the overall measure of risk for a given country, calculated by using all 17 risk components from the PRS Methodology including turmoil, financial transfer, direct investment, and export markets. The Index provides a basic convenient way to compare countries directly as well as demonstrating changes over the last five years, [5].

The rankings in the following tables are from low to high.

Table 2: OECD natural disaster and political stability rankings

Country	Natural Disaster	Political Stability	Country	Natural Disaster	Political Stability
Australia	34	17	Korea, Rep.	28	23
Austria	7	14	Latvia	13	22
Belgium	19	24	Lithuania	14	18
Canada	33	12	Luxembourg	1	3
Chile	30	32	Mexico	36	34
Czech Rep.	3	9	Netherlands	18	13
Denmark	5	10	New Zealand	29	1
Estonia	11	19	Norway	16	5
Finland	8	8	Poland	20	29
France	24	30	Portugal	22	11
Germany	17	20	Slovak Rep.	4	27
Greece	25	31	Slovenia	9	21
Hungary	2	15	Spain	27	26
Iceland	10	2	Sweden	12	7
Ireland	15	16	Switzerland	6	4
Israel	21	35	Turkey	31	36
Italy	26	25	United Kingdom	23	28
Japan	32	6	United States	35	33

Let M and S be the fuzzy similarity measures of Example 2.2. Here $n = 36$. We have that $S(\mu_A, \mu_B) = 1 - \frac{290}{1332} = 0.782$. Thus $M(\mu_A, \mu_B) = \frac{S(\mu_A,\mu_B)}{2-S(\mu_A,\mu_B)} = \frac{0.782}{1.218} = 0.642$. The smallest M can be is $\frac{n+2}{3n+2} = \frac{38}{110} = 0.345$.

Hence $\frac{0.642-0.345}{1-0.345} = 0.453$. Therefore, the similarity is medium. The smallest S can be is $\frac{n/2+1}{n+1} = \frac{19}{37} = 0.514$. Thus $\frac{0.782-0.514}{1-0.514} = 0.551$. Hence the similarity is medium.

$S_L(\mu_A, \mu_B) = 1 - \frac{1}{36^2}(149 + 141) = 1 - \frac{1}{1296}(290) = 0.7762$. The smallest $S_L(\mu_A, \mu_B)$ can be is 0.5. Thus $\frac{0.776-0.5}{1-0.5} = \frac{0.276}{0.5} = 0.552$. Hence the fuzzy similarity measure is medium.

$S_1(\mu_A, \mu_B) = \frac{\sum_{x\in X} \mu_A(x) \overline{\wedge} \mu_B(x)}{n} = \frac{549/36}{36} = 0.424$ and $S_2 \approx S_1(\mu_A, \mu_B) + \ln 2 - \frac{5}{8} = 0.424 + 0.068 = 0.492$. The smallest S_1 can be is $\frac{1}{4} + \frac{1}{n} = 0.25 + 0.028 = 0.278$ since $n = 36$ is even. Thus $\frac{0.492-0.278}{1-0.278} = \frac{0.214}{0.722} = 0.296$. Hence the similarity is low.

We find that $S_2(\mu_A, \mu_B) = \frac{22}{36} = 0.611$. The smallest S_2 can be is $\approx 0.386 + \frac{2}{36}(0.693) = 0.442$. Thus $\frac{0.611-0.442}{1-0.442} = \frac{0.169}{0.558} = 0.303$. Once again the similarity is low.

We have that $\frac{1}{3}S_1 + \frac{1}{3}S_2 + \frac{1}{3}S_L \approx \frac{1}{3}(0.424 + 0.611 + 0.776) = \frac{1}{3}(1.811) = 0.604$. The smallest $\frac{1}{3}S_1 + \frac{1}{3}S_2 + \frac{1}{3}S_L$ can be is $\approx \frac{1}{3}(0.278 + 0.442 + 0.500) = \frac{1}{3}(1.220) = 0.407$.

Now $\frac{0.604-0.407}{1-0.402} = \frac{0.197}{0.598} = 0.329$. The average similarity is low.

6 Conclusion

We used fuzzy implication operators to define the fuzzy similarity between the two rankings of countries involving health security and health care. We then found a fuzzy similarity involving the rankings of countries with respect to national disaster and political disaster. In each case, we found the similarity measures to be medium for S_L, M, and S and low for S_1 and S_2. Future research could involve other regions in the world other than the OECD countries. It was shown in ([2], Theorem 3.6) that $M \subseteq S_L \subseteq S$. It is clear that $S_1 \subseteq S_2$. Another potential project is to determine the relationship between S_2 and M. Further reading on implication operators can be found in [11].

Conflict of Interest: The authors declare no conflict of interest.

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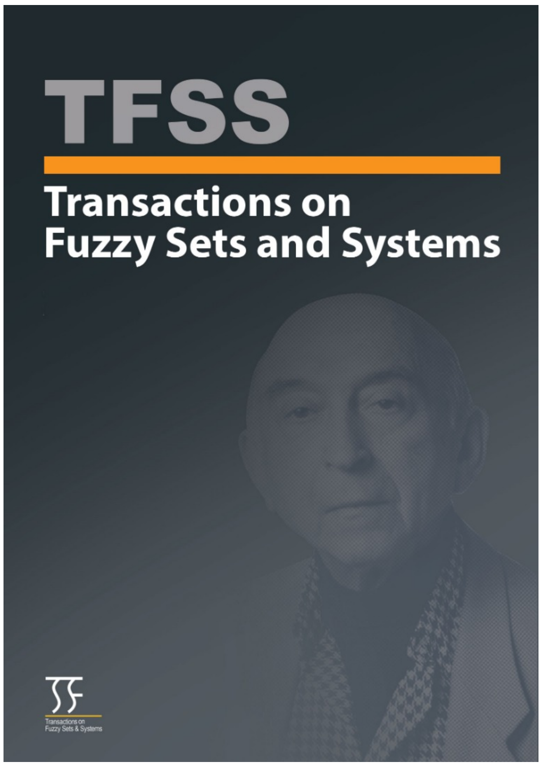

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A New Approach to Define the Number of Clusters for Partitional Clustering Algorithms

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Abstract. Data clustering consists of grouping similar objects according to some characteristic. In the literature, there are several clustering algorithms, among which stands out the Fuzzy C-Means (FCM), one of the most discussed algorithms, being used in different applications. Although it is a simple and easy to manipulate clustering method, the FCM requires as its initial parameter the number of clusters. Usually, this information is unknown, beforehand and this becomes a relevant problem in the data cluster analysis process. In this context, this work proposes a new methodology to determine the number of clusters of partitional algorithms, using subsets of the original data in order to define the number of clusters. This new methodology, is intended to reduce the side effects of the cluster definition phase, possibly making the processing time faster and decreasing the computational cost. To evaluate the proposed methodology, different cluster validation indices will be used to evaluate the quality of the clusters obtained by the FCM algorithms and some of its variants, when applied to different databases. Through the empirical analysis, we can conclude that the results obtained in this article are promising, both from an experimental point of view and from a statistical point of view.

AMS Subject Classification 2020: 68T01; 68T05; 68T27
Keywords and Phrases: Partitional clustering algorithms, Clustering fuzzy, Number of cluster.

1 Introduction

The concept of data clustering consists of clustering similar objects together into groups, which are called clusters, taking into account one or more common features [1, 2]. In this context, several clustering techniques have been proposed in the literature, including hierarchical and partitional algorithms, which are widely used in several applications of different areas of knowledge. Most clustering algorithms require the number of clusters in the partition as an input parameter. However, the ideal number of clusters that represents the dataset, most of the time, is unknown. Thus, the definition of the number of clusters is one of the fundamental problems in the process of clustering data [3].

In the literature, several approaches have been proposed to determine the number of clusters [4, 5, 6, 7, 8, 9]. In spite of the potential of these approaches, there is no universal approach that performs well for all clustering problems [10]. The definition of the number of clusters in the data clustering process can take time and have a high computational cost, especially when a dataset has a large number of instances and attributes. In general, it is necessary to run an algorithm several times with different cluster settings, to

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evaluate the result of each run, and then to select the best number of clusters. In addition, most applications have large datasets, which demand computational time and resources for processing. Therefore, the size of the dataset can become an aggravating factor in the definition of the number of clusters and, consequently, in the clustering process.

Despite presenting satisfactory results, the different approaches proposed in the literature to define the number of clusters [4, 5, 7, 8, 11, 12, 13, 14, 15] use the complete dataset, which can lead to problems such as high processing time and high computational cost. Consequently, the use of large datasets with high dimensions limits the application of a clustering algorithm, which can lead to an inefficient clustering process. Therefore, the main motivation of this work started from the need to try to improve the definition of the number of clusters of partitional algorithms, by proposing a new methodology to determine the number of clusters. The idea is to use a subset of the original data to define the number of clusters, in an attempt to mitigate the side effects of the cluster definition phase, possibly making processing time faster and lowering the computational cost.

2 The Proposed Approach

As previously mentioned, the phase of finding the best number of clusters for a dataset demands time and has a high computational cost, since it is performed by running an algorithm with different cluster configurations, evaluating each execution, and then evaluating each result to select the best number of clusters. In addition, the use of large datasets is an aggravating factor at this stage, due to the time and computational resources that this processing demands. In order to accelerate the definition of the number of clusters in the clustering process, this work presents a proposal for data partitioning applied to clustering algorithms, based on validation indices to determine the ideal subset of a dataset. The main objective of this proposal is to find, for each clustering algorithm considered in this research, a percentage p of the data that, for any dataset X we can choose a $p\%$ of the data in X to determine the best number of clusters. Our goal is to define an expressive p value in a way that its performance is similar to when the whole dataset is considered.

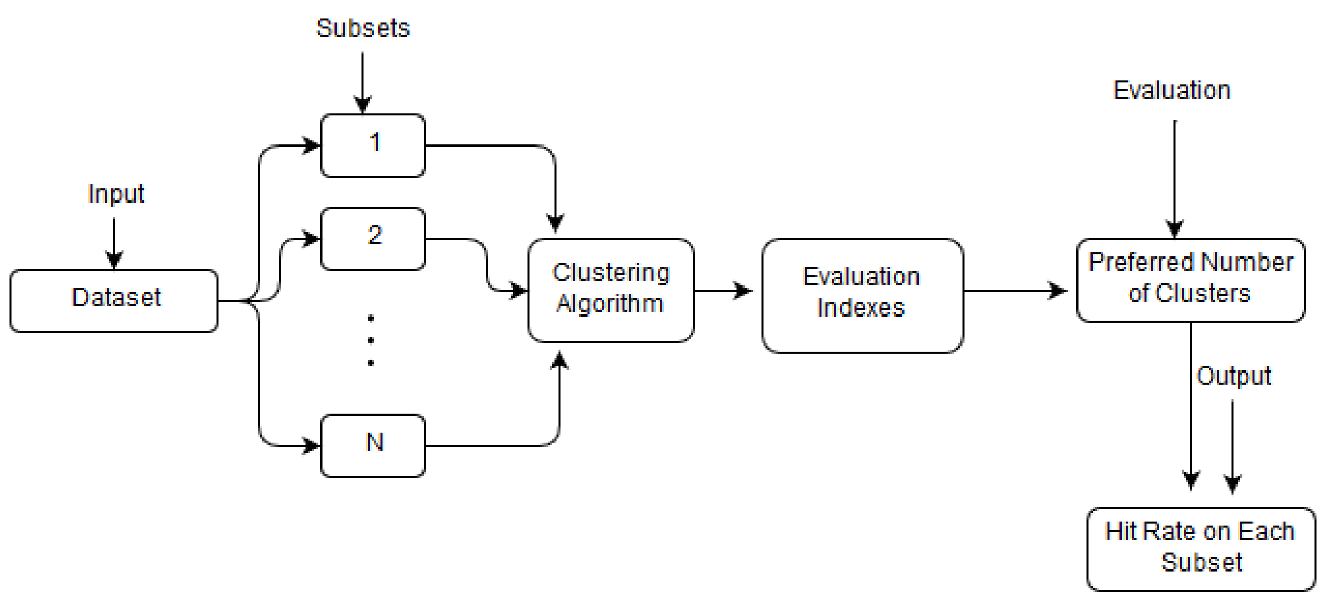

Figure 1: Structure of the proposed approach

Figure 1 presents the steps of the approach proposed in this work. A dataset is partitioned into N subsets.

Each subset has different sizes in relation to the total set, i.e., each subset has a particular percentage of the total set (p). The instances of each subset are randomly selected. and each subset has different sizes (with a variation of 10% each). The clustering algorithm is applied on all subsets and also on the full dataset ($p = 100\%$). Each clustering is evaluated based on validation indices and the best number of clusters for each index is defined. Finally, an analysis is performed on the number of clusters selected by each index, in an attempt to identify the minimum percentage of data sufficient to infer the number of clusters.

As an illustrative example, consider a given dataset with 1000 instances and 5 attributes. We need to cluster the instances in k clusters that best represent this dataset. Initially, the dataset is divided into N subsets, each of which has a size corresponding to a different percentage of the full dataset. Usually, we start with a low percentage. Then, there is a gradual increase of this percentage in the following subsets, until it reaches 100% ($p = 100\%$). For example, we can start with 10% of the original data and increase the percentage by 10%, until it reaches 100% of the data. In this case, we will have 10 subsets with 10, 20, ..., 100% of the original dataset. The next step is to apply clustering algorithms on each subset, varying the number of clusters k, $k_{min} \leq k \leq k_{max}$. Suppose we use $k_{min} = 2$ and $k_{max} = 10$. In this case, each clustering algorithm is applied to each subset of the original dataset, varying the number of clusters from 2 to 10. Then, the validation stage of the clusters is performed based on validation indices, in order to identify the subset that best represents the entire dataset. Each subset has the best cluster number defined according to the used validation index. For example, suppose that the best cluster number is $k = 3$ for $p = 10\%$ and $p = 100\%$ and $k = 4$ for $p = 50\%$, according to an index I. Note that the number of clusters set for 10% and 100% of the data is the same. In this example, the use of 10% of the data would be sufficient to infer the number of clusters that best represent the dataset. Finally, it is possible to identify the minimum data subset sufficient to infer the number of clusters that best represent the dataset, after analyzing the number of clusters defined by an index I in each subset.

3 Experimental Setting up

In order to evaluate the feasibility of the proposed method, an empirical analysis will be conducted. This section describes the main aspects of this empirical analysis.

3.1 Methods and Material

As previously discussed, the methodology proposed in this paper aims to investigate the minimum amount of data sufficient to infer the number of clusters in a dataset using partitional algorithms.

Initially, each dataset was normalized and partitioned into subsets of different sizes. The experiments carried out compared the performance of each subset in relation to the original dataset, through the performance obtained by the clustering validation indices (described in 3.4). The quality of a cluster is related to the ability to find the ideal number of clusters.

In order to evaluate the quality of the obtained partitions, an experimental methodology was proposed in order to investigate and identify the best number of clusters for the dataset. Each clustering algorithm runs on each subset and on the original dataset for each number of clusters k, $k_{min} \leq k \leq k_{max}$, aiming to find the best number of clusters in each subset as well as in the original dataset. Then, for each validation index, the number of clusters selected is stored. This experiment is repeated 31 times, with different initialization in each run [16]. The purpose of performing different initializations is to get as close as possible to the most likely number of clusters for a given dataset. The number of clusters is obtained after checking the most frequent value resulting from the 31 iterations. The variation in the number of clusters, k, being $k_{min} = 2$ and $k_{max} = 10$, took place based on the number of classes of all analyzed datasets.

3.2 Datasets

In this analysis, 30 datasets were imported from UCI Machine Learning Repository [17], Kaggle Datasets [18] and GitHub. The main features of these datasets are described in Table 1. All datasets had been preprocessed, with the goal of correcting some issues, such as attributes in different scales and missing values. In addition, it is noteworthy that they are supervised classification datasets and, for this reason, the class attribute was removed for the clustering process.

Table 1: Dataset features

Datasets	Instances	Attributes	Classes
Balance Scale	625	4	3
Banknote Authentication (BA)	1372	4	2
Bupa	345	6	2
Climate Model Simulation Crashes CMSC)	540	20	2
Cnae-9	1080	856	9
Column 3C	310	6	3
Contraceptive	1473	9	3
Ecoli	336	7	8
Glass Identification (GI)	214	9	6
Heart Statlog	270	13	2
Haberman	306	3	2
Ionosphere	351	34	2
Iris	150	4	3
Lymphography	148	18	4
Molecular Biology (MB)	3190	60	3
Multiple Features (MF)	2000	64	10
Parkinsons	195	22	2
Pima	768	8	2
Planning Relax	182	12	2
QSAR Biodegradation (QSAR-B)	1055	41	2
Robot Failure lp4	117	90	3
Seeds	210	7	3
Semeion	1593	256	10
Sonar	208	60	2
Steel Plates Faults (SPF)	1941	33	2
Thoracic Surgery (TS)	470	16	2
User Knowledge Modeling (UKM)	403	5	5
Vehicle	846	18	4
Voice	3168	20	2
Wine	178	13	3

3.3 Clustering Algorithms

Partitional algorithms can be divided into two approaches: *crisp* and *fuzzy*. In the *crisp* approach, each observation in the dataset belongs exclusively to a single cluster, while in the *fuzzy* approach, each object can belong to more than one cluster with a degree of relevance $u_{ij} \in [0, 1]$. In this article, the focus is on the *fuzzy* approach, specifically the following algorithms: FCM [19, 20], ckMeans [21] and FCMσ [22].

Based on the concept of fuzzy logic, the Fuzzy C-means algorithm [19, 20] divides a dataset $X = \{x_1, x_2, ..., x_n\}$ into k clusters, resulting in a fuzzy partition matrix $[\mu_{ij}]_{n\times k}$, called a membership matrix [23].

The Fuzzy C-Means algorithm searches for the fuzzy partition of the dataset, which minimizes the objective function, given in Equation (1).

$$J = \sum_{i=1}^{n}\sum_{j=1}^{k} \mu_{ij}^{m} d(x_i; c_j)^2 \tag{1}$$

where m is the fuzzification parameter, [2] which defines the allowed distance between an object (point) and cluster centers; x_i is i-th data object; c_j is the center of the j-th cluster; μ_{ij} is the membership degree of x_i to j-th cluster; $d(x_i; c_j)$ is the distance between x_i and c_j. Note that c_j does not necessarily belong to the dataset, but has similar composition (same attributes) of the elements in the dataset.

The Fuzzy C-Means algorithm receives as input a dataset $X = \{x_1, x_2, ..., x_n\}$, the number of clusters k and the value of m. Then, it initializes the membership matrix μ; calculating the initial fuzzy membership matrix μ according to Equation (2).

$$\mu_{ij} = \frac{1}{\sum_{l=1}^{k}\left(\frac{d(x_i;c_j)}{d(x_i;c_l)}\right)^{\frac{2}{m-1}}} \tag{2}$$

After **calculating** J using Equation (1); the center of the cluster j is calculated using Equation (3).

$$c_j = \frac{\sum_{i=1}^{n} \mu_{ij}^{m} x_i}{\sum_{i=1}^{n} \mu_{ij}^{m}} \tag{3}$$

The algorithm continues to update of the fuzzy membership matrix according to Equation (2), as well as the centers of the clusters, according to Equation (3). This process continues until a stopping condition is reached. The two most usual stop conditions of clustering algorithms are: defining a fixed number of iterations and defining a threshold $\varepsilon > 0$, stopping the process when $||J^t - J^{(t-1)}|| \leq \varepsilon$, where J^t is the objective function calculated in the current iteration and $J^{(t-1)}$ is the objective function calculated in the previous iteration.

The other clustering algorithms considered in this research are variants from the FCM. The ckMeans algorithm follows the same FCM framework, however, it differs in how to calculate the cluster centers [24, 25]. Finally, the FCMσ changes the distance metric used in the conventional FCM to a new metric, taking into account a distance variation in each cluster [22].

3.4 Validation Indices

In clustering tasks, it is important to evaluate the resulting partition to determine the quality of the obtained solution, as well as whether it is satisfactory for the desired goal. The evaluation of the partition can be done by clustering validation indices, which can be broadly divided into two main categories: internal and external. The internal indices measure the similarity of a partition using the instances of the clusters obtained by the clustering algorithm. External indices use external information related to the partition, usually the class labels to evaluate the obtained partition [26, 27].

[2] The value of m influences directly on the resulting cluster. High values of m result in less well defined clusters [28].

In this article, three cluster validation indices will be used: (i) Modified Partition Coefficient (MPC); (ii) Sugeno; e (iii) Xie and Beni (XB). The first index involves only the membership matrix, while the second and third ones use membership matrix and information about the dataset. These are well-known indices and the formal definitions of these validation indices can be found easily in the literature[29, 30, 31].

4 Experimental Results

Tables 2, 3 and 4 illustrate the results obtained for all analyzed datasets, using the FCM algorithm and the MPC, XB and Sugeno indices, respectively. This experiment was replicated for the other clustering algorithms and validation indices. Tables 5, 6 and 7 present the results obtained for all analyzed datasets, using the ckMeans algorithm and the MPC, XB and Sugeno indices, respectively. Tables 8, 9 and 10 present the results obtained for all datasets analyzed, using the FCMσ algorithm and the MPC, XB and Sugeno indices, respectively. The rows of each table correspond to each dataset and the columns correspond to the percentage of the data of the corresponding subset. The values presented in each cell correspond to the number of groups selected for each dataset and the number of times this number was selected (in parentheses), out of 31 repetitions. Finally, the shaded cells indicate the cases in which the algorithm-index configuration selected the same number of groups in all analyzed percentages and in the original dataset.

Table 2: FCM-MPC: Number of clusters selected for multiple subsets of data.

Datasets	Percentages of Data									
	10%	**20%**	**30%**	**40%**	**50%**	**60%**	**70%**	**80%**	**90%**	**100%**
Balance	10(10)	4(9)	6(7)	4(6)	6(13)	6(18)	6(15)	6(25)	6(25)	6(30)
BA	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)
Bupa	2(23)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)
CMSC	2(19)	2(28)	2(29)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)
Cnae-9	2(5)3(5)	3(5)5(5)8(5)	3(9)	3(9)	9(8)	2(9)	10(6)	2(7)	2(6)	3(10)
Column 3C	2(28)	2(29)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)
Contraceptive	2(25)	2(27)	2(28)	2(30)	2(30)	2(30)	2(31)	2(31)	2(31)	2(31)
Ecoli	10(7)	3(24)	3(22)	3(27)	3(28)	3(29)	3(31)	3(30)	3(31)	3(31)
GI	2(20)	2(24)	2(27)	2(21)	2(17)	2(19)	3(18)	2(16)	3(19)	3(23)
Haberman	10(15)	3(7)	4(9)	4(14)	4(16)	4(19)	4(16)	3(14)4(14)	4(15)	3(19)
Heart Statlog	2(11)	2(22)	2(28)	2(27)	2(31)	2(31)	2(31)	2(31)	2(30)	2(31)
Ionosphere	2(15)	2(30)	2(29)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)
Iris	2(29)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)
Lymph	10(31)	10(20)	2(18)	2(14)	2(15)	2(16)	2(16)	2(18)	2(19)	2(28)
MB	10(23)	10(15)	10(21)	10(13)	10(11)	10(9)	10(12)	9(8)	10(8)	10(11)
MF	3(16)	2(21)	2(17)	2(23)	2(22)	2(20)	2(24)	2(24)	2(20)	2(23)
Parkinsons	2(14)	3(19)	3(23)	3(25)	3(28)	3(31)	3(30)	3(30)	3(27)	3(28)
Pima	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)
Planning Relax	10(31)	10(26)	2(12)	3(11)	2(18)	2(15)	3(14)	3(12)	3(12)	2(22)
QSAR-B	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)
Robot Failure	10(31)	10(28)	2(23)	2(24)	2(27)	2(28)	2(29)	2(28)	2(29)	2(26)
Seeds	10(15)	2(28)	2(30)	2(30)	2(31)	2(31)	2(30)	2(31)	2(30)	2(31)
Semeion	3(9)	2(8)	2(10)	2(13)	2(11)	2(18)	2(12)	2(19)	2(16)	2(18)
Sonar	10(29)	2(9)	2(19)	2(18)	2(23)	2(18)	2(20)	2(15)3(15)	2(19)	2(27)
SPF	2(18)	2(25)	2(29)	2(30)	2(30)	2(31)	2(31)	2(31)	2(31)	2(31)
TS	2(26)	2(23)	2(31)	2(30)	2(30)	2(31)	2(31)	2(29)	2(31)	2(31)
UKM	10(21)	10(7)	2(17)	2(21)	2(21)	2(26)	2(29)	2(30)	2(31)	2(29)
Vehicle	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)
Voice	2(27)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)
Wine	10(12)	4(11)	3(12)	4(13)	3(13)4(13)	3(12)4(12)	4(15)	4(17)	4(20)	4(19)

Table 3: FCM-XB: Number of clusters selected for multiple subsets of data

Datasets	Percentages of Data									
	10%	**20%**	**30%**	**40%**	**50%**	**60%**	**70%**	**80%**	**90%**	**100%**
Balance	10(11)	9(11)10(11)	10(12)	10(17)	10(13)	10(17)	10(22)	10(22)	10(24)	10(18)
GI	2(7)	2(12)	2(11)	2(9)5(9)6(9)	2(11)	2(11)	2(9)	5(18)	2(12)	5(30)
Bupa	2(10)	2(19)	2(16)	2(25)	2(26)	2(23)	2(28)	2(28)	2(29)	2(31)
CMSC	2(23)	2(29)	2(29)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)
Cnae-9	2(8)3(8)	2(13)	2(13)	2(9)3(9)	2(14)	2(13)	2(15)	2(16)	2(16)	2(13)
Column 3C	10(13)	2(24)	2(28)	2(28)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)
Contraceptive	2(8)	3(20)	3(26)	3(20)	3(22)	3(24)	3(29)	3(26)	3(27)	3(30)
Ecoli	10(7)	9(7)	3(9)	3(12)	3(8)	3(14)	3(12)	3(14)	3(9)9(9)	8(12)
GI	2(15)	2(17)	2(25)	2(21)	2(17)	2(18)	3(19)	2(16)	3(19)	3(23)
Haberman	3(6)8(6)10(6)	3(10)	8(7)	3(13)	3(14)	7(8)	3(12)	3(13)	3(11)	3(18)
Heart Statlog	10(8)	9(6)10(6)	5(9)	7(6)8(6)9(6)10(6)	10(10)	9(7)10(7)	10(9)	8(11)	9(9)	9(14)
Ionosphere	2(16)	2(25)	2(25)	2(30)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)
Iris	2(28)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)
Lymph	10(29)	2(9)	2(17)	3(16)	3(17)	3(19)	3(16)	3(19)	3(17)	3(18)
MB	10(26)	10(25)	10(23)	10(26)	10(20)	10(28)	10(22)	10(22)	10(21)	10(23)
MF	2(12)	2(20)	2(15)	2(21)	2(22)	2(13)	2(22)	2(21)	2(15)	2(25)
Parkinsons	2(16)	3(17)	3(24)	3(24)	3(28)	3(31)	3(31)	3(30)	3(30)	3(29)
Pima	2(28)	2(30)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)
Planning Relax	10(20)	10(9)	2(14)	2(11)	2(16)	2(16)	2(12)	2(12)	2(18)	2(21)
QSAR-B	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)
Robot Failure	10(31)	10(17)	2(21)	2(25)	2(26)	2(26)	2(29)	2(28)	2(29)	2(26)
Seeds	10(9)	2(7)	5(5)7(5)	7(9)	6(8)	6(6)7(6)10(6)	6(9)	9(9)	6(8)9(8)	10(14)
Semeion	10(9)	10(10)	10(7)	9(8)	10(12)	10(7)	10(7)	10(7)	10(10)	10(9)
Sonar	10(18)	3(15)	2(17)	3(14)	2(17)	3(16)	3(15)	3(20)	3(22)	3(25)
SPF	2(23)	2(28)	2(25)	2(29)	2(31)	2(30)	2(31)	2(31)	2(31)	2(31)
TS	2(26)	2(24)	2(31)	2(30)	2(30)	2(31)	2(31)	2(29)	2(31)	2(31)
UKM	9(11)	9(10)10(10)	10(15)	9(12)	10(9)	7(8)	10(8)	6(8)	7(15)	7(13)
Vehicle	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)
Voice	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)
Wine	3(11)	3(13)	4(13)	3(17)	3(16)	3(15)4(15)	4(16)	4(22)	4(19)	4(19)

Table 4: FCM-Sugeno: Number of clusters selected for multiple subsets of data

Datasets	**Percentages of Data**									
	10%	**20%**	**30%**	**40%**	**50%**	**60%**	**70%**	**80%**	**90%**	**100%**
Balance	10(22)	10(19)	10(25)	10(27)	10(25)	10(26)	10(29)	10(26)	10(29)	10(27)
GI	10(11)	9(10)10(10)	10(11)	9(15)	9(16)	9(19)	9(19)	9(27)	9(16)	9(17)
Bupa	10(13)	6(6)	10(7)	5(7)	5(6)	5(9)	7(8)	5(12)	5(10)7(10)	5(21)
CMSC	8(8)	9(9)	4(10)	5(8)	5(15)	5(10)	10(7)	5(9)	5(10)	5(18)
Cnae-9	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)
Column 3C	10(23)	10(10)	4(10)	5(10)	5(16)	5(14)	5(14)	4(15)	5(23)	5(20)
Contraceptive	10(19)	10(24)	10(28)	10(28)	10(28)	10(29)	10(29)	10(30)	10(31)	10(31)
Ecoli	10(24)	10(17)	10(17)	10(10)	10(10)	9(10)	10(11)	10(11)	7(13)	7(14)
GI	10(13)	3(10)	4(14)	3(10)	4(14)	3(13)	3(18)	3(15)	3(19)	3(23)
Haberman	10(12)	10(14)	10(10)	10(9)	10(13)	10(12)	10(13)	10(14)	10(22)	10(20)
Heart Statlog	10(17)	10(15)	10(16)	9(16)	10(15)	10(15)	9(11)	8(13)	10(14)	9(10)
Ionosphere	10(22)	10(20)	10(15)	10(19)	10(18)	10(16)	10(22)	10(20)	10(23)	10(20)
Iris	10(15)	2(29)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)
Lymph	10(31)	10(29)	10(29)	10(26)	10(28)	10(30)	10(31)	10(31)	10(30)	10(31)
MB	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)
MF	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)
Parkinsons	10(8)	3(19)	3(23)	3(25)	3(28)	3(31)	3(31)	3(30)	3(29)	3(29)
Pima	8(6)	3(10)	2(7)	3(9)	3(14)	3(15)	3(14)	3(22)	3(22)	3(28)
Planning Relax	10(31)	10(30)	10(30)	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)
QSAR-B	5(8)	5(16)	6(11)	5(15)	5(19)	6(16)	5(16)	5(18)	5(21)	5(30)
Robot Failure	10(31)	10(31)	10(29)	10(27)	10(25)	10(29)	10(31)	10(31)	10(31)	10(31)
Seeds	10(23)	10(15)	7(8)9(8)10(8)	10(12)	10(11)	9(12)	9(9)10(9)	9(11)	9(10)10(10)	10(14)
Semeion	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)
Sonar	10(31)	10(25)	10(26)	10(29)	10(29)	10(29)	10(30)	10(30)	10(31)	10(31)
SPF	3(9)	5(11)	5(16)	5(17)	5(19)	5(18)	5(20)	7(16)	7(19)	7(27)
TS	2(21)	2(19)	2(29)	2(29)	2(30)	2(31)	2(31)	2(29)	2(31)	2(31)
UKM	10(27)	10(22)	10(20)	10(21)	10(25)	10(29)	10(26)	10(25)	10(27)	10(25)
Vehicle	4(12)	2(18)	2(15)	2(16)	2(21)	2(22)	2(19)	2(21)	2(27)	2(31)
Voice	4(13)	4(13)	4(19)	4(16)	4(25)	4(21)	4(21)	4(22)	4(21)	4(31)
Wine	10(15)	10(10)	4(9)	4(10)	4(12)	4(13)	4(15)	4(19)	4(18)	4(19)

Table 5: ckMeans-MPC: Number of clusters selected for multiple subsets of data

Datasets	Percentages of Data									
	10%	**20%**	**30%**	**40%**	**50%**	**60%**	**70%**	**80%**	**90%**	**100%**
Balance	4(8)	2(8)	4(12)	4(13)	3(14)	4(12)	4(17)	4(14)	3(15)	3(13)4(13)
GI	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)
Bupa	2(22)	2(29)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)
CMSC	2(24)	2(30)	2(30)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)
Cnae-9	3(12)	3(11)	3(18)	3(22)	3(22)	3(17)	3(22)	3(23)	3(22)	3(23)
Column 3C	2(30)	2(30)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)
Contraceptive	2(29)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(30)	2(31)
Ecoli	3(15)	3(20)	3(24)	3(20)	3(24)	3(23)	3(24)	3(29)	3(30)	3(30)
GI	2(18)	2(28)	2(29)	2(29)	2(31)	2(31)	2(31)	2(30)	2(30)	2(31)
Haberman	10(7)	3(13)	3(13)	3(19)	4(16)	3(17)	3(21)	3(25)	3(28)	3(20)
Heart Statlog	2(13)	2(23)	2(25)	2(27)	2(29)	2(29)	2(31)	2(31)	2(31)	2(31)
Ionosphere	2(25)	2(27)	2(31)	2(30)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)
Iris	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)
Lymph	10(30)	10(11)	2(25)	2(21)	3(18)	2(18)	3(23)	3(21)	3(21)	3(29)
MB	2(22)	2(27)	2(28)	2(30)	2(31)	2(31)	2(30)	2(31)	2(30)	2(30)
MF	2(24)	2(26)	2(28)	2(30)	2(31)	2(31)	2(29)	2(31)	2(31)	2(31)
Parkinsons	2(15)	2(19)	2(23)	2(24)	2(24)	2(21)	2(28)	2(28)	2(30)	2(31)
Pima	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)
Planning Relax	10(26)	10(15)	2(11)	2(14)	3(11)	2(16)	3(15)	2(16)	2(21)	2(27)
QSAR-B	2(30)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)
Robot Failure	10(31)	10(19)	2(17)	2(28)	2(30)	2(31)	2(31)	2(31)	2(31)	2(31)
Seeds	10(11)	2(24)	2(30)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)
Semeion	2(14)	2(20)	2(16)	2(22)	2(24)	2(20)	2(19)	3(19)	3(18)	3(24)
Sonar	10(21)	2(15)	2(15)3(15)	2(15)3(15)	2(17)	2(17)	2(18)	2(19)	2(16)	2(29)
SPF	2(23)	2(26)	2(29)	2(31)	2(30)	2(30)	2(31)	2(31)	2(31)	2(31)
TS	2(31)	2(30)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)
UKM	10(8)	2(18)	2(27)	2(22)	2(23)	2(29)	2(27)	2(28)	2(29)	2(31)
Vehicle	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)
Voice	2(26)	2(30)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)
Wine	10(14)	4(12)	3(15)	3(15)	3(17)	3(18)	3(13)	3(14)	3(16)	4(30)

Table 6: ckMeans-XB: Number of clusters selected for multiple subsets of data.

Datasets	Percentages of Data									
	10%	**20%**	**30%**	**40%**	**50%**	**60%**	**70%**	**80%**	**90%**	**100%**
Balance	10(10)	10(16)	10(21)	10(22)	10(26)	10(24)	10(26)	10(23)	10(27)	10(27)
GI	8(10)	9(11)	10(8)	9(11)	10(10)	9(12)	10(10)	10(12)	10(12)	10(13)
Bupa	10(13)	2(10)	10(12)	10(11)	10(14)	10(15)	10(14)	10(11)	10(10)	10(8)
CMSC	2(28)	2(30)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)
Cnae-9	10(27)	10(25)	10(27)	10(24)	10(24)	10(23)	10(26)	10(23)	10(22)	10(26)
Column 3C	10(13)	10(16)	10(14)	10(11)	10(21)	10(19)	10(20)	10(19)	10(21)	10(21)
Contraceptive	10(15)	10(17)	10(21)	10(18)	10(20)	10(20)	10(16)	9(14)10(14)	10(19)	10(17)
Ecoli	10(11)	10(11)	9(12)	10(11)	10(13)	10(12)	10(12)	10(11)	10(11)	10(12)
GI	2(12)	2(27)	2(28)	2(27)	2(30)	2(30)	2(31)	2(30)	2(30)	2(31)
Haberman	4(8)	4(7)	3(11)	3(16)	3(12)	3(10)	3(15)	3(17)	3(22)	3(17)
Heart Statlog	10(13)	10(8)	10(10)	9(9)	8(8)	10(11)	10(13)	9(11)10(11)	10(10)	9(17)
Ionosphere	10(8)	6(8)7(8)	8(8)	9(9)10(9)	10(9)	8(9)9(9)	7(9)	8(11)	7(9)8(9)	8(7)10(7)
Iris	2(29)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)
Lymph	10(27)	10(20)	10(19)	10(25)	10(26)	10(20)	10(23)	10(24)	10(18)	10(20)
MB	10(29)	10(31)	10(26)	10(28)	10(29)	10(29)	10(26)	10(29)	10(28)	10(28)
MF	10(23)	10(18)	10(24)	10(28)	10(26)	10(29)	10(21)	10(27)	10(24)	10(23)
Parkinsons	2(15)	2(19)	2(22)	2(24)	2(24)	2(21)	2(28)	2(28)	2(30)	2(31)
Pima	2(27)	2(28)	2(29)	2(26)	2(28)	2(29)	2(30)	2(31)	2(31)	2(30)
Planning Relax	10(20)	10(22)	10(21)	10(23)	10(20)	10(19)	10(22)	10(20)	10(22)	10(24)
QSAR-B	10(13)	9(11)	10(15)	10(12)	10(11)	10(12)	9(12)	10(15)	10(18)	10(11)
Robot Failure	10(30)	10(19)	10(16)	10(13)	10(13)	10(10)	10(14)	9(8)10(8)	10(11)	8(11)
Seeds	7(8)	5(6)	10(8)	9(7)10(7)	9(8)	10(8)	10(9)	9(9)	8(12)	10(13)
Semeion	10(27)	10(26)	10(28)	10(27)	10(24)	10(28)	10(31)	10(26)	10(28)	10(26)
Sonar	10(20)	10(17)	10(22)	10(18)	10(20)	10(23)	10(20)	10(20)	10(21)	10(21)
SPF	2(28)	2(30)	2(29)	2(31)	2(31)	2(30)	2(31)	2(31)	2(31)	2(31)
TS	2(26)	2(30)	2(31)	2(30)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)
UKM	10(18)	10(15)	10(19)	10(18)	10(18)	10(20)	10(16)	10(22)	10(15)	10(17)
Vehicle	2(30)	2(31)	2(29)	2(31)	2(31)	2(28)	2(30)	2(31)	2(31)	2(31)
Voice	2(28)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)
Wine	4(7)10(7)	3(15)	3(15)	3(19)	3(16)	3(19)	4(15)	4(14)	3(15)	4(30)

Table 7: ckMeans-Sugeno: Number of clusters selected for multiple subsets of data

Datasets	**Percentages of Data**									
	10%	**20%**	**30%**	**40%**	**50%**	**60%**	**70%**	**80%**	**90%**	**100%**
Balance	10(17)	10(15)	10(19)	10(18)	10(15)	10(24)	10(22)	10(22)	10(26)	10(25)
GI	9(11)10(11)	10(14)	10(18)	10(14)	10(15)	9(15)	9(19)	9(17)	9(18)	9(17)
Bupa	9(8)10(8)	2(12)	2(14)	2(15)	2(15)	2(12)	2(14)	2(13)	2(17)	4(24)
CMSC	9(7)	6(9)	9(9)	5(14)	5(9)	5(15)	5(20)	5(23)	5(25)	5(27)
Cnae-9	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)
Column 3C	10(12)	10(7)	4(15)	4(15)	4(13)	4(16)	4(16)	4(16)	3(19)	3(29)
Contraceptive	10(11)	10(17)	10(16)	10(22)	10(17)	10(19)	10(20)	10(17)	10(21)	10(20)
Ecoli	10(16)	10(12)	10(12)	6(9)	7(7)8(7)10(7)	7(10)	8(9)	7(8)9(8)	9(10)	7(10)
GI	10(9)	10(8)	2(7)4(7)	4(10)	2(8)	4(10)	4(13)	4(16)	4(13)	4(24)
Haberman	10(13)	10(9)	10(11)	9(11)	10(9)	10(11)	9(11)	10(13)	10(18)	10(11)
Heart Statlog	10(15)	10(12)	8(8)	8(7)	8(7)9(7)	8(7)	7(7)8(7)	8(9)	8(9)	5(10)
Ionosphere	10(19)	10(14)	8(8)	10(11)	8(8)9(8)	8(11)	7(10)	8(10)	9(9)	8(9)
Iris	2(20)	2(29)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)
Lymph	10(29)	10(19)	10(27)	10(14)	10(18)	10(22)	10(26)	10(23)	10(19)	10(22)
MB	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)
MF	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)
Parkinsons	3(7)	3(11)	4(11)	4(11)	4(15)	4(12)	4(16)	4(19)	4(26)	4(31)
Pima	6(9)	4(7)	5(11)	5(11)	5(12)	5(14)	5(17)	5(19)	5(21)	5(23)
Planning Relax	10(26)	10(23)	10(24)	10(25)	10(25)	10(28)	10(27)	10(29)	10(29)	10(29)
QSAR-B	9(6)	5(8)	7(9)	6(9)	6(9)	5(15)	7(9)	5(9)6(9)	6(11)	5(16)
Robot Failure	10(31)	10(24)	10(23)	10(18)	10(18)	10(14)	10(16)	10(13)	9(13)	10(16)
Seeds	10(19)	10(12)	10(9)	10(9)	6(6)7(6)10(6)	10(11)	8(8)	6(9)	8(7)	10(10)
Semeion	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)
Sonar	10(24)	10(24)	10(21)	10(21)	10(23)	10(16)	10(23)	10(28)	10(20)	10(25)
SPF	4(17)	4(22)	4(15)	4(19)	5(17)	5(15)	5(18)	5(28)	5(29)	5(31)
TS	2(27)	2(30)	2(31)	2(30)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)
UKM	10(20)	10(15)	10(19)	10(23)	10(19)	10(17)	10(22)	10(24)	10(27)	10(23)
Vehicle	2(12)	2(17)	4(14)	4(16)	4(20)	4(22)	4(22)	4(21)	4(26)	4(28)
Voice	4(10)	6(15)	6(16)	6(22)	6(18)	6(21)	6(23)	6(22)	6(28)	6(31)
Wine	10(19)	4(8)	5(7)10(7)	5(10)	5(11)	6(10)	5(13)	5(16)	5(12)	4(30)

Table 8: FCMσ-MPC: Number of clusters selected for multiple subsets of data

Datasets	**Percentages of Data**									
	10%	**20%**	**30%**	**40%**	**50%**	**60%**	**70%**	**80%**	**90%**	**100%**
Balance	3(15)	3(15)	2(19)	2(18)	2(24)	2(27)	2(27)	2(28)	2(31)	2(30)
GI	2(28)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)
Bupa	2(19)	2(23)	2(13)	2(15)	2(14)	2(15)	2(11)	2(8)	3(7)4(7)	5(8)
CMSC	3(7)5(7)	4(10)	3(15)	3(19)	3(19)	3(25)	3(27)	3(31)	3(31)	3(31)
Cnae-9	7(6)	10(8)	8(8)10(8)	10(7)	8(8)	9(10)	9(10)	10(8)	8(8)	8(8)
Column 3C	2(30)	2(31)	2(30)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)
Contraceptive	2(24)	2(21)	2(22)	2(24)	2(25)	2(28)	2(24)	2(20)	2(26)	2(25)
Ecoli	2(12)	10(8)	2(12)	2(14)	2(16)	3(15)	2(18)	2(18)	3(16)	2(17)
GI	2(23)	2(30)	2(30)	2(29)	2(30)	2(31)	2(30)	2(30)	2(31)	2(31)
Haberman	2(10)	2(23)	2(27)	2(24)	2(30)	2(29)	2(31)	2(31)	2(31)	2(31)
Heart Statlog	2(14)	2(23)	2(27)	2(29)	2(31)	2(29)	2(30)	2(31)	2(31)	2(31)
Ionosphere	2(13)	2(22)	2(24)	2(25)	2(24)	2(25)	2(27)	2(29)	2(28)	2(28)
Iris	2(16)	2(20)	2(20)	2(18)	2(18)	2(22)	2(19)	2(20)	2(22)	2(31)
Lymph	3(14)	2(13)	3(14)	2(13)3(13)	2(15)	3(18)	2(14)	2(14)	2(15)	2(17)
MB	5(18)	7(12)	7(13)	7(13)	8(11)	9(10)	8(10)	7(10)	7(12)	8(8)
MF	3(7)	2(9)	2(8)	2(10)	2(12)	2(14)	2(13)	2(15)	2(13)	2(14)
Parkinsons	2(26)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)
Pima	2(31)	2(31)	2(31)	2(30)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)
Planning Relax	3(9)	3(12)	3(10)	2(11)	2(11)	2(11)	2(9)	2(9)	2(8)3(8)	3(11)
QSAR-B	2(29)	2(31)	2(31)	2(30)	2(30)	2(31)	2(31)	2(31)	2(31)	2(31)
Robot Failure	10(31)	2(16)	2(15)	2(17)	2(24)	2(25)	2(17)	2(24)	2(22)	2(22)
Seeds	2(16)	2(26)	2(27)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(30)
Semeion	10(24)	9(16)	10(19)	10(18)	10(17)	10(15)	10(15)	10(17)	9(14)	9(12)
Sonar	3(16)	2(13)	3(11)	3(13)	2(12)3(12)	2(14)	2(9)	2(11)	3(10)	2(8)
SPF	2(26)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)
TS	2(12)	2(14)	10(15)	10(16)	10(24)	10(21)	10(22)	10(28)	10(29)	10(31)
UKM	2(13)3(13)	2(14)	3(13)	2(14)	2(14)	2(13)	2(12)3(12)	2(13)	2(13)	3(10)
Vehicle	2(30)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)
Voice	2(27)	2(30)	2(30)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)
Wine	2(13)	2(23)	2(20)	2(24)	3(16)	2(22)	2(20)	2(18)	2(21)	2(31)

Table 9: FCMσ-XB: Number of clusters selected for multiple subsets of data

Datasets	**Percentages of Data**									
	10%	**20%**	**30%**	**40%**	**50%**	**60%**	**70%**	**80%**	**90%**	**100%**
Balance	3(8)	10(7)	10(8)	10(13)	9(11)	10(15)	10(17)	10(15)	10(11)	10(13)
GI	2(30)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)
Bupa	2(20)	2(26)	2(19)	2(24)	2(22)	2(24)	2(21)	2(20)	2(21)	2(25)
CMSC	2(23)	2(24)	2(29)	2(27)	2(30)	2(31)	2(31)	2(31)	2(31)	2(31)
Cnae-9	2(8)	9(6)10(6)	9(7)	10(6)	2(8)	9(6)10(6)	10(8)	10(7)	8(7)	9(9)
Column 3C	2(31)	2(31)	2(30)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)
Contraceptive	2(18)	2(19)	2(19)	2(16)	2(21)	2(19)	2(20)	2(19)	2(24)	2(23)
Ecoli	2(16)	2(16)	2(16)	2(15)	2(23)	2(21)	2(24)	2(22)	2(24)	2(21)
GI	2(26)	2(31)	2(31)	2(29)	2(30)	2(31)	2(30)	2(30)	2(31)	2(31)
Haberman	2(13)	2(15)	2(18)	2(17)	2(21)	2(23)	2(20)	2(19)	2(16)	2(28)
Heart Statlog	2(17)	2(21)	2(23)	2(25)	2(28)	2(25)	2(30)	2(28)	2(27)	2(28)
Ionosphere	2(22)	2(21)	2(17)	2(23)	2(21)	2(22)	2(25)	2(26)	2(27)	2(22)
Iris	2(28)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)
Lymph	2(8)	2(9)	2(11)	2(10)	2(9)	2(10)3(10)	2(12)	2(12)	2(13)	2(16)
MB	5(19)	6(12)7(12)	7(13)	8(16)	8(12)	9(11)	8(12)9(12)	7(10)8(10)	7(11)	8(12)
MF	2(7)	2(15)	2(8)	2(12)	2(13)	2(13)	2(11)	2(11)	2(13)	2(18)
Parkinsons	2(27)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)
Pima	2(31)	2(31)	2(31)	2(30)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)
Planning Relax	3(8)	4(9)	9(7)	2(6)9(6)	10(12)	10(10)	9(8)	7(8)	9(10)	10(12)
QSAR-B	2(29)	2(31)	2(30)	2(30)	2(28)	2(30)	2(31)	2(31)	2(31)	2(31)
Robot Failure	10(31)	2(19)	2(13)	2(16)	2(24)	2(19)	2(20)	2(20)	2(22)	2(25)
Seeds	2(18)	2(31)	2(29)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)
Semeion	10(14)	10(16)	9(14)	10(17)	10(19)	10(17)	10(18)	9(14)	10(13)	10(16)
Sonar	2(12)	2(11)	2(8)	2(11)	3(8)	2(9)	3(9)	3(8)	2(11)	2(11)
SPF	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)
TS	2(24)	2(24)	2(27)	2(25)	2(28)	2(27)	2(29)	2(28)	2(25)	2(30)
UKM	2(12)	2(8)4(8)	2(12)	2(9)	2(7)3(7)	2(14)	2(9)	2(11)	2(10)	3(10)
Vehicle	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)	2(31)
Voice	2(23)	2(28)	2(27)	2(30)	2(29)	2(31)	2(31)	2(31)	2(31)	2(31)
Wine	2(15)	2(14)	3(17)	3(15)	3(25)	3(28)	3(30)	3(30)	3(29)	3(31)

Table 10: FCMσ-Sugeno: Number of clusters selected for multiple subsets of data

Datasets	**Percentages of Data**									
	10%	**20%**	**30%**	**40%**	**50%**	**60%**	**70%**	**80%**	**90%**	**100%**
Balance	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)
GI	10(25)	10(24)	10(26)	10(30)	10(31)	10(29)	10(29)	10(28)	10(29)	10(28)
Bupa	10(28)	10(30)	10(30)	10(31)	10(30)	10(31)	10(31)	10(31)	10(31)	10(31)
CMSC	6(8)	6(7)	5(8)7(8)	5(11)	5(15)	5(14)	5(23)	5(17)	5(26)	5(31)
Cnae-9	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)
Column 3C	10(16)	10(25)	10(25)	10(26)	10(27)	10(26)	10(29)	10(30)	10(26)	10(30)
Contraceptive	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)
Ecoli	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)
GI	10(22)	10(20)	10(24)	10(28)	10(28)	10(31)	10(30)	10(30)	10(31)	10(31)
Haberman	10(21)	10(18)	10(20)	10(23)	10(24)	10(26)	10(28)	10(31)	10(31)	10(30)
Heart Statlog	10(27)	10(30)	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)
Ionosphere	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)
Iris	2(12)	3(15)	3(24)	3(23)	3(28)	3(27)	3(29)	3(31)	3(31)	3(31)
Lymph	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)
MB	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)
MF	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)
Parkinsons	10(9)	2(9)	3(8)	2(8)	3(8)	3(8)	5(8)	5(7)10(7)	5(17)	5(25)
Pima	2(15)	2(17)	2(17)	2(13)	2(15)	2(25)	2(22)	2(25)	2(28)	2(31)
Planning Relax	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)
QSAR-B	10(29)	10(31)	10(30)	10(30)	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)
Robot Failure	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)
Seeds	10(16)	10(13)	10(14)	10(22)	10(25)	10(22)	10(24)	10(26)	10(28)	10(28)
Semeion	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)
Sonar	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)
SPF	6(11)	6(18)	5(16)	5(15)6(15)	6(19)	6(17)	6(22)	6(23)	6(24)	6(31)
TS	10(22)	10(31)	10(29)	10(30)	10(31)	10(31)	10(30)	10(31)	10(31)	10(31)
UKM	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)	10(31)
Vehicle	3(17)	3(24)	3(30)	3(30)	3(30)	3(31)	3(31)	3(31)	3(31)	3(31)
Voice	3(23)	3(23)	3(25)	3(25)	3(23)	3(30)	3(31)	3(30)	3(31)	3(31)
Wine	10(10)	4(10)	4(16)	4(19)	4(23)	4(24)	4(28)	4(30)	4(31)	4(31)

From Tables 2, 3 and 4, we can observe that the FCM algorithm showed similar behavior with 15, 12 and 15 datasets, respectively. In other words, the same number of clusters was chosen in all subsets and in the total set, when using the MPC, XB and Sugeno index for evaluation. The ckMeans algorithm showed similar behavior with 19, 22 and 15 datasets, when using the MPC, XB and Sugeno index, respectively, as shown in Tables 5 to 7. The FCMσ algorithm showed similar behavior with 15, 20 and 25 datasets, considering the MPC, XB and Sugeno index, respectively, according to Tables 8 to 10. Considering that FCM, ckMeans and FCMσ are non-deterministic algorithms, in some cases there was a difference in the number of times the number of clusters was chosen.

From Tables 2 to 10, we can observe that, in a general perspective, the analyzed algorithms showed similar behavior, selecting the same number of clusters for almost all the data subsets as well as for the original dataset. It is important to emphasize that there was a discrepancy in some scenarios when selecting the number of clusters. Therefore, it is necessary to apply statistical tests in order to assess the performance delivered by the analyzed scenarios, from a statistical point of view.

4.1 Statistical Test

In the statistical analysis, the Kruskal-Wallis test as well as the Mann-Whitney test were used. The Kruskal-Wallis test was used to compare the behavior of the algorithms in each data subset and in the original dataset. Therefore, it is applied directly to the classification result, i.e., over a vector of 31 positions, where each position refers to the number of clusters selected in the corresponding execution. This test serves to evaluate the hypothesis that the different percentages of data have the same distribution.

The results of the Kruskal-Wallis test for the FCM, ckMeans and FCM σ algorithms are presented in Table 11. The values presented in each cell correspond to the p-value of this test for each dataset, algorithm and validation index.

Table 11: p-value result from the Kruskal Wallis test

Datasets	**FCM**			**ckMeans**			**FCMσ**		
	MPC	**XB**	**Sugeno**	**MPC**	**XB**	**Sugeno**	**MPC**	**XB**	**Sugeno**
Balance	0.0094	0.0191	0.0255	0.0314	<0.0001	0.0027	<0.0001	<0.0001	0.5000
GI	0.5000	0.4687	0.9397	0.5000	0.5543	0.4453	0.0013	0.4373	0.0526
Bupa	<0.0001	<0.0001	<0.0001	<0.0001	0.5248	<0.0001	0.0001	0.3967	0.1144
CMSC	<0.0001	<0.0001	0.0996	<0.0001	0.0122	<0.0001	0.0002	<0.0001	<0.0001
Cnae-9	0.3136	0.3520	0.5000	0.2736	0.7996	0.5000	0.0867	0.1878	0.5000
Column 3C	0.0115	<0.0001	<0.0001	0.5315	0.0014	<0.0001	0.5315	0.4373	<0.0001
Contraceptive	0.0027	0.0512	<0.0001	0.1313	0.7323	0.0814	0.3865	0.6837	0.5000
Ecoli	<0.0001	0.2498	<0.0001	0.0270	0.9997	<0.0001	0.0002	0.0195	0.5000
GI	<0.0001	0.0005	<0.0001	<0.0001	<0.0001	<0.0001	<0.0001	0.0078	<0.0001
Haberman	<0.0001	0.0436	0.0002	<0.0001	0.1271	0.1774	<0.0001	0.0002	<0.0001
Heart Statlog	<0.0001	0.5155	0.0025	<0.0001	0.0160	<0.0001	<0.0001	0.0002	0.0006
Ionosphere	<0.0001	<0.0001	0.1281	<0.0001	0.2875	0.0002	<0.0001	0.2473	0.5000
Iris	0.0345	0.0013	<0.0001	0.5000	0.0345	<0.0001	0.0035	0.0013	0.0303
Lymph	<0.0001	<0.0001	0.0318	<0.0001	0.0706	0.0002	0.1517	0.5758	0.5000
MB	<0.0001	0.3280	0.5000	<0.0001	0.5213	0.5000	<0.0001	<0.0001	0.5000
MF	0.0519	0.0010	0.5000	0.0001	0.0214	0.5000	<0.0001	0.0180	0.5000
Parkinsons	0.0004	<0.0001	<0.0001	<0.0001	<0.0001	0.1569	<0.0001	<0.0001	0.2060
Pima	0.5000	0.0116	0.0024	0.5000	0.1914	0.0594	0.4373	0.4373	<0.0001
Planning Relax	<0.0001	<0.0001	0.5315	<0.0001	0.9457	0.2321	0.6555	<0.0001	0.5000
QSAR-B	0.5000	0.5000	0.0463	0.4373	0.4376	0.0114	0.2684	0.1981	0.2684
Robot Failure	<0.0001	<0.0001	0.0004	<0.0001	<0.0001	<0.0001	<0.0001	<0.0001	0.5000
Seeds	<0.0001	<0.0001	0.0006	<0.0001	0.0073	<0.0001	<0.0001	<0.0001	<0.0001
Semeion	0.0002	0.4946	0.5000	0.0004	0.4341	0.5000	0.0005	0.6987	0.5000
Sonar	<0.0001	<0.0001	0.0088	<0.0001	0.9374	0.0964	0.1381	0.4885	0.5000
SPF	<0.0001	<0.0001	<0.0001	<0.0001	0.0974	<0.0001	<0.0001	0.5000	<0.0001
TS	<0.0001	<0.0001	<0.0001	0.4373	0.0002	0.0034	<0.0001	0.2740	<0.0001
UKM	<0.0001	<0.0001	0.0974	<0.0001	0.4562	0.0407	0.6245	0.5412	0.5000
Vehicle	0.5000	0.5000	<0.0001	0.5000	0.1034	0.1338	0.4373	0.5000	<0.0001
Voice	<0.0001	0.5000	<0.0001	<0.0001	0.0013	0.0085	0.0037	<0.0001	<0.0001
Wine	0.0007	0.2636	<0.0001	<0.0001	<0.0001	<0.0001	<0.0001	0.0045	<0.0001

Based on the values presented in Table 11, it can be observed that there is a statistically significant difference in all validation algorithms and indices (p-value ≤ 0.05). Therefore, it is necessary to identify which subsets showed a statistically significant difference in relation to the total set. In this context, Mann-Whitney test was applied for all algorithms and validation indices. Mann-Whitney test is a non-parametric method well known in the literature, which compares two paired samples, and can identify pairs that are statistically different.

Table 12 presents the results of Mann-Whitney test when comparing the behavior obtained by the algorithms in each subset of data with the total set. In this analysis, we took into account the number of datasets that did not present a statistically significant difference by percentage of data, for each validation index and for each clustering algorithm. For example, using the FCM algorithm, MPC index and 40% of the data from the total set, we have 21 datasets (out of 30), which did not differ from a statistical point of view. Therefore, values outside parentheses indicate the number of datasets that did not show a statistically significant difference, while values inside parentheses indicate otherwise.

Table 12: Result of the Mann-Whitney test when comparing each subset of data with the total set

Algorithms	**Indices**	**10%**	**20%**	**30%**	**40%**	**50%**	**60%**	**70%**	**80%**	**90%**
FCM	**MPC**	7(23)	14(16)	19(11)	21(9)	24(6)	23(7)	25(5)	25(5)	27(3)
	XB	12(18)	17(13)	18(12)	21(9)	25(5)	22(8)	24(6)	23(7)	27(3)
	Sugeno	13(17)	14(16)	16(14)	17(13)	15(15)	19(11)	19(11)	20(10)	22(8)
ckMeans	**MPC**	11(19)	15(15)	19(11)	20(10)	21(9)	23(7)	22(8)	24(6)	25(5)
	XB	18(12)	19(11)	23(7)	22(8)	25(5)	26(4)	26(4)	27(3)	28(2)
	Sugeno	10(20)	17(13)	18(12)	20(10)	20(10)	25(5)	24(6)	26(4)	23(7)
FCMσ	**MPC**	6(24)	15(15)	19(11)	19(11)	22(8)	23(7)	25(5)	26(4)	27(3)
	XB	11(19)	19(11)	22(8)	21(9)	26(4)	26(4)	26(4)	27(3)	28(2)
	Sugeno	17(13)	20(10)	20(10)	20(10)	22(8)	22(8)	26(4)	27(3)	26(4)

Table 13 presents the percentage of datasets that did not show any difference from the statistical point of view, when comparing each data subset (column) with the original dataset. The percentage was calculated based on the value obtained in each index for each algorithm. For example, in order to calculate the percentage of datasets that did not show statistical difference taking into account $p = 10\%$ and the FCM algorithm, we took into consideration the values presented in each index and algorithm ($7 + 12 + 13 = 32$) and the percentage of the value is calculated, taking into account all 30 datasets used in the experiments, for each index and algorithm (3 indices x 30 datasets = 90). Therefore, the percentage of datasets that did not show statistical difference for the FCM algorithm when 10% of the data was used is equal to 35.56% ((32x100)/90 = 35.56).

Table 13: Percentage of Datasets that did not show statistical difference

Algorithms	**Percentage of Data**								
	10%	**20%**	**30%**	**40%**	**50%**	**60%**	**70%**	**80%**	**90%**
FCM	35.56%	50.00%	58.89%	65.56%	71.11%	71.11%	75.56%	75.56%	84.44%
ckMeans	43.33%	56.67%	66.67%	68.89%	73.33%	82.22%	80.00%	85.56%	84.44%
FCMσ	37.78%	60.00%	67.78%	66.67%	77.78%	78.89%	85.56%	88.89%	90.00%

In this analysis, let consider percentages equal or higher than 70% was considered as a strong percentage (more than 2/3 of the subsets that did not present a statistically significant difference). The use of the strong percentage concept aims to identify the smallest data subset that delivers similar perfomance than the orginal dataset (reaches the strong percentage). Therefore, from Table 13, we can observe that the strong percentage was reached with $p \geq 50\%$, for all the analyzed algorithms (shadded values).

Based on this analysis, it is possible to infer that the number of clusters in a dataset can be done using a data subset higher than 50% ($p \geq 50\%$). It is important to highlight that this reduction in the number of instances in a dataset allows to optimize the computational time and processing cost in the data clustering process, especially when using large datasets with no previous knowledge about the data.

5 Final Remarks

Several approaches in the literature investigate the problem of defining the number of clusters in a dataset. In general, these approaches use the original dataset as a way to determine the number of clusters. In this work, we investigated the use of data subsets in the definition of the number of clusters. It is a smaller sample, but still able to infer the number of clusters of a dataset. In this investigation, three clustering algorithms (FCM, ckMeans and FCMσ) were applied and assessed using three validation indices. In order to assess the performance of this proposal, we performed an empirical analysis with 30 datasets. Each dataset was partitioned into 9 data subsets, starting with 10% ($p = 10\%$) and increasing with 10% intervals untils it reaches the original dataset size (p=10, 20, ...,90%).

Through the empirical analysis, we can conclude that the results obtained in this article are promising, both from an experimental point of view and from a statistical point of view. These results show that the use of a smaller percentage of a dataset can be used to infer the number of clusters with an efficient performance. More specifically, data subsets higher than 50% ($p \geq 50\%$) present results similar to the original dataset when defining the best number of clusters of a dataset.

In future work, we can use a larger number of datasets and characterize them categorically, in an attempt to find the data percentage to infer the number of clusters, according to the characteristics of each dataset. In addition, we can use other clustering algorithms, such as k-Means and initialization of initial centers as in[32], and perform a similar study for interval fuzzy clustering algorithms [33] or use this methodology applied to the context of clustering ensembles optimization [34].

Conflict of Interest: The authors declare no conflict of interest.

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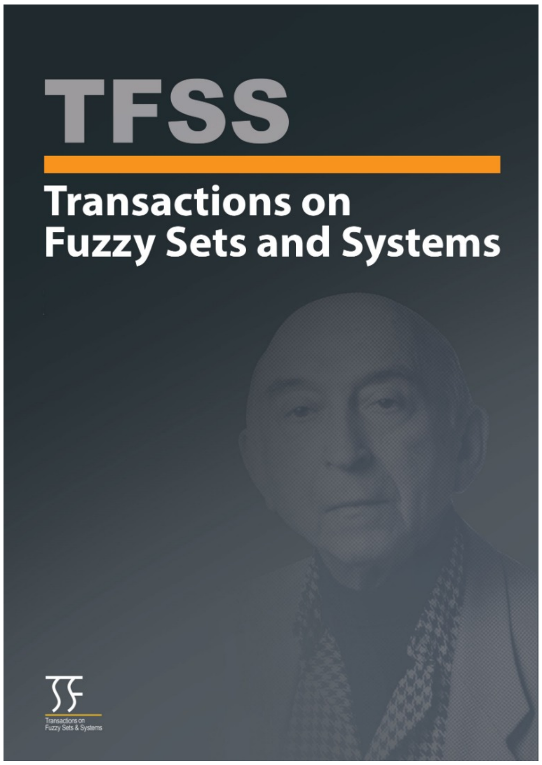

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Enhancing Big Data Governance Framework Implementation Using Novel Fuzzy Frank Operators: An Application to MADM Process

Hamza Iftikhar, **Faisal Mehmood***

Abstract. In today's data-driven landscape, to ensure continuous survival and betterment, the implementation of a robust Big Data Governance Framework (BDGF) is imperative for organizations to effectively manage and harness the potential of their vast data resources. The BDGF serves no purpose when implemented in a random manner. This article delves into the complex decision-making challenges that emerge in the context of implementation of the BDGF under uncertain conditions. Specifically, we aim to analyze and evaluate the BDGF performance using the Multi-Attribute Decision-Making (MADM) techniques aiming to address the intricacies of big data governance uncertainties. To achieve our objectives, we explore the application of Frank operators within the framework of complex picture fuzzy (CPF) sets (CPFs). We introduce complex picture fuzzy Frank weighted averaging (CPFFWA) and complex picture fuzzy Frank ordered weighted averaging (CPFFOWA) operators to enable more accurate implementation of the BDGF. Additionally, we rigorously examine the reliability of these newly proposed fuzzy Frank (FF) operators (FFAOs), taking into consideration essential properties such as idempotency, monotonicity, and boundedness. To illustrate the practical applicability of our approach, we present a case study that highlights the decision-making challenges encountered in the implementation of the BDGF. Subsequently, we conduct a comprehensive numerical example to assess various BDGF implementation options using the MADM technique based on complex picture fuzzy Frank aggregation (CPFFA) operators. Furthermore, we perform a comprehensive comparative assessment of our proposed methodology, emphasizing the significance of the novel insights and results derived. In conclusion, this research article offers a unique and innovative perspective on decision-making within the realm of the BDGF. By employing the CPFFWA and the CPFFOWA operators, organizations can make well-informed decisions to optimize their BDGF implementations, mitigate uncertainties, and harness the full potential of their data assets in an ever-evolving data landscape. This work contributes to the advancement of decision support systems for big data governance (BDG), providing valuable insights for practitioners and scholars alike.

AMS Subject Classification 2020: 03B52; 03B80; 03B50
Keywords and Phrases: Picture fuzzy set, Complex picture fuzzy set, Frank operations, Averaging operators, Geometric operators.

1 Introduction

Although having a BDGF is crucial for companies, a judicious and optimized implementation of the BDGF is a lot more important. The BDGF is like a set of rules and plans to make sure that when a lot of information (Big Data) is collected and used, it's done in a smart and responsible way. It's about making sure the data is accurate, safe, and used in a way that helps rather than causes problems. Just like traffic rules help everyone drive safely on the roads, Big Data Governance rules help manage information in a sensible and secure manner.

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In today's fiercely competitive business environment, companies are constantly challenged to not only attain profitability but also to allocate their funds judiciously. The imperative arises when organizations seek to implement a robust BDGF to manage and harness the potential of their ever-expanding data assets. However, the challenge they face lies in determining how to deploy these financial resources effectively, particularly when confronted with multiple options for their allocation. This problem is further worsened by the inherent complexity of the BDGF, where cost attributes and profit attributes are pivotal components, rendering it a multi-attribute decision-making problem. This study underscores the critical need for a suitable and comprehensive implementation of the BDGF within companies to ensure their continuous survival in competitiveness and achieve long-term profitability. To address this challenge, our research article introduces an innovative approach based on CPFSs and FFA Operators. This novel method offers a powerful tool to tackle the intricacies of multi-attribute decision-making in the context of data governance, ensuring that organizations make well-informed, data-driven choices when allocating resources for optimal outcomes. Through powerful empirical and academic analysis, our research article provides a convincing argument for the implementation of a BDGF in a manner that strictly goes in the companys favor. By leveraging CPFS and FFAOs, organizations can not only navigate the complexities of data governance but also make informed decisions that enhance profitability, reduce risks, and fortify their competitive position in an increasingly data-centric world. This research article serves as a valuable resource for executives, practitioners, and scholars seeking to fortify their organizations' data governance practices in the pursuit of sustainable success.

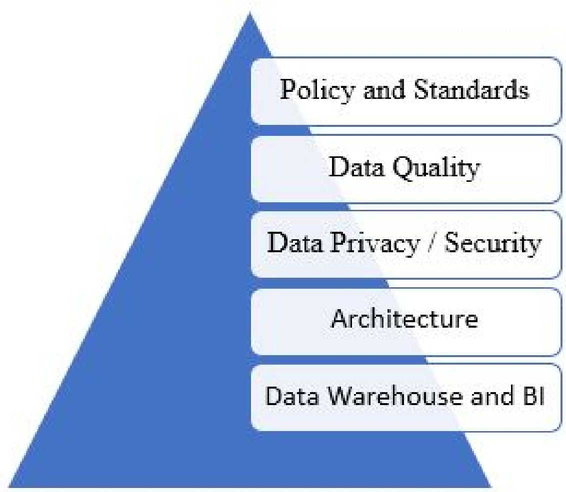

Figure 1: Big data governance framework of a company.

The MADM is a method of organizing and resolving planning and judgmental problems by determining the most appropriate alternative based on an expert's judgment in line with predetermined criteria [1]. This decision-making process is very important and has drawn the attention of many academicians [2]. Typically, professionals or decision-makers resolve the problems to assess the data using fuzzy information [3]. To give just a few examples, numerous researchers have made substantial contributions to the fields of fuzzy sets (FSs) [4], intuitionistic fuzzy sets (IFSs) [3], cubic intuitionistic fuzzy sets [5], linguistic interval-valued IFSs [6], and other generalized sets [7, 8, 9, 10, 11, 12], as well as Pythagorean fuzzy set (Py-FS) [13], which relaxes the IFS limitation $0 \leq \mu + \nu \leq 1$ into $0 \leq \mu^2 + \nu^2 \leq 1$. PFSs are more broadly applicable than IFSs. Using various aggregation operators (AOs), several scholars have presented different sorts of models in these generalized environments [14, 15, 16]. A mathematical function known as an aggregate operator turns a collection of inputs or data into a single datum. In the course of making decisions, aggregation operators are crucial [17, 18] and [19], respectively, creating the ordered weighted averaging (OWA) and ordered weighted geometric (OWG) operators, which, according to their ranking order, applied weights to all values in the collection or data. The weighted averaging operator was developed by [20] for use with IF data, while the geometric aggregation operators were provided by [21]. The FSs and IFSs environments, however, only provide partial information on the data set's components. Picture fuzzy set (PFS) is a novel concept introduced by [22]. It is distinguished by neutral membership, membership, and non-membership features that show data on human decisions like yes, refrain, no, and rejection. [23] Also provided some PFS results. In a PFS environment, [24] proposed WA, OWA, and hybrid averaging operators. The Einstein operations on PFSs were first introduced by [25]. In [26] the most recently used the Hamachar operators in PFSs.

A noteworthy extended form of Lukasiewicz as well as probabilistic t-norm and t-conorm [27] have emerged as Frank t-norm and t-conorm [28]. Moreover, they constitute a sufficiently flexible type of the continuous triangular norm. The Frank models, along with the process of fusion of information, became more adaptable owing to the fact that a certain parameter is used in them, and the literature is replete with numerous works [17, 23, 29, 30, 31, 32] related to these models. Frank operators have gained the attention of researchers in a great number recently [33] Ullah et al used Frank operators to evaluate electric motor cars, and Milosevic et al [34] used those operators for IFS-IBA logical aggregation. Also to improve the MADM process Seikh et al [35, 36] used Frank operators in a very efficient manner. For two types comprising commutative, associative, and growing binary operators, the Alsina and Frank functional equations have skillfully been examined by [37]. Yager [38] developed a paradigm for approximate reasoning using Frank t-norms by examining the additive-generating function of these norms. The scalar cardinality related to Frank t-norms was studied in a novel way by Casasnovas and Torrens [17], who also further established the characteristics of other common t-norms. Sarkoci [39] came to the conclusion that two separate t-norms, Frank and Hamacher t-norms, are actually members of the same family. In order to address the MADM issues, Xing et al. [40] presented aggregation operators pertaining to the PyFs depending on Frank models. Zhou et al. [41] examined a case study of choosing agriculture socialization and looked into several Frank aggregation operations of interval-valued neutron-sophic numbers. Aggregation operators of Frank pertaining to triangular interval type2 FSs were presented by Qin and Liu [42]. Based on Frank t-norm procedures, Qin et al. [43] created more hesitant fuzzy aggregation operators.

It has been determined that the MADM problems addressed in the aforementioned studies in FSs, IFSs, and PFSs environments only handle ambiguity as well as vagueness. All of these models are unable to cope with data insensitivity and periodicity, but a complex data set potentially addresses data periodicity, its continually changing nature and uncertainty at the same time. To deal with these circumstances, Ramot [44] and Ramot et al. [45] proposed the noteworthy notion known as a complex fuzzy set (CFS). In a complex plane, he suggested that a CFS membership degree is expressed as $\mu e^{i\phi\mu}$ the range of which is expanded to a unit disk, where $\mu \in [0, 1]$ and $\phi \in [0, 2]$, Zhang et al. presented certain CFS operating rules and characteristics [46]. Since CFSs first appeared in a variety of real-world sectors, such as biometric procedures, medical investigations, etc., a broad range of applications have been well established. Using the CF data, Bashir and Akram [47] put out the novel idea of ordered weighted quadratic averaging operators. In the realm of the CF, Luqman et al.[48], [49] produced outstanding work. Some notable works can also be found in [50, 51, 52, 53, 54, 55].

The CFS is insufficient to reveal a data set elements inconsistency. Then, Garg and Rani [15], [56] produced the complex intuitionistic fuzzy set (CIFS) and their aggregation operators, which is a generalization of a CFS, in which both an element's membership as well as non-membership were embedded. They provided definitions for the CIFSs intersection, complement, and union. In order to overcome the MADM challenges later, AOs utilizing the CIFSs information were presented by Rani and Garg [57]. Ali et al [17] used complex T-Spherical fuzzy Frank aggregation operators, and yang et al [58] used complex intuitionistic fuzzy Frank (CIFF) aggregation operators for decision making, However, CIFS is unable to cope with data that has been somewhat neglected, such as the fact that it only displays the membership and non-membership degrees of

Table 1: Superiority of CPFS over existing models in literature.

Model	Fuzzy set	Intuitionistic fuzzy set	Picture fuzzy set	Complex fuzzy set	Complex intuitionistic fuzzy set	Complex picture fuzzy set
Uncertainty	✓	✓	✓	✓	✓	✓
Falsity	×	✓	✓	×	✓	✓
Hesitation	×	×	✓	×	×	✓
Periodicity	×	×	×	✓	✓	✓
Handles 2-D data	×	×	×	×	✓	✓
Handles 3-D data	×	×	×	×	×	✓

a data set's elements in a complex plane and is unable to convey whether a choice was made to abstain (neutral value) or reject a piece of information. We introduce the complex picture fuzzy set (CPFS), which is distinguished by membership, neutral membership, and non-membership values in a unit disc of a complex plane, in response to the absence of information in CIFS theory. By criteria $[0, 1]$ and $[0, 2\pi]$, respectively, the amplitude terms and corresponding phase terms of a CPFS are constrained. The set's applicability is increased by complex neutral membership grade, which also makes it simpler for a decision maker to be aware of greater depth as compared to a CIFS. The readers are directed to [59], [60, 61, 49, 48], [22, 2, 43, 44, 57], and [62] for any additional and useful discussions related to AOs as well as the MADM techniques.

The presence of neutral membership gives a CPFS leverage over a CFS and CIFS: The CPFS has a larger range as compared to them. Compared to previous models, CPFSs can address uncertainty and periodicity concurrently and provide significantly more detailed and insightful about an object. The following is a description of the suggested model's motivation:

1. A CPFSs membership degrees have complex values made up of terms for amplitude and phase. The amplitude component of the neutral membership function denotes the abstinence degree. The phase term for this function, however, offers further details, usually regarding periodicity. In essence, the neutral degree has boosted the adaptability of CPFS by providing additional details about an object being evaluated.

2. The CPFS is distinct from the conventional ideas of PFS due to the innovative phase term conception. This is caused by the fact that PFS only works with one-dimensional data, which causes data loss. However, when dealing with issues in real-world occurrences, the second dimension must be taken into account. To remedy it, we incorporated the phase term.

3. The T-norm and t-conorm of Frank [24] appear to be fascinating generalizations extracted from the t-norm and t-conorm of probabilistics and Lukasiewicz [6], and they constitute a common as well as a sufficiently compromising branch of these models. Usage of a certain parameter, the robustness of the Frank models and the information fusion process increase manifold.

The structure of this manuscript can be viewed as follows: In Section 2, several essential ideas that are important to comprehending this manuscript have been given. The novel conception of a CPFS with several CPFS attributes and operating rules as well as score and accuracy functions are presented in Section 3. After applying, in the earlier part of section 3, the Frank AOs to the innovative CPFS concept, we introduced the idea of the CPFHWA and the CPFHWG, complex picture fuzzy Frank hybrid averaging (CPFFHA), complex picture fuzzy Frank weighted geometric averaging (CPFFWGA), complex picture fuzzy Frank ordered weighted geometric averaging (CPFFOWG), and complex picture fuzzy Frank hybrid geometric averaging (CPFFHGA) operators in the later parts of this section. For these operators, we demonstrate a few features and outcomes. An MADM technique, along with the method to determining attributes weights, is put forth in section 4 to determine the most preferable alternative using a specific example and the creditworthiness to show the relevance of the proposed work. After that, section 5 deals with a numerical illustration of our proposed work with the help of a real-world problem. To demonstrate our manuscript's superiority and influence over other prevalent approaches, we compare the model we have offered in Section 6 of our paper. Finally, in Section 7, we have concluded our approach and have provided some recommendations for future work as far as this wider and profound area of study is concerned.

2 Preliminaries

This section deals with some of the fundamental definitions and preliminaries.

Definition 2.1. *[37] Consider X as a universal set. The CPFS $\bar{R}$ over X is defined as*

$$\bar{R}=\left\{\left\langle x,\mu_{\bar{R}}(x)e^{i2\pi\mu_{\bar{R}}(x)},\eta_{\bar{R}}(x)e^{i2\pi\eta_{\bar{R}}(x)},\nu_{\bar{R}}(x)e^{i2\pi\nu_{\bar{R}}(x)}\right\rangle \mid x\in X\right\},$$

where $\mu_{\bar{R}}(x)e^{i2\pi\mu_{\bar{R}}(x)}: X\to[0,1], \eta_{\bar{R}}(x)e^{i2\pi\eta_{\bar{R}}(x)}: X\to[0,1]$ and $\nu_{\bar{R}}(x)e^{i2\pi\nu_{\bar{R}}(x)}: X\to[0,1]$ are reffered to as positive, neutral and negative degrees respectively, such that $0\leq\mu_{\bar{R}}(x)e^{i2\pi\mu_{\bar{R}}(x)}+\eta_{\bar{R}}(x)e^{i2\pi\eta_{\bar{R}}(x)}+\nu_{\bar{R}}(x)e^{i2\pi\nu_{\bar{R}}(x)}\leq 1$ for every $x\in X$.

Moreover, $\varkappa_{\bar{R}}(x)e^{i2\pi\mu_{\bar{R}}(x)}=\left(1-\mu_{\bar{R}}(x)-\eta_{\bar{R}}(x)-\nu_{\bar{R}}(x)\right)e^{i2\pi\mu_{\bar{R}}(x)}$ is reffered to as the degree of hesitancy for $\in X$. For our convenience, we denote $\bar{R}=\left(\mu_{\bar{R}}(x)e^{i2\pi\mu_{\bar{R}}(x)},\eta_{\bar{R}}e^{i2\pi\eta_{\bar{R}}(x)},\nu_{\bar{R}}e^{i2\pi\nu_{\bar{R}}(x)}\right)$ as the representation of a CPFN.

Definition 2.2. *Let $\bar{R}=\left(\mu_{\bar{R}}e^{i2\pi\mu_{\bar{R}}(x)},\eta_{\bar{R}}e^{i2\pi\eta_{\bar{R}}(x)},\nu_{\bar{R}}e^{i2\pi\nu_{\bar{R}}(x)}\right)$ and $\bar{S}=\left(\mu_{\bar{S}}e^{i2\pi\mu_{\bar{S}}(x)},\eta_{\bar{S}}e^{i2\pi\eta_{\bar{S}}(x)},\nu_{\bar{S}}e^{i2\pi\nu_{\bar{S}}(x)}\right)$ be CPFNs over a universal set X and $\zeta>0$ belonging to real number, then following are defined some notable operations:*

1. $\bar{R}\leq\bar{S}$, *if* $\mu_{\bar{R}}e^{i2\pi\mu_{\bar{R}}(x)}\leq\mu_{\bar{S}}e^{i2\pi\mu_{\bar{S}}(x)},\eta_{\bar{R}}e^{i2\pi\eta_{\bar{R}}(x)}\leq\eta_{\bar{S}}e^{i2\pi\eta_{\bar{S}}(x)}$ *and* $\nu_{\bar{R}}e^{i2\pi\nu_{\bar{R}}(x)}\geq\nu_{\bar{S}}e^{i2\pi\nu_{\bar{S}}(x)}$.
2. $\bar{R}\bigvee\bar{S}=\left(\begin{array}{c}\max\left\{\mu_{\bar{R}}e^{i2\pi\mu_{\bar{R}}(x)},\mu_{\bar{S}}e^{i2\pi\mu_{\bar{S}}(x)}\right\},\min\left\{\eta_{\bar{R}}e^{i2\pi\eta_{\bar{R}}(x)},\eta_{\bar{S}}e^{i2\pi\eta_{S}(x)}\right\},\\ \min\left\{\nu_{\bar{R}}e^{i2\pi\nu_{\bar{R}}(x)},\nu_{\bar{S}}e^{i2\pi\nu_{\bar{S}}(x)}\right\}\end{array}\right)$.
3. $\bar{R}\wedge\bar{S}=\left(\begin{array}{c}\min\left\{\mu_{\bar{R}}e^{i2\pi\mu_{\bar{R}}(x)},\mu_{\bar{S}}e^{i2\pi\mu_{\bar{S}}(x)}\right\},\max\left\{\eta_{\bar{R}}e^{i2\pi\eta_{\bar{R}}(x)},\eta_{\bar{S}}e^{i2\pi\eta_{\bar{S}}(x)}\right\}\\ ,\max\left\{\nu_{\bar{R}}e^{i2\pi\nu_{\bar{R}}(x)},\nu_{\bar{S}}e^{i2\pi\nu_{\bar{S}}(x)}\right\}\end{array}\right)$.
4. $\bar{R}^{c}=\left(\nu_{\bar{R}}e^{i2\pi\mu_{\bar{R}}(x)},\eta_{\bar{R}}e^{i2\pi\eta_{\bar{R}}(x)},\mu_{\bar{R}}e^{i2\pi\nu_{\bar{R}}(x)}\right)$.
5. $\bar{R}\wedge\bar{S}=\left(\begin{array}{c}\min\left\{\mu_{\bar{R}}e^{i2\pi\mu_{\bar{R}}(x)},\mu_{\bar{S}}e^{i2\pi\mu_{\bar{R}}(x)}\right\},\max\left\{\eta_{\bar{R}}e^{i2\pi\eta_{\bar{R}}(x)},\eta_{\bar{S}}e^{i2\pi\eta_{\bar{R}}(x)}\right\},\\ \max\left\{\nu_{\bar{R}}e^{i2\pi v_{\bar{R}}(x)},\nu_{\bar{S}}e^{i2\pi\nu_{\bar{R}}(x)}\right\}\end{array}\right)$.
6. $\bar{R}\vee\bar{S}=\left(\begin{array}{c}\max\left\{\mu_{\bar{R}}e^{i2\pi\mu_{\bar{R}}(x)},\mu_{\bar{S}}e^{i2\pi\mu_{\bar{S}}(x)}\right\},\min\left\{\eta_{\bar{R}}e^{i2\pi\eta_{\bar{R}}(x)},\eta_{\bar{S}}e^{i2\pi\eta_{\bar{S}}(x)}\right\},\\ \min\left\{\nu_{\bar{R}}e^{i2\pi\nu_{\bar{R}}(x)},\nu_{\bar{S}}e^{i2\pi\nu_{\bar{S}}(x)}\right\}\end{array}\right)$.
7. $\bar{R}\oplus\bar{S}=\left(\begin{array}{c}\mu_{\bar{R}}e^{i2\pi\mu_{\bar{R}}(x)}+\mu_{\bar{S}}e^{i2\pi\mu_{\bar{S}}(x)}-\mu_{\bar{R}}e^{i2\pi\mu_{\bar{R}}(x)}\mu_{\bar{S}}e^{i2\pi\mu_{\bar{S}}(x)},\eta_{\bar{R}}e^{i2\pi\eta_{\bar{R}}(x)}\eta_{\bar{S}}e^{i2\pi\eta_{S}(x)},\\ \nu_{\bar{R}}e^{i2\pi\nu_{\bar{R}}(x)}\nu_{\bar{S}}e^{i2\pi\nu_{\bar{S}}(x)}\end{array}\right)$.
8. $\bar{R}\otimes\bar{S}=\left(\begin{array}{c}\mu_{\bar{R}}e^{i2\pi\mu_{\bar{R}}(x)}\mu_{\bar{S}}e^{i2\pi\mu_{\bar{S}}(x)},\eta_{\bar{R}}e^{i2\pi\eta_{\bar{R}}(x)}+\eta_{\bar{S}}e^{i2\pi\eta_{\bar{S}}(x)}-\eta_{\bar{R}}e^{i2\pi\eta_{\bar{R}}(x)}\eta_{\bar{S}}e^{i2\pi\eta_{\bar{S}}(x)},\\ \nu_{\bar{R}}e^{i2\pi\nu_{\bar{R}}(x)}+\nu_{\bar{S}}e^{i2\pi\nu_{S}(x)}-\nu_{\bar{R}}e^{i2\pi\nu_{\bar{R}}(x)}\nu_{\bar{S}}e^{i2\pi\nu_{S}(x)}\end{array}\right)$.

9. $\zeta\bar{R} = \left(1-\left(1-\mu_{\bar{R}}e^{i2\pi\mu_{\bar{R}}(x)}\right)^{\zeta}, \left(\eta_{\bar{R}}^{\zeta}\right)e^{i2\pi\eta_{\bar{R}}(x)}, \nu_{\bar{R}}^{\zeta}e^{i2\pi\nu_{\bar{R}}(x)}\right)$.

10. $\bar{R}^{\zeta} = \left(\mu_{\bar{R}}^{\zeta}e^{i2\pi\mu_{\bar{R}}(x)}, 1-\left(1-\eta_{\bar{R}}e^{i2\pi\eta_{\bar{R}}(x)}\right)^{\zeta}, 1-\left(1-\nu_{\bar{R}}e^{i2\pi\nu_{\bar{R}}(x)}\right)^{\zeta}\right)$.

Definition 2.3. *For a $CPFN\bar{R} = \left(\mu_{\bar{R}}(x)e^{i2\pi\mu_{\bar{R}}(x)}, \eta_{\bar{R}}e^{i2\pi\eta_{\bar{R}}(x)}, \nu_{\bar{R}}e^{i2\pi\nu_{\bar{R}}(x)}\right)$*

$$\Delta(p) = \frac{2+\mu_{\bar{R}}e^{i2\pi\mu_{\bar{R}}(x)} - \nu_{\bar{R}}e^{i2\pi\nu_{\bar{R}}(x)}}{4},$$

is defined as score function, where $\Delta(p) \in [0,1]$.

Definition 2.4. *For a $CPFN\bar{R} = \left(\mu_{\bar{R}}(x)e^{i2\pi\mu_{\bar{R}}(x)}, \eta_{\bar{R}}e^{i2\pi\eta_{\bar{R}}(x)}, \nu_{\bar{R}}e^{i2\pi\nu_{\bar{R}}(x)}\right)$*

$$\nabla(p) = \mu_{\bar{R}}e^{i2\pi\mu_{\bar{R}}(x)} + \nu_{\bar{R}}e^{i2\pi\nu_{\bar{R}}(x)},$$

is defined as accuracy function, where $\Psi(p) \in [-1,1]$.

According to Definitions 2.3 and 2.4, if $\bar{R} = \left(\mu_{\bar{R}}e^{i2\pi\mu_{\bar{R}}(x)}, \eta_{\bar{R}}e^{i2\pi\eta_{\bar{R}}(x)}, \nu_{\bar{R}}e^{i2\pi\nu_{\bar{R}}(x)}\right)$ and $\bar{S} = \left(\mu_{\bar{S}}e^{i2\pi\mu_{\bar{R}}(x)}, \eta_{\bar{S}}e^{i2\pi\eta_{\bar{R}}(x)}, \nu_{\bar{S}}e^{i2\pi\nu_{\bar{R}}(x)}\right)$ are any two CPFNs then

1. If $\Delta(\bar{R}) > \Delta(\bar{S})$ then $\bar{R} > \bar{S}$.
2. If $\Delta(\bar{R}) < \Delta(\bar{S})$ then $\bar{R} < \bar{S}$.
3. If $\Delta(\bar{R}) = \Delta(\bar{S})$, then
   (a) If $\nabla(\bar{R}) > \nabla(\bar{S})$, then $\bar{R} > \bar{S}$.
   (b) If $\nabla(\bar{R}) = \nabla(\bar{S})$, then $\bar{R} = \bar{S}$.

Definition 2.5. *Let $\bar{R}_k = \left(\mu_{\bar{R}_k}e^{i2\pi\mu_{\bar{R}_k}(x)}, \eta_{\bar{R}_k}e^{i2\pi\eta_{\bar{R}_k}(x)}, \nu_{\bar{R}_k}e^{i2\pi\nu_{\bar{R}_k}(x)}\right)(k=1,2,\ldots,n)$ be CPFNs' collection. Then, by using the CPFWAOs, their aggregated value is also a CPFN and $CPFWA\left(\bar{R}_1, \bar{R}_2, \ldots, \bar{R}_n\right) = \oplus_{k=1}^{n}\left(w_k\bar{R}_k\right)$*

$$= \begin{pmatrix} 1-\prod_{k=1}^{n}\left(1-\mu_{\bar{R}_k}e^{i2\pi\mu_{\bar{R}_k}(x)}\right)^{w_k}, \left(\prod_{k=1}^{n}\eta_{\bar{R}_k}e^{i2\pi\eta_{\bar{R}_k}(x)}\right)^{w_k}, \\ \left(\prod_{k=1}^{n}\nu_{\bar{R}_k}e^{i2\pi\nu_{\bar{R}_k}(x)}\right)^{w_k} \end{pmatrix},$$

here $w = (w_1, w_2, \ldots, w_n)^t$ denotes the weight-vector of $p_k(k=1,2,\ldots,n), w_k \in [0,1]$ and $\sum_{k=1}^{n} w_k = 1$.

Definition 2.6. *Let $\bar{R}_k = \left(\mu_{\bar{R}_k}e^{i2\pi\mu_{\bar{R}_k}(x)}, \eta_{\bar{R}_k}e^{i2\pi\eta_{\bar{R}_k}(x)}, \nu_{\bar{R}_k}e^{i2\pi\nu_{\bar{R}_k}(x)}\right)(k=1,2,\ldots,n)$ be a collection of CPFNs. A structure $p^n \to p$ such that,*

$$CPFOWA\left(\bar{R}_1, \bar{R}_2, \ldots, \bar{R}_n\right) = \bigoplus_{k=1}^{n}\left(w_k\bar{R}_{\bar{R}(k)}\right) = \left(1-\prod_{k=1}^{n}\left(1-\mu_{\bar{R}_{\bar{R}(k)}}\right)^{w_i}, \prod_{k=1}^{n}\eta_{\bar{R}_{\bar{R}(k)}}w_k, \prod_{k=1}^{n}\nu_{\bar{R}_{\bar{R}(k)}}w_k\right),$$

is known as CPFOWA operator and,$(\rho(1), \rho(2), \ldots, \rho(n))$ denotes the permutation related to $(k=1,2,\ldots,n)$, satisfying $p_{\rho(i-1)} \geq p_{\rho(k)}; k=1,2,\ldots,n$.

The definition of Frank t-norm and t-conorm is provided as follows:

Definition 2.7. *[28] For a and b as two real numbers, the functions*

$$\text{Fra}(a,b) = \log_r\left(1 + \frac{\left(r^a - 1\right)\left(r^b - 1\right)}{r-1}\right),$$

and

$$\text{Fra}'(a,b) = 1 - \log_r\left(1 + \frac{\left(r^{1-a} - 1\right)\left(r^{1-b} - 1\right)}{r-1}\right),$$

are defined as Frank t-norm and Frank t-conorm respectively, where $(a,b) \in [0,1] \times [0,1]$ and $r \neq 1$.

Following observations should be considered here [63]:

1. $\text{Fra}'(a,b) \to a+b-ab$, when $r \to 1$, also Fra $(a,b) \to ab$ when $r \to 1$. Therefore, we conclude that sum and product of Frank change into sum and product of probabilistic when $r \to 1$.
2. $\text{Fra}'(a,b) \to \min\{a+b,1\}$ when $r \to \infty$ and Fra $(a,b) \to \max\{0, a+b-1\}$ when $r \to \infty$. Therefore, we conclude that the sum as well as product of Frank change into sum as well as product of Lukasiewicz, when $r \to \infty$.

Example 2.8. Let $a = 0.33, b = 0.98$ and $r = 5$, then,

$$\text{Fra}(0.33, 0.98) = \log_4\left(1 + \frac{\left(4^{0.33} - 1\right)\left(4^{0.98} - 1\right)}{5-1}\right) = 0.3197.$$

$$\text{Fra}'(0.33, 0.98) = 1 - \log_4\left(1 + \frac{\left(4^{1-0.33} - 1\right)\left(4^{1-0.98} - 1\right)}{5-1}\right) = 0.9902.$$

3 Complex Picture Fuzzy Frank Aggregation Operators

By skillfully using the t-norm and t-conorm of Frank, notable operating rules for the CPF environment have been created in this part. We also recommend the CPFFWA, CPFFOWA, CPFFHWA, CPFFWG, CPFFOWG, and CPFFHWA aggregation operators utilizing the operational principles we have defined.

Definition 3.1. *Let $\bar{R} = \left(\mu_{\bar{R}} e^{i2\pi\mu_{\bar{R}}(x)}, \eta_{\bar{R}} e^{i2\pi\eta_{\bar{R}}(x)}, \nu_{\bar{R}} e^{i2\pi\nu_{\bar{R}}(x)}\right), \bar{S} = \left(\mu_{\bar{S}} e^{i2\pi\mu_{\bar{J}}(x)}, \eta_{\bar{S}} e^{i2\pi\eta_{\bar{J}}(x)}, \nu_{\bar{S}} e^{i2\pi\nu_{\bar{J}}(x)}\right)$ and $\bar{T} = \left(\mu_{\bar{T}} e^{i2\pi\mu_{\bar{T}}(x)}, \eta_{\bar{T}} e^{i2\pi\eta_{\bar{T}}(x)}, \nu_{\bar{T}} e^{i2\pi\nu_{\bar{T}}(x)}\right)$ be CPFNs, $r > 1$, and $\zeta > 0$ is a real number.*

Frank t-norm and t-conorm operations for CPFNs are provided as follows:

1. $\bar{R} \oplus \bar{S} =$

$$= \left(\begin{array}{c} 1 - \log_r\left(1 + \frac{\left(\begin{array}{c}\left(r^{1-\mu_{\bar{R}} e^{i2\pi\mu_{\bar{R}}(x)}} - 1\right) \\ \left(r^{1-\mu_{\bar{S}} e^{i2\pi\mu_{\bar{S}}(x)}} - 1\right)\end{array}\right)}{r-1}\right), \log_r\left(1 + \frac{\left(\begin{array}{c}\left(r^{\eta_{\bar{R}} e^{i2\pi\eta_{\bar{R}}(x)}} - 1\right) \\ \left(r^{\eta_{\bar{S}} e^{i2\pi\eta_{\bar{S}}(x)}} - 1\right)\end{array}\right)}{r-1}\right), \\ \log_r\left(1 + \frac{\left(\begin{array}{c}\left(r^{\nu_{\bar{R}} e^{i2\pi\nu_{\bar{R}}(x)}} - 1\right) \\ \left(r^{\nu_{\bar{S}} e^{i2\pi\nu_{\bar{S}}(x)}} - 1\right)\end{array}\right)}{r-1}\right) \end{array}\right).$$

2. $\bar{R}\otimes\bar{S}=$

$$= \left(\begin{array}{c} \log_r\left(1+\frac{\left(\begin{array}{c}\left(r^{\mu_{\bar{R}}e^{i2\pi\mu_{\bar{R}}(x)}}-1\right)\\ \left(r^{\mu_{\bar{S}}e^{i2\pi\mu_{\bar{S}}(x)}}-1\right)\end{array}\right)}{r-1}\right), 1-\log_r\left(1+\frac{\left(\begin{array}{c}\left(r^{1-\eta_{\bar{R}}e^{i2\pi\eta_{\bar{R}}(x)}}-1\right)\\ \left(r^{1-\eta_{\bar{S}}e^{i2\pi\eta_{\bar{S}}(x)}}-1\right)\end{array}\right)}{r-1}\right), \\ 1-\log_r\left(1+\frac{\left(\begin{array}{c}\left(r^{1-\nu_{\bar{R}}e^{i2\pi\nu_{\bar{R}}(x)}}-1\right)\\ \left(r^{1-\nu_{\bar{S}}e^{i2\pi\nu_{\bar{S}}(x)}}-1\right)\end{array}\right)}{r-1}\right) \end{array} \right).$$

3. $\zeta\bar{R}$

$$= \left(\begin{array}{c} 1-\log_r\left(1+\frac{\left(r^{1-\mu_{\bar{R}}e^{i2\pi\mu_{\bar{R}}(x)}}-1\right)^{\zeta}}{(r-1)^{\zeta-1}}\right), \log_r\left(1+\frac{\left(r^{\eta_{\bar{R}}e^{i2\pi\eta_{\bar{R}}(x)}}-1\right)^{\zeta}}{(r-1)^{\zeta-1}}\right), \\ \log_r\left(1+\frac{\left(r^{\nu_{\bar{R}}e^{i2\pi\nu_{\bar{R}}(x)}}-1\right)^{\zeta}}{(r-1)^{\zeta-1}}\right) \end{array} \right).$$

4. $\bar{R}^{\zeta}$

$$= \left(\begin{array}{c} \log_r\left(1+\frac{\left(r^{\mu_{\bar{R}}e^{i2\pi\mu_{\bar{R}}(x)}}-1\right)^{\zeta}}{(r-1)^{\zeta-1}}\right), 1-\log_r\left(1+\frac{\left(r^{1-\eta_{\bar{R}}e^{i2\pi\eta_{\bar{R}}(x)}}-1\right)^{\zeta}}{(r-1)^{\zeta-1}}\right), \\ 1-\log_r\left(1+\frac{\left(r^{1-\nu_{\bar{R}}e^{i2\pi\nu_{\bar{R}}(x)}}-1\right)^{\zeta}}{(r-1)^{\zeta-1}}\right) \end{array} \right).$$

Example 3.2. Let $\bar{R}=\left(0.5e^{i2\pi(0.5)},0.3e^{i2\pi(0.1)},0.2e^{i2\pi(0.4)}\right)$ and $\bar{S}=\left(0.1e^{i2\pi(0.4)},0.5e^{i2\pi(0.4)},0.4e^{i2\pi(0.2)}\right)$ be any two CPFNs. Let $r=2$ and $\zeta=3$ in definition 3.1, we get

1. $\bar{R}\oplus\bar{S}=\left(0.5577e^{i2\pi(0.7206)},0.1319e^{i2\pi(0.0327)},0.0669e^{i2\pi(0.06)}\right)$.
2. $\bar{R}\otimes\bar{S}=\left(0.0422e^{i2\pi(0.1793)},0.6680e^{i2\pi(0.4672)},0.0.2243e^{i2\pi(0.3330)}\right)$.
3. $5\bar{R}=\left(0.9348e^{i2\pi(0.9860)},0.0012e^{i2\pi(0.00001)},0.000015e^{i2\pi(0.000015)}\right)$.
4. $\bar{R}^4=\left(0.00003e^{i2\pi(0.0034)},0.0975e^{i2\pi(0.0876)},0.9216e^{i2\pi(0.9216)}\right)$.

Theorem 3.3. *Let* $\bar{R}=\left(\mu_{\bar{R}}e^{i2\pi\mu_{\bar{R}}(x)},\eta_{\bar{R}}e^{i2\pi\eta_{\bar{R}}(x)},\nu_{\bar{R}}e^{i2\pi\nu_{\bar{R}}(x)}\right)$, *and* $\bar{S}=\left(\mu_{\bar{S}}e^{i2\pi\mu_{\bar{S}}(x)},\eta_{\bar{S}}e^{i2\pi\eta_{\bar{S}}(x)},\nu_{\bar{S}}e^{i2\pi\nu_{\bar{S}}(x)}\right)$ *and* $\bar{T}=\left(\mu_{\bar{T}}e^{i2\pi\mu_{\bar{T}}(x)},\eta_{\bar{T}}e^{i2\pi\eta_{\bar{T}}(x)},\nu_{\bar{T}}e^{i2\pi\nu_{\bar{T}}(x)}\right)$ *be any three CPFNs. Let* $r>1$ *and* ζ,ζ_1,ζ_2 *are positive real numbers, then*

1. $\bar{R}\oplus\bar{S}=\bar{S}\oplus\bar{R}$.
2. $\bar{R}\otimes\bar{S}=\bar{S}\otimes\bar{R}$.
3. $\zeta(\bar{R}\oplus\bar{S})=\zeta\bar{R}\oplus\zeta\bar{S}$.

4. $\zeta_1\bar{R}\oplus\zeta_2\bar{R}=(\zeta_1+\zeta_2)\,\bar{R}$.

5. $(\bar{R}\otimes\bar{S})^{\zeta}=\bar{R}^{\zeta}\otimes\bar{S}^{\zeta}$.

6. $\bar{R}^{\zeta_1}\otimes\bar{R}^{\zeta_2}=\bar{R}^{\zeta_1+\zeta_2}$.

Proof. For three CPFNs $\bar{R}=\left(\mu_{\bar{R}}e^{i2\pi\mu_{\bar{R}}(x)},\eta_{\bar{R}}e^{i2\pi\eta_{\bar{R}}(x)},\nu_{\bar{R}}e^{i2\pi\nu_{\bar{R}}(x)}\right)$, $\bar{S}=\left(\mu_{\bar{S}}e^{i2\pi\mu_{\bar{S}}(x)},\eta_{\bar{S}}e^{i2\pi\eta_{\bar{S}}(x)},\nu_{\bar{S}}e^{i2\pi\nu_{\bar{S}}(x)}\right)$ and $\bar{T}=\left(\mu_{\bar{T}}e^{i2\pi\mu_{\bar{T}}(x)},\eta_{\bar{T}}e^{i2\pi\eta_{\bar{T}}(x)},\nu_{\bar{T}}e^{i2\pi\nu_{\bar{T}}(x)}\right)$ with $\zeta,\zeta_1,\zeta_2>0$. According to Definition 3.1, we can obtain

1. $$\bar{R}\oplus\bar{S}=\left(\begin{array}{c}1-\log_r\left(1+\frac{\left(\begin{array}{c}\left(r^{1-\mu_{\bar{R}}e^{i2\pi\mu_{\bar{R}}(x)}}-1\right)\\\left(r^{1-\mu_{\bar{S}}e^{i2\pi\mu_{\bar{S}}(x)}}-1\right)\end{array}\right)}{r-1}\right),\log_r\left(1+\frac{\left(\begin{array}{c}\left(r^{\eta_{\bar{R}}e^{i2\pi\eta_{\bar{R}}(x)}}-1\right)\\\left(r^{\eta_{\bar{S}}e^{i2\pi\eta_{\bar{S}}(x)}}-1\right)\end{array}\right)}{r-1}\right),\\\log_r\left(1+\frac{\left(r^{\nu_{\bar{R}}e^{i2\pi\nu_{\bar{R}}(x)}}-1\right)\left(r^{\nu_{\bar{S}}e^{i2\pi\nu_{\bar{S}}(x)}}-1\right)}{r-1}\right)\end{array}\right)$$

$$=\left(\begin{array}{c}1-\log_r\left(1+\frac{\left(\begin{array}{c}\left(r^{1-\mu_{\bar{R}}e^{i2\pi\mu_{\bar{R}}(x)}}-1\right)\\\left(r^{1-\mu_{\bar{S}}e^{i2\pi\mu_{\bar{S}}(x)}}-1\right)\end{array}\right)}{r-1}\right),\log_r\left(1+\frac{\left(\begin{array}{c}\left(r^{\eta_{\bar{R}}e^{i2\pi\eta_{\bar{R}}(x)}}-1\right)\\\left(r^{\eta_{\bar{S}}e^{i2\pi\eta_{\bar{S}}(x)}}-1\right)\end{array}\right)}{r-1}\right),\\\log_r\left(1+\frac{\left(r^{\nu_{\bar{R}}e^{i2\pi\nu_{\bar{R}}(x)}}-1\right)\left(r^{\nu_{\bar{S}}e^{i2\pi\nu_{\bar{S}}(x)}}-1\right)}{r-1}\right)\end{array}\right)=\bar{S}\oplus\bar{R}.$$

2. $$\bar{R}\otimes\bar{S}=\left(\begin{array}{c}\log_r\left(1+\frac{\left(\begin{array}{c}\left(r^{\mu_{\bar{S}}e^{i2\pi\mu_{\bar{S}}(x)}}-1\right)\\\left(r^{\mu_{\bar{R}}e^{i2\pi\mu_{\bar{R}}(x)}}-1\right)\end{array}\right)}{r-1}\right),1-\log_r\left(1+\frac{\left(\begin{array}{c}\left(r^{1-\eta_{\bar{R}}e^{i2\pi\eta_{\bar{R}}(x)}}-1\right)\\\left(r^{1-\eta_{\bar{S}}e^{i2\pi\eta_{\bar{S}}(x)}}-1\right)\end{array}\right)}{r-1}\right)\\1-\log_r\left(1+\frac{\left(r^{1-\nu_{\bar{R}}e^{i2\pi\nu_{\bar{R}}(x)}}-1\right)\left(r^{1-\nu_{\bar{S}}e^{i2\pi\nu_{\bar{S}}(x)}}-1\right)}{r-1}\right)\end{array}\right)$$

$$=\left(\begin{array}{c}\log_r\left(1+\frac{\left(\begin{array}{c}\left(r^{\mu_{\bar{S}}e^{i2\pi\mu_{\bar{S}}(x)}}-1\right)\\\left(r^{\mu_{\bar{R}}e^{i2\pi\mu_{\bar{R}}(x)}}-1\right)\end{array}\right)}{r-1}\right),1-\log_r\left(1+\frac{\left(\begin{array}{c}\left(r^{1-\eta_{\bar{R}}e^{i2\pi\eta_{\bar{R}}(x)}}-1\right)\\\left(r^{1-\eta_{\bar{S}}e^{i2\pi\eta_{\bar{S}}(x)}}-1\right)\end{array}\right)}{r-1}\right)\\1-\log_r\left(1+\frac{\left(r^{1-\nu_{\bar{R}}e^{i2\pi\nu_{\bar{R}}(x)}}-1\right)\left(r^{1-\nu_{\bar{S}}e^{i2\pi\nu_{\bar{S}}(x)}}-1\right)}{r-1}\right)\end{array}\right)=\bar{S}\otimes\bar{R}.$$

$$3.\ \zeta(\bar{R}\oplus\bar{S}) = \zeta\left(\begin{array}{c} 1-\log_r\left(1+\frac{\left(\begin{array}{c}\left(r^{1-\mu_{\bar{R}}}e^{i2\pi\mu_{\bar{R}}(x)}-1\right)\\ \left(r^{1-\mu_{\bar{S}}}e^{i2\pi\mu_{\bar{S}}(x)}-1\right)\end{array}\right)}{r-1}\right), \log_r\left(1+\frac{\left(\begin{array}{c}\left(r^{\eta_{\bar{R}}}e^{i2\pi\eta_{\bar{R}}(x)}-1\right)\\ \left(r^{\eta_{\bar{S}}}e^{i2\pi\eta_{\bar{S}}(x)}-1\right)\end{array}\right)}{r-1}\right), \\ \log_r\left(1+\frac{\left(r^{\nu_{\bar{R}}e^{i2\pi\nu_{\bar{R}}(x)}}-1\right)\left(r^{\nu_{\bar{S}}e^{i2\pi\nu_{\bar{S}}(x)}}-1\right)}{r-1}\right)\end{array}\right)$$

$$= \left(\begin{array}{c} 1-\log_r\left(1+\frac{\left(\begin{array}{c}\left(r^{1-\mu_{\bar{R}}}e^{i2\pi\mu_{\bar{R}}(x)}-1\right)^{\zeta}\\ \left(r^{1-\mu_{\bar{S}}}e^{i2\pi\mu_{\bar{S}}(x)}-1\right)^{\zeta}\end{array}\right)}{(r-1)^{2\zeta-1}}\right), \log_r\left(1+\frac{\left(\begin{array}{c}\left(r^{\eta_{\bar{R}}}e^{i2\pi\eta_{\bar{R}}(x)}-1\right)^{\zeta}\\ \left(r^{\eta_{\bar{S}}}e^{i2\pi\eta_{\bar{S}}(x)}-1\right)^{\zeta}\end{array}\right)}{(r-1)^{2\zeta-1}}\right), \\ \log_r\left(1+\frac{\left(r^{\nu_{\bar{R}}e^{i2\pi\nu_{\bar{R}}(x)}}-1\right)^{\zeta}\left(r^{\nu_{\bar{S}}e^{i2\pi\nu_{\bar{S}}(x)}}-1\right)^{\zeta}}{(r-1)^{2\zeta-1}}\right)\end{array}\right).$$

Now

$$\zeta\bar{R}\oplus\zeta\bar{S} = \left(\begin{array}{c} 1-\log_r\left(1+\frac{\left(r^{1-\mu_{\bar{R}}e^{i2\pi\mu_{\bar{R}}(x)}}-1\right)^{\zeta}}{(r-1)^{\zeta}}\right), \log_r\left(1+\frac{\left(r^{\eta_{\bar{R}}e^{i2\pi\eta_{\bar{R}}(x)}}-1\right)^{\zeta}}{(r-1)^{\zeta}}\right), \\ \log_r\left(1+\frac{\left(r^{\nu_{\bar{R}}e^{i2\pi\nu_{\bar{R}}(x)}}-1\right)^{\zeta}}{(r-1)^{\zeta}}\right)\end{array}\right)$$

$$\oplus\left(\begin{array}{c} 1-\log_r\left(1+\frac{\left(r^{1-\mu_{\bar{S}}e^{i2\pi\mu_{\bar{S}}(x)}}-1\right)^{\zeta}}{(r-1)^{\zeta}}\right), \log_r\left(1+\frac{\left(r^{\eta_{\bar{S}}e^{i2\pi\eta_{\bar{S}}(x)}}-1\right)^{\zeta}}{(r-1)^{\zeta}}\right), \\ \log_r\left(1+\frac{\left(r^{\nu_{\bar{S}}e^{i2\pi\nu_{\bar{S}}(x)}}-1\right)^{\zeta}}{(r-1)^{\zeta}}\right)\end{array}\right)$$

$$= \left(\begin{array}{c} 1-\log_r\left(1+\frac{\left(\begin{array}{c}\left(r^{1-\mu_{\bar{R}}}e^{i2\pi\mu_{\bar{R}}(x)}-1\right)^{\zeta}\\ \left(r^{1-\mu_{\bar{S}}}e^{i2\pi\mu_{\bar{S}}(x)}-1\right)^{\zeta}\end{array}\right)}{(r-1)^{2\zeta-1}}\right), \log_r\left(1+\frac{\left(\begin{array}{c}\left(r^{\eta_{\bar{R}}}e^{i2\pi\eta_{\bar{R}}(x)}-1\right)^{\zeta}\\ \left(r^{\eta_{\bar{S}}}e^{i2\pi\eta_{\bar{S}}(x)}-1\right)^{\zeta}\end{array}\right)}{(r-1)^{2\zeta-1}}\right), \\ \log_r\left(1+\frac{\left(r^{\nu_{\bar{R}}e^{i2\pi\nu_{\bar{R}}(x)}}-1\right)^{\zeta}\left(r^{\nu_{\bar{S}}e^{i2\pi\nu_{\bar{S}}(x)}}-1\right)^{\zeta}}{(r-1)^{2\zeta-1}}\right)\end{array}\right).$$

Therefore,

$$\zeta(\bar{R} \oplus \bar{S}) = \zeta\bar{R} \oplus \zeta\bar{S}.$$

4. $$\zeta_1\bar{R} \oplus \zeta_2\bar{R} = \begin{pmatrix} 1-\log_r\left(1+\frac{\left(r^{1-\mu_{\bar{R}}e^{i2\pi\mu_{\bar{R}}(x)}}-1\right)^{\zeta_1}}{(r-1)^{\zeta_1}}\right), \log_r\left(1+\frac{\left(r^{\eta_{\bar{R}}e^{i2\pi\eta_{\bar{R}}(x)}-1}\right)^{\zeta_1}}{(r-1)^{\zeta_1}}\right) \\ \log_r\left(1+\frac{\left(r^{\nu_{\bar{R}}e^{i2\pi\nu_{\bar{R}}(x)}-1}\right)^{\zeta_1}}{(r-1)^{\zeta_1}}\right) \end{pmatrix}$$

$$\oplus \begin{pmatrix} 1-\log_r\left(1+\frac{\left(r^{1-\mu_{\bar{R}}e^{i2\pi\mu_{\bar{R}}(x)}}-1\right)^{\zeta_2}}{(r-1)^{\zeta_2}}\right), \log_r\left(1+\frac{\left(r^{\eta_{\bar{R}}e^{i2\pi\eta_{\bar{R}}(x)}-1}\right)^{\zeta_2}}{(r-1)^{\zeta_2}}\right) \\ \log_r\left(1+\frac{\left(r^{\nu_{\bar{R}}e^{i2\pi\nu_{\bar{R}}(x)}-1}\right)^{\zeta_2}}{(r-1)^{\zeta_2}}\right) \end{pmatrix}$$

$$= \begin{pmatrix} 1-\log_r\left(1+\frac{\left(r^{1-\mu_{\bar{R}}e^{i2\pi\mu_{\bar{R}}(x)}}-1\right)^{\zeta_1+\zeta_2}}{(r-1)^{\zeta_1+\zeta_2}}\right), \log_r\left(1+\frac{\left(r^{\eta_{\bar{R}}e^{i2\pi\eta_{\bar{R}}(x)}}\right)^{\zeta_1+\zeta_2}}{(r-1)^{\zeta_1+\zeta_2}}\right) \\ , \log_r\left(1+\frac{\left(r^{\nu_{\bar{R}}e^{i2\pi\nu_{\bar{R}}(x)}-1}\right)^{\zeta_1+\zeta_2}}{(r-1)^{\zeta_1+\zeta_2}}\right) \end{pmatrix}.$$

$$= (\zeta_1+\zeta_2)\,\bar{R}.$$

5. $$\left(\bar{R}_1 \otimes \bar{R}_2\right)^\zeta = \begin{pmatrix} \log_r\left(1+\frac{\left(\left(r^{\mu_{\bar{R}_1}e^{i2\pi\mu_{\bar{R}_1}(x)}}-1\right)\left(r^{\mu_{\bar{R}_2}e^{i2\pi\mu_{\bar{R}_2}(x)}}-1\right)\right)}{r-1}\right), 1-\log_r\left(1+\frac{\left(\left(r^{1-\eta_{\bar{R}_1}e^{i2\pi\eta_{\bar{R}_1}(x)}}-1\right)\left(r^{1-\eta_{\bar{R}_2}e^{i2\pi\eta_{\bar{R}_2}(x)}}-1\right)\right)}{r-1}\right) \\ 1-\log_r\left(1+\frac{\left(r^{1-\nu_{\bar{R}_1}e^{i2\pi\nu_{\bar{R}_1}(x)}}-1\right)\left(r^{1-\nu_{\bar{R}_2}e^{i2\pi\nu_{\bar{R}_2}(x)}}-1\right)}{r-1}\right) \end{pmatrix}^\zeta$$

$$= \left(\begin{array}{c} \log_r\left(1+\frac{\left(\left(r^{\mu_{\bar{R}_1}e^{i2\pi\mu_{\bar{R}_1}(x)}}-1\right)\left(r^{\mu_{\bar{R}_2}e^{i2\pi\mu_{\bar{R}_2}(x)}}-1\right)\right)^{\zeta}}{(r-1)^{2\zeta-1}}\right), 1-\log_r\left(1+\frac{\left(\left(r^{1-\eta_{\bar{R}_1}e^{i2\pi\eta_{\bar{R}_1}(x)}}-1\right)\left(r^{1-\eta_{\bar{R}_2}e^{i2\pi\eta_{\bar{R}_2}(x)}}-1\right)\right)^{\zeta}}{(r-1)^{2\zeta-1}}\right) \\ 1-\log_r\left(1+\frac{\left(\left(r^{1-\nu_{\bar{R}_1}e^{i2\pi\nu_{\bar{R}_1}(x)}}-1\right)\left(r^{1-\nu_{\bar{R}_2}e^{i2\pi\nu_{\bar{R}_2}(x)}}-1\right)\right)^{\zeta}}{(r-1)^{2\zeta-1}}\right) \end{array} \right)$$

$$= \left(\begin{array}{c} \log_r\left(1+\frac{\left(r^{\mu_{\bar{R}_1}e^{i2\pi\mu_{\bar{R}_1}(x)}}-1\right)^{\zeta}}{(r-1)^{\zeta}}\right), 1-\log_r\left(1+\frac{\left(r^{1-\eta_{\bar{R}_1}e^{i2\pi\eta_{\bar{R}_1}(x)}}-1\right)^{\zeta}}{(r-1)^{\zeta}}\right), \\ 1-\log_r\left(1+\frac{\left(r^{1-\nu_{\bar{R}_1}e^{i2\pi\nu_{\bar{R}_1}(x)}}-1\right)}{(r-1)^{\zeta}}\right) \end{array} \right).$$

6. $$\bar{R}^{\zeta_1}\otimes\bar{R}^{\zeta_2} = \left(\begin{array}{c} \log_r\left(1+\frac{\left(r^{\mu_{\bar{R}}e^{i2\pi\mu_{\bar{R}}(x)}}-1\right)^{\zeta_1}}{(r-1)^{\zeta_1-1}}\right), 1-\log_r\left(1+\frac{\left(r^{1-\eta_{\bar{R}}e^{i2\pi\eta_{\bar{R}}(x)}}-1\right)^{\zeta_1}}{(r-1)^{\zeta_1-1}}\right), \\ 1-\log_r\left(1+\frac{\left(r^{1-\nu_{\bar{R}}e^{i2\pi\nu_{\bar{R}}(x)}}-1\right)^{\zeta_1}}{(r-1)^{\zeta_1-1}}\right) \end{array} \right)$$

$$\otimes \left(\begin{array}{c} \log_r\left(1+\frac{\left(r^{\mu_{\bar{R}}e^{i2\pi\mu_{\bar{R}}(x)}}-1\right)^{\zeta_2}}{(r-1)^{\zeta_2-1}}\right), 1-\log_r\left(1+\frac{\left(r^{1-\eta_{\bar{R}}e^{i2\pi\eta_{\bar{R}}(x)}}-1\right)^{\zeta_2}}{(r-1)^{\zeta_2-1}}\right), \\ 1-\log_r\left(1+\frac{\left(r^{1-\nu_{\bar{R}}e^{i2\pi\nu_{\bar{R}}(x)}}-1\right)^{\zeta_2}}{(r-1)^{\zeta_2-1}}\right) \end{array} \right)$$

$$= \left(\begin{array}{c} \log_r\left(1+\frac{\left(r^{\mu}e^{i2\pi\mu_{\bar{R}}(x)}-1\right)^{\zeta_1+\zeta_2}}{(r-1)^{\zeta_1+\zeta_2-1}}\right), 1-\log_r\left(1+\frac{\left(r^{1-\eta_{\bar{R}}e^{i2\pi\eta_{\bar{R}}(x)}}\right)^{\zeta_1+\zeta_2}}{(r-1)^{\zeta_1+\zeta_2-1}}\right), \\ 1-\log_r\left(1+\frac{\left(r^{1-\nu_{\bar{R}}e^{i2\pi\nu_{\bar{R}}(x)}}-1\right)^{\zeta_1+\zeta_2}}{(r-1)^{\zeta_1+\zeta_2-1}}\right) \end{array} \right) = \bar{R}^{\zeta_1+\zeta_2}.$$

□

3.1 Complex Picture Fuzzy Frank Arithmetic Aggregation Operators

Definition 3.4. *Let* $\bar{R}_k = \left(\mu_{\bar{R}_k} e^{i2\pi\mu_{\bar{R}_k}(x)}, \eta_{\bar{R}_k} e^{i2\pi\eta_{\bar{R}_k}(x)}, \nu_{\bar{R}_k} e^{i2\pi\nu_{\bar{R}_k}(x)}\right)(k = 1, 2, \ldots, n)$ *be CPFNs' collection. Then, a function* $p^n \to p$

$$CPFFWA\left(\bar{R}_1, \bar{R}_2, \ldots, \bar{R}_n\right) = \bigoplus_{k=1}^{n} \left(w_k \bar{R}_k\right),$$

is known as the CPFFWA operator with $w = (w_1, w_2, \ldots, w_n)^t$ *as the weight vector of* $\bar{R}_k (k = 1, 2, \ldots, n), w_k \in [0, 1]$ *and* $\sum_{k=1}^{n} w_k = 1$.

As a result, the following consequential theorem is obtained.

Theorem 3.5. *Let* $\bar{R}_k = \left(\mu_{\bar{R}_k} e^{i2\pi\mu_{\bar{R}_k}(x)}, \eta_{\bar{R}_k} e^{i2\pi\eta_{\bar{R}_k}(x)}, \nu_{\bar{R}_k} e^{i2\pi\nu_{\bar{R}_k}(x)}\right)(k = 1, 2, \ldots, n)$ *be CPFNs' collection, then the aggregated value is also a CPFN, and*

$$CPFFWA\left(\bar{R}_1, \bar{R}_2, \ldots, \bar{R}_n\right) = \bigoplus_{k=1}^{n} \left(w_k \bar{R}_k\right)$$

$$= \begin{pmatrix} 1 - \log_r\left(1 + \prod_{k=1}^{n}\left(r^{1-\mu_{\bar{R}_k} e^{i2\pi\mu_{\bar{R}}(x)}} - 1\right)^{w_k}\right), \log_r\left(1 + \prod_{k=1}^{n}\left(r^{\eta_{\bar{R}_k} e^{i2\pi\eta_{\bar{R}}(x)}} - 1\right)^{w_k}\right), \\ \log_r\left(1 + \prod_{k=1}^{n}\left(r^{\nu_{\bar{R}_k} e^{i2\pi\nu_{\bar{R}}(x)}}\right)^{w_k}\right) \end{pmatrix}.$$

Proof. Method of mathematical induction would be used for proving this theorem. We take $n = 2$, and by using Frank operations for CPFNs, we get

$$CPFFWA\left(\bar{R}_1, \bar{R}_2\right) = \bigoplus_{k=1}^{2} w_k = w_1 \bar{R}_1 \bigoplus w_2 \bar{R}_2$$

$$= \begin{pmatrix} 1 - \log_r\left(1 + \frac{\left(r^{1-\mu_{\bar{R}_1} e^{i2\pi\mu_{\bar{R}_1}(x)}} - 1\right)^{w_1}}{(r-1)^{w_1 - 1}}\right), \log_r\left(1 + \frac{\left(r^{\eta_{\bar{R}_1} e^{i2\pi\eta_{\bar{R}}(x)}} - 1\right)^{w_1}}{(r-1)^{w_1 - 1}}\right), \\ \log_r\left(1 + \frac{\left(r^{\nu_{\bar{R}_1} e^{i2\pi\nu_{\bar{R}_1}(x)}}\right)^{w_1}}{(r-1)^{w_1 - 1}}\right) \end{pmatrix}$$

$$\oplus \begin{pmatrix} 1 - \log_r\left(1 + \frac{\left(r^{1-\mu_{\bar{R}_2} e^{i2\pi\mu_{\bar{R}_2}(x)}}\right)^{w_2}}{(r-1)^{w_2 - 1}}\right), \log_r\left(1 + \frac{\left(r^{\eta_{\bar{R}_2} e^{i2\pi\eta_{\bar{R}_2}(x)}}\right)^{w_2} - 1}{(r-1)^{w_2 - 1}}\right), \\ \log_r\left(1 + \frac{\left(r^{\nu_{\bar{R}_2} e^{i2\pi\nu_{\bar{R}_2}(x)}} - 1\right)^{w_2}}{(r-1)^{w_2 - 1}}\right) \end{pmatrix}$$

$$= \left(\begin{array}{c} 1 - \log_r \left(1 + \prod_{k=1}^{2} \left(r^{1-\mu_{\bar{R}_k} e^{i2\pi\mu_{\bar{R}_k}(x)}} - 1\right)^{w_k}\right), \log_r \left(1 + \prod_{k=1}^{2} \left(r^{\eta_{\bar{R}_k} e^{i2\pi\mu_{\bar{R}_k}(x)}} - 1\right)^{w_k}\right), \\ \log_r \left(1 + \prod_{k=1}^{2} \left(r^{\nu_{\bar{R}_k} e^{i2\pi\nu_{\bar{R}_k}(x)}} - 1\right)^{w_k}\right) \end{array} \right).$$

$$\left[\because \sum_{k=1}^{2} w_k = 1\right]$$

Therefore, for $n = 2$, the result is true.
By considering the given result as true for $n = s$, we have,

$$CPFFWA\left(\bar{R}_1, \bar{R}_2, \ldots, \bar{R}_n\right) = \bigoplus_{k=1}^{s} w_k \bar{R}_k$$

$$= \left(\begin{array}{c} 1 - \log_r \left(1 + \prod_{k=1}^{s} \left(r^{1-\mu_{\bar{R}_k} e^{i2\pi\mu_{\bar{R}_k}(x)}} - 1\right)^{w_k}\right), \log_r \left(1 + \prod_{k=1}^{s} \left(r^{\eta_{\bar{R}_k} e^{i2\pi\eta_{\bar{R}_k}(x)}} - 1\right)^{w_k}\right), \\ \log_r \left(1 + \prod_{k=1}^{s} \left(r^{\nu_{\bar{R}_k} e^{i2\pi\nu_{\bar{R}_k}(x)}} - 1\right)^{w_k}\right) \end{array} \right).$$

Now, for $n = s + 1$, we have,

$$CPFFWA\left(\bar{R}_1, \bar{R}_2, \ldots, \bar{R}_{n+1}\right) = \bigoplus_{k=1}^{s+1} w_k \bar{R}_k = \bigoplus_{i=1}^{s} w_k \bar{R}_k \bigoplus w_{s+1} \bar{R}_{s+1}$$

$$= \left(\begin{array}{c} 1 - \log_r \left(1 + \frac{\prod_{k=1}^{s} \left(r^{1-\mu_{\bar{R}_k} e^{i2\pi\mu_{\bar{R}_k}(x)}} - 1\right)^{w_k}}{(r-1)^{\sum_{i=1}^{s} w_i - 1}}\right), \\ \log_r \left(1 + \frac{\prod_{k=1}^{s} \left(r^{\eta_{\bar{R}_k} e^{i2\pi\eta_{\bar{R}_k}(x)}} - 1\right)^{w_k}}{(r-1)^{\sum_{i=1}^{s} w_i - 1}}\right), \\ \log_r \left(1 + \frac{\prod_{k=1}^{s} \left(r^{\nu_{\bar{R}_k} e^{i2\pi\nu_{\bar{R}_k}(x)}} - 1\right)^{w_k}}{(r-1)^{\sum_{i=1}^{s} w_i - 1}}\right) \end{array} \right)$$

$$\oplus\begin{pmatrix} 1-\log_r\left(1+\frac{\left(r^{1-\mu_{\bar{R}_{S+1}}e^{i2\pi\mu_{\bar{R}_{S+1}}(x)}}-1\right)^{w_{S+1}}}{(r-1)^{w_{S+1}-1}}\right) \\ \log_r\left(1+\frac{\left(r^{\eta_{\bar{R}_{S+1}}e^{i2\pi\eta_{\bar{R}_{S+1}}(x)}}-1\right)^{w_{S+1}}}{(r-1)^{w_{S+1}-1}}\right) \\ \log_r\left(1+\frac{\left(r^{\nu_{\bar{R}_{S+1}}e^{i2\pi\nu_{\bar{R}_{S+1}}(x)}}-1\right)^{w_{S+1}}}{(r-1)^{w_{S+1}-1}}\right) \end{pmatrix}$$

$$=\begin{pmatrix} 1-\log_r\left(1+\prod_{i=1}^{s+1}\left(r^{1-\mu_{\bar{R}_k}e^{i2\pi\mu_{\bar{R}_k}(x)}}-1\right)^{w_k}\right), \\ \log_r\left(1+\prod_{i=1}^{s+1}\left(r^{\eta_{\bar{R}_k}e^{i2\pi\eta_{\bar{R}_k}(x)}}-1\right)^{w_k}\right), \\ \log_r\left(1+\prod_{i=1}^{s+1}\left(r^{\nu_{\bar{R}_k}e^{i2\pi\nu_{\bar{R}_k}(x)}}-1\right)^{w_k}\right) \end{pmatrix}.$$

$$\text{as }\sum_{k=1}^{s+1}w_k=1.$$

Which shows the result is valid for $n = s+1$, if it is valid for $n = s$. Hence, method of induction shows the validity of our result, no matter what natural number n is. □

Example 3.6. For $\bar{R}_1=\left(0.5e^{i2\pi(0.5)},0.3e^{i2\pi(0.1)},0.1e^{i2\pi(0.4)}\right),\bar{R}_2=\left(0.1e^{i2\pi(0.4)},0.5e^{i2\pi(0.4)},0.4e^{i2\pi(0.2)}\right),\bar{R}_3=\left(0.1e^{i2\pi(0.1)},0.1e^{i2\pi(0.2)},0.7e^{i2\pi(0.5)}\right),\bar{R}_4=\left(0.6e^{i2\pi(0.2)},0.1e^{i2\pi(0.2)},0.3e^{i2\pi(0.3)}\right)$ with weights $=(0.2,0.3,0.1,0.4)$ and $r=2$, step by step working of the operator is given as follows:

$$CPFFWA\left(\bar{R}_1,\bar{R}_2,\ldots,\bar{R}_n\right)=\begin{pmatrix} 1-\log_r\left(1+\prod_{k=1}^{n}\left(r^{1-\mu_{\bar{R}_k}}e^{i2\pi\mu_{\bar{R}}(x)}-1\right)^{w_k}\right), \\ \log_r\left(1+\prod_{k=1}^{n}\left(r^{\eta_{\bar{R}_k}}e^{i2\pi\eta_{\bar{R}}(x)}-1\right)^{w_k}\right), \\ \log_r\left(1+\prod_{k=1}^{n}\left(r^{\nu_{\bar{R}_k}}e^{i2\pi\nu_{\bar{R}}(x)}-1\right)^{w_k}\right) \end{pmatrix}$$

$$=\begin{pmatrix} 1-\log_2\begin{pmatrix} 1+\left(2^{1-(0.5)e^{i2\pi(0.5)}}-1\right)^{0.2}+\left(2^{1-(0.1)e^{i2\pi(0.4)}}-1\right)^{0.3} \\ +\left(2^{1-(0.1)e^{i2\pi(0.1)}}-1\right)^{0.1}+\left(2^{1-(0.6)e^{i2\pi(0.2)}}-1\right)^{0.4} \end{pmatrix}, \\ \log_2\begin{pmatrix} 1+\left(2^{(0.3)e^{i2\pi(0.1)}}-1\right)^{0.2}+\left(2^{(0.5)e^{i2\pi(0.4)}}-1\right)^{0.3} \\ +\left(2^{(0.1)e^{i2\pi(0.1)}}-1\right)^{0.1}+\left(2^{(0.1)e^{i2\pi(0.2)}}-1\right)^{0.4} \end{pmatrix}, \\ \log_2\begin{pmatrix} 1+\left(2^{(0.1)e^{i2\pi(0.4)}}-1\right)^{0.2}+\left(2^{(0.4)e^{i2\pi(0.2)}}-1\right)^{0.3} \\ +\left(2^{(0.7)e^{i2\pi(0.5)}}-1\right)^{0.1}+\left(2^{(0.3)e^{i2\pi(0.3)}}-1\right)^{0.4} \end{pmatrix} \end{pmatrix}$$

$$=\left(0.321e^{i2\pi(0.321)},0.205e^{i2\pi(0.216)},0.330e^{i2\pi(0.297)}\right).$$

Theorem 3.7. *(Idempotent). Let* $\bar{R}_k = \left(\mu_{\bar{R}_k} e^{i2\pi\mu_{\bar{R}_k}(x)}, \eta_{\bar{R}_k} e^{i2\pi\eta_{\bar{R}_k}(x)}, \nu_{\bar{R}_k} e^{i2\pi\nu_{\bar{R}_k}(x)}\right)$ $(k = 1, 2, \ldots, n)$ *be a collection of CPFNs which are all identical, i.e.,* $\bar{R}_k = \bar{R}$ *for all* k, *where* $R = \left(\mu_{\bar{R}} e^{i2\pi\mu_{\bar{R}}(x)}, \eta_{\bar{R}} e^{i2\pi\eta_{\bar{R}}(x)}, v_{\bar{R}} e^{i2\pi\nu_{\bar{R}}(x)}\right)$, *then* $CPFFWA\left(\bar{R}_1, \bar{R}_2, \ldots, \bar{R}_n\right) = \bar{R}$.

Proof. As for every $\bar{R}_k = \bar{R}$, therefore,

$$CPFFWA\left(\bar{p}_1, \bar{p}_2, \ldots, \bar{p}_n\right) =$$

$$= \begin{pmatrix} 1 - \log_r \left(1 + \prod_{k=1}^n \left(r^{1-\mu_{\bar{R}} e^{i2\pi\mu_{\bar{R}}(x)}} - 1\right)^{w_i}\right), \log_r \left(1 + \prod_{k=1}^n \left(r^{\eta_{\bar{R}} e^{i2\pi\eta_{\bar{R}}(x)}} - 1\right)^{w_k}\right), \\ \log_r \left(1 + \prod_{k=1}^n \left(r^{\nu_{\bar{R}} e^{i2\pi\nu_{\bar{R}}(x)}} - 1\right)^{w_k}\right) \end{pmatrix}$$

$$= \begin{pmatrix} 1 - \log_r \left(1 + \prod_{k=1}^n \left(r^{1-\mu_{\bar{R}} e^{i2\pi\mu_{\bar{R}}(x)}} - 1\right)^{w_k}\right), \log_r \left(1 + \prod_{k=1}^n \left(r^{\eta_{\bar{R}} e^{i2\pi\eta_{\bar{R}}(x)}} - 1\right)^{w_k}\right), \\ \log_r \left(1 + \prod_{k=1}^n \left(r^{\nu_{\bar{R}} e^{i2\pi\nu_{\bar{R}}(x)}} - 1\right)^{w_k}\right) \end{pmatrix}$$

$$= \begin{pmatrix} 1 - \log_r \left(1 + \prod_{k=1}^n \left(r^{1-\mu_{\bar{R}} e^{i2\pi\mu_{\bar{R}}(x)}} - 1\right)^{\sum_{k=1}^n w_k}\right), \log_r \left(1 + \prod_{k=1}^n \left(r^{1-\eta_{\bar{R}} e^{i2\pi\eta_{\bar{R}}(x)}} - 1\right)^{\sum_{k=1}^n w_k}\right), \\ \log_r \left(1 + \prod_{k=1}^n \left(r^{1-\nu_{\bar{R}} e^{i2\pi\nu_{\bar{R}}(x)}} - 1\right)^{\sum_{k=1}^n w_k}\right) \end{pmatrix}$$

$$= \begin{pmatrix} 1 - \log_r \left(1 + \prod_{k=1}^n \left(r^{1-\mu_{\bar{R}} e^{i2\pi\mu_{\bar{R}}(x)}} - 1\right)\right), \log_r \left(1 + \prod_{k=1}^n \left(r^{1-\eta_{\bar{R}} e^{i2\pi\eta_{\bar{R}}(x)}} - 1\right)\right), \\ \log_r \left(1 + \prod_{k=1}^n \left(r^{1-\nu_{\bar{R}} e^{i2\pi\nu_{\bar{R}}(x)}} - 1\right)\right) \end{pmatrix}$$

$$= \left(\mu_{\bar{R}} e^{i2\pi\mu_{\bar{R}}(x)}, \eta_{\bar{R}} e^{i2\pi\eta_{\bar{R}}(x)}, v_{\bar{R}} e^{i2\pi v_{\bar{R}}(x)}\right) = \bar{R}.$$

Hence, it completes the proof. □

Theorem 3.8. *(Boundedness). Let* $\bar{R}_k = \left(\mu_{\bar{R}_k} e^{i2\pi\mu_{\bar{R}_k}(x)}, \eta_{\bar{R}_k} e^{i2\pi\eta_{\bar{R}_k}(x)}, \nu_{\bar{R}_k} e^{i2\pi\nu_{\bar{R}_k}(x)}\right)$ $(k = 1, 2, \ldots, n)$ *be CPFNs'collection. Let* $\bar{R}^- = \min\left\{\bar{R}_1, \bar{R}_2, \ldots, \bar{R}_n\right\}$, *and* $\bar{R}^+ = \max\left\{\bar{R}_1, \bar{R}_2, \ldots, \bar{R}_n\right\}$. *Then,*

$$\bar{R}^- \leq CPFFWA\left(\bar{R}_1, \bar{R}_2, \ldots, \bar{R}_n\right) \leq \bar{R}^+.$$

Proof. Let $\bar{R}_k = \left(\mu_{\bar{R}_k} e^{i2\pi\mu_{\bar{R}_k}(x)}, \eta_{\bar{R}_k} e^{i2\pi\eta_{\bar{R}_k}(x)}, \nu_{\bar{R}_k} e^{i2\pi\nu_{\bar{R}_k}(x)}\right)$ $(k = 1, 2, \ldots, n$ is a collection of CPFN. Let

$$\bar{R}^- = \min\left\{\bar{R}_1, \bar{R}_2, \ldots, \bar{R}_n\right\} = \left(\mu^-{}_{\bar{R}_k} e^{i2\pi\mu_{\bar{R}_k}(x)}, \eta^-{}_{\bar{R}_k} e^{i2\pi\eta_{\bar{R}_k}(x)}, \nu^-{}_{\bar{R}_k} e^{i2\pi\nu_{\bar{R}_k}(x)}\right),$$

and

$$\bar{R}^+ = \max\left\{\bar{R}_1, \bar{R}_2, \ldots, \bar{R}_n\right\} = \left(\mu^+{}_{\bar{R}_k} e^{i2\pi\mu_{\bar{R}_k}(x)}, \eta^+{}_{\bar{R}_k} e^{i2\pi\eta_{\bar{R}_k}(x)}, \nu^+{}_{\bar{R}_k} e^{i2\pi\nu_{\bar{R}_k}(x)}\right).$$

Then, we have,

$$\begin{aligned} \mu^- \bar{R}_k e^{i2\pi\mu_{\bar{R}_k}(x)} &= \min_k \left\{\mu_{\bar{R}_k} e^{i2\pi\mu_{\bar{R}_k}(x)}\right\}, \eta^-{}_{\bar{R}_k} e^{i2\pi\eta_{\bar{R}_k}(x)} \\ &= \max_k \left\{\eta_{\bar{R}_k} e^{i2\pi\eta_{\bar{R}_k}(x)}\right\}, \nu^-_{\bar{R}_k} e^{i2\pi\nu_{\bar{R}_k}(x)} = \max_k \left\{\nu_{\bar{R}_k} e^{i2\pi\nu_{\bar{R}_k}(x)}\right\} \\ \mu^+{}_{\bar{R}_k} e^{i2\pi\mu_{\bar{R}_k}(x)} &= \max_k \left\{\mu_{\bar{R}_k} e^{i2\pi\mu_{\bar{R}_k}(x)}\right\}, \eta^+{}_{\bar{R}_k} e^{i2\pi\eta_{\bar{R}_k}(x)} \\ &= \min_k \left\{\eta_{\bar{R}_k} e^{i2\pi\eta_{\bar{R}_k}(x)}\right\}, \nu^+{}_{\bar{R}_k} e^{i2\pi\nu_{\bar{R}_k}(x)} = \min_k \left\{\nu_{\bar{R}_k} e^{i2\pi\nu_{\bar{R}_k}(x)}\right\}. \end{aligned}$$

Now,

$$1-\log_r\left(1+\prod_{k=1}^{n}\left(r^{1-\left(\mu^-{}_{\bar{R}_k}e^{i2\pi\mu_{\bar{R}_k}(x)}\right)}-1\right)^{w_k}\right)$$
$$\leq 1-\log_r\left(1+\prod_{k=1}^{n}\left(r^{1-\mu_{\bar{R}_k}e^{i2\pi\mu_{\bar{R}_k}(x)}}-1\right)^{w_k}\right)$$
$$\leq 1-\log_r\left(1+\prod_{k=1}^{n}\left(r^{1-\left(\mu^+{}_{\bar{R}_k}e^{i2\pi\mu_{\bar{R}_k}(x)}\right)}-1\right)^{w_k}\right).$$
$$\log_r\left(1+\prod_{k=1}^{n}\left(r^{\left(\eta^+{}_{\bar{R}_k}e^{i2\pi\eta_{\bar{R}_k}(x)}\right)}-1\right)^{w_k}\right)\bigg)$$
$$\leq \log_r\left(1+\prod_{k=1}^{n}\left(r^{\eta_{\bar{R}_k}e^{i2\pi\eta_{\bar{R}_k}(x)}-1}\right)^{w_k}\right)$$
$$\leq \log_r\left(1+\prod_{k=1}^{n}\left(r^{\left(\eta^-_{\bar{R}_k}e^{i2\pi\eta_{\bar{R}_k}(x)}-1\right)}\right)^{w_k}\right).$$

and

$$\log_r\left(1+\prod_{k=1}^{n}\left(r^{\left(\nu^+{}_{\bar{R}_k}e^{i2\pi\nu_{\bar{R}_k}(x)}\right)}-1\right)^{w_k}\right)\bigg)$$
$$\leq \log_r\left(1+\prod_{k=1}^{n}\left(r^{\nu_{\bar{R}_k}e^{i2\pi\nu_{\bar{R}_k}(x)}}-1\right)^{w_k}\right)\bigg)$$
$$\leq \log_r\left(1+\prod_{k=1}^{n}\left(r^{\left(\nu^-{}_{\bar{R}_k}^{i2\pi\nu_{\bar{R}_k}(x)}\right)}-1\right)^{k}\right)\Bigg).$$

Therefore,

$$\bar{R}^- \leq CPFFWA\left(\bar{R}_1,\bar{R}_2,\ldots,\bar{R}_n\right)\leq \bar{R}^+.$$

□

Theorem 3.9. *(Monotonicity) Let the two sets $\bar{R}_k$ and $\bar{R}'_k(k=1,2,\ldots,n)$ of CPFNs, if for all k $\bar{R}_k\leq\bar{R}'_k$, then $CPFFWA\left(\bar{R}_1,\bar{R}_2,\ldots,\bar{R}_n\right)\leq$ CPFFWA $\left(\bar{R}'_1,\bar{R}'_2,\ldots,\bar{R}'_n\right)$.*

Proof. Since $\bar{R}_k \leq \bar{R}'_k$ for all $k=1,2,\ldots,n$, then, $\mu_{\bar{R}_k}e^{i2\pi\mu_{\bar{R}_k}(x)} \leq \mu'_{\bar{R}_k}e^{i2\pi\mu'\bar{R}_k(x)}, r^{\eta_{\bar{R}_k}}e^{i2\pi\eta_{\bar{R}_k}(x)} \leq \eta'_{\bar{R}_k}e^{i2\pi\eta'_{\bar{R}_k}(x)}$ and $\nu_{\bar{R}_k}e^{i2\pi\nu_{\bar{R}_k}(x)}\geq\nu'_{\bar{R}_k}e^{i2\pi\nu'\bar{R}_k(x)}$ for all $k=1,2,\ldots,n$. Now

$$\left(r^{1-\mu_{\bar{R}_k}e^{i2\pi\mu_{\bar{R}_k}(x)}}-1\right)^{w_k} \geq \left(r^{1-\mu'_{\bar{R}_k}e^{i2\pi\mu'_{\bar{R}_k}(x)}}-1\right)^{w_k}$$
$$\Rightarrow \log_r\left(1+\prod_{k=1}^{n}\left(r^{1-\mu_{\bar{R}_k}e^{i2\pi\mu_{\bar{R}_k}(x)}}-1\right)^{w_k}\right)$$
$$\geq \log_r\left(1+\prod_{i=\mathrm{k}}^{n}\left(r^{1-\mu'\bar{R}_k e^{i2\pi\mu'\bar{R}_k(x)}}-1\right)^{w'_k}\right)$$
$$\Rightarrow 1-\log_r\left(1+\prod_{k=1}^{n}\left(r^{1-\mu_{\bar{R}_k}e^{i2\pi\mu_{\bar{R}_k}(x)}}-1\right)^{w_k}\right)$$
$$\leq 1-\log_r\left(1+\prod_{k=1}^{n}\left(r^{1-\mu'_{\bar{R}_k}e^{i2\pi\mu'\bar{R}_k(x)}}-1\right)^{w_k}\right).$$

Similarly, it can be shown that

$$\log_r\left(1+\prod_{k=1}^{n}\left(r^{\eta_{\bar{R}_k}e^{i2\pi\eta_{\bar{R}_k}(x)}}-1\right)^{w_k}\right)$$
$$\leq \log_r\left(1+\prod_{k=1}^{n}\left(r^{\eta'_{\bar{R}_k}e^{i2\pi\eta'_{\bar{R}_k}(x)}}-1\right)^{w_k}\right).$$

And

$$\geq \log_r\left(1+\prod_{k=1}^{n}\left(r^{\nu'\bar{R}_k e^{i2\pi\nu'_{\bar{R}_k}(x)}}-1\right)^{w_i}\right).$$

Therefore,

$$CPFFWA\left(\bar{R}_1,\bar{R}_2,\ldots,\bar{R}_n\right) \leq CPFFWA\left(\bar{R}'_1,\bar{R}'_2,\ldots,\bar{R}'_n\right).$$

Now, the $CPFFOWA$ operator will be introduced.

□

Definition 3.10. *Let* $\bar{R}_k = \left(\mu_{\bar{R}_k}e^{i2\pi\mu_{\bar{R}_k}(x)}, \eta_{\bar{R}_k}e^{i2\pi\eta_{\bar{R}_k}(x)}, \nu_{\bar{R}_k}e^{i2\pi\nu_{\bar{R}_k}(x)}\right)(k=1,2,\ldots,n)$ *be a collection of CPFNs. The function* $\bar{R}^n \to \bar{R}$ *such that,*

$$CPFFOWA\left(\bar{R}_1,\bar{R}_2,\ldots\bar{R}_n\right) = \bigoplus_{k=1}^{n} w_k\bar{R}_{\bar{R}(k)},$$

is defined as the $CPFFOWA$ *operator of dimension* n *with weight vector* $w=(w_1,w_2,\ldots,w_n)^t$ *of* $\bar{R}_k(k=1,2,\ldots,n); w_k\in[0,1]$ *and* $\sum_{k=1}^{n} w_k = 1, (\bar{R}(1),\bar{R}(2),\ldots,\bar{R}(n))$ *represents permutation of* $(k=1,2,\ldots,n)$*, such that for every* $k=1,2,\ldots,n, \bar{R}_{\bar{R}(k-1)} \geq \bar{R}_{\bar{R}(k)}$*.*

By using the above definition, we get the following theorem.

Theorem 3.11. *Let* $\bar{R}_k = \left(\mu_{\bar{R}_k}e^{i2\pi\mu_{\bar{R}_k}(x)}, \eta_{\bar{R}_k}e^{i2\pi\eta_{\bar{R}_k}(x)}, \nu_{\bar{R}_k}e^{i2\pi\nu_{\bar{R}_k}(x)}\right)(k=1,2,\ldots,n)$ *be a collection of CPFNs. Then, a function* $\bar{R}^n \to \bar{R}$ *containing a weight vector* $w=(w_1,w_2,\ldots,w_n)^t; w_k\in[0,1]$ *and* $\sum_{k=1}^{n} w_k = 1$*.*

So,

$$CPFFOWA\left(\bar{R}_1, \bar{R}_2, \ldots, \bar{R}_n\right) = \oplus_{\mathrm{w}_{k=1}}^{n} w_k \bar{R}_{\bar{R}(k)}$$

$$= \left(\begin{array}{c} 1-\log_r\left(1+\prod_{k=1}^{n}\left(r^{1-\mu_{\bar{R}_{\bar{R}(k)}}e^{i2\pi\mu_{\bar{R}_k}(x)}}-1\right)^{w_k}\right), \\ \log_r\left(1+\prod_{k=1}^{n}\left(r^{\eta_{\bar{R}(k)}e^{i2\pi\eta_{\bar{R}_k}(x)}}-1\right)^{w_k}\right), \\ \log_r\left(1+\prod_{k=1}^{n}\left(r^{\nu_{\bar{R}_k}e^{i2\pi\nu_{\bar{R}_k}(x)}}-1\right)^{w_k}\right)\right) \end{array} \right),$$

is defined as the CPFFOWA operator of dimension n *with* $(\bar{R}(1), \bar{R}(2), \ldots, \bar{R}(n))$ *represents permutation of* $(k = 1, 2, \ldots, n)$*, such that for every* $k = 1, 2, \ldots, n, \bar{R}_{\bar{R}(k-1)} \geq \bar{R}_{\bar{R}(k)}$.

Proof. By using the $CPFFOWA$ operators, the following properties can be easily proved. □

Theorem 3.12. *(Idempotent). Let* $\bar{R}_k = \left(\mu_{\bar{R}_k}e^{i2\pi\mu_{\bar{R}_k}(x)}, \eta_{\bar{R}_k}e^{i2\pi\eta_{\bar{R}_k}(x)}, \nu_{\bar{R}_k}e^{i2\pi\nu_{\bar{R}_k}(x)}\right)(k = 1, 2, \ldots, n)$ *be a collection of identical CPFNs, i.e.,* $\bar{R}_i = \bar{R}$ *for all* k*. Then,* $CPFFOWA\left(\bar{R}_1, \bar{R}_2, \ldots, \bar{R}_n\right) = \bar{R}$.

Theorem 3.13. *(Boundedness). Let* $\bar{R}_k = \left(\mu_{\bar{R}_k}e^{i2\pi\mu_{\bar{R}_k}(x)}, \eta_{\bar{R}_k}e^{i2\pi\eta_{\bar{R}_k}(x)}, \nu_{\bar{R}_k}e^{i2\pi\nu_{\bar{R}_k}(x)}\right)(k = 1, 2, \ldots, n)$ *be a collection of CPFNs. Let* $\bar{R}^- = \min\left\{\bar{R}_1, \bar{R}_2, \ldots, \bar{R}_n\right\}$ *and* $\bar{R}^+ = \max\left\{\bar{R}_1, \bar{R}_2, \ldots, \bar{R}_n\right\}$*. Then,* $\bar{R}^- \leq CPFFOWA\left(\bar{R}_1, \bar{R}_2, \ldots, \bar{R}_n\right) \leq \bar{R}^+$.

Theorem 3.14. *(Monotonicity). Let* $\bar{R}_k$ *and* $\bar{R}'_k(k = 1, 2, \ldots, n)$ *be any two sets of CPFNs, if* $\bar{R}_k \leq \bar{R}'_k$ *for all* k*. Then,* $CPFFOWA\left(\bar{R}_1, \bar{R}_2, \ldots, \bar{R}_n\right) \leq CPFFOWA\left(\bar{R}'_1, \bar{R}'_2, \ldots, \bar{R}'_n\right)$.

Theorem 3.15. *(Commutativity). Let* $\bar{R}_k$ *and* $\bar{R}'_k(k = 1, 2, \ldots, n)$ *be any two sets of CPFNs, then CPF-FOWA* $\left(\bar{R}_1, \bar{R}_2, \ldots, \bar{R}_n\right) = CPFFOWA\left(\bar{R}'_1, \bar{R}'_2, \ldots, \bar{R}'_n\right)$*, where* $\bar{R}'_k$ *denotes the permutation of* $\bar{R}_k(k = 1, 2, \ldots, n)$.

The weights associated with the CPFFWA operator in Definition 3.4 are in the most basic form of a CPF value, but the weights associated with the CPFFOWG operator in Definition 3.10 are not so. Which tells the weights associated with the CPFFWAO as well as the CPFFOWAO convey different viewpoints that are conflicting with one another. However, both viewpoints are intended to be similar in a broad sense. Only to overcome such a shortcoming, we are now introducing the CPFFHA operator.

Definition 3.16. *Let* $\bar{R}_k = \left(\mu_{\bar{R}_k}e^{i2\pi\mu_{\bar{R}_k}(x)}, \eta_{\bar{R}_k}e^{i2\pi\eta_{\bar{R}_k}(x)}, \nu_{\bar{R}_k}e^{i2\pi\nu_{\bar{R}_k}(x)}\right)(k = 1, 2, \ldots, n)$ *is a collection of CPFNs. Then, a function* $\bar{R}^n \to \bar{R}$ *such that,*

$$CPFFHA\left(\bar{R}_1, \bar{R}_2, \ldots, \bar{R}_n\right) = \oplus_{k=1}^{n} \bar{w}_k \dot{\bar{R}}_{\bar{R}(k)}$$

$$\left(\begin{array}{c} 1-\log_r\left(1+\prod_{k=1}^{n}\left(r^{1-\dot{\mu}_{\bar{R}_{\bar{R}(k)}}e^{i2\pi\mu_{\bar{R}_k}(x)}}-1\right)^{\bar{w}_k}\right), \\ \log_r\left(1+\prod_{k=1}^{n}\left(r^{\dot{\eta}_{\bar{R}_{\bar{R}(k)}}e^{i2\pi\eta_{\bar{R}_k}(x)}}-1\right)^{\bar{w}_k}\right), \\ \log_r\left(1+\prod_{k=1}^{n}\left(r^{\dot{\nu}_{\bar{R}_{\bar{R}(k)}}e^{i2\pi\nu_{\bar{R}_k}(x)}}-1\right)^{\bar{w}_k}\right)\right) \end{array} \right),$$

is defined as the CPFFHA operator of dimension n *with* $\bar{w} = (\bar{w}_1, \bar{w}_2, \ldots, \bar{w}_n)^t$ *as aggregation associated weight vector,* $\sum_{k=1}^{n} \bar{w}_k = 1, w = (w_1, w_2, \ldots, w_n)^t$ *as weight vector of* $\bar{R}_k(k = 1, 2, \ldots, n), w_k \in [0, 1]$ *and*

$\sum_{k=1}^{n} w_k = 1$. $\bar{R}_{\bar{R}(k)}$ *represents* k^{th} *weighted greatest CPF value for* $\dot{p}_k\,(\dot{p}_k = nw_k p_k, k = 1, 2, \ldots, n)$, *and* n *being the balancing coefficient.*

Remark 3.17. *When* $w = \left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)^t$, *then* $\bar{R}_k = n \times \frac{1}{n} \times \bar{R}_k = \bar{R}_k$ *for* $k = 1, 2, \ldots, n$. *When this happens the CPFFHA operator becomes the CPFFOWA operator. CPFFHA operator becomes the CPFFWA operator, if* $\bar{w} = \left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)^t$. *As a result, the CPFFWA and the CPFFOWA operators are particular varieties of the CPFFHA operators. Therefore, the CPFFHA operator, which indicates the magnitude of the stated disagreements and their structured situations, appears to be a generalization of both the CPFFWA and the CPFFOWA operators.*

3.2 Complex Picture Fuzzy Frank Geometric Aggregation Operators

Definition 3.18. *Let* $\bar{R}_k = \left(\mu_{\bar{R}_k} e^{i2\pi\mu_{\bar{R}_k}(x)}, \eta_{\bar{R}_k} e^{i2\pi\eta_{\bar{R}_k}(x)}, \nu_{\bar{R}_k} e^{i2\pi\nu_{\bar{R}_k}(x)}\right)(k = 1, 2, \ldots, n)$ *be a collection of CPFNs. Then a function* $p^n \to p$ *such that*

$$CPFFWG\,(R_1, R_2, \ldots, R_n) = \bigotimes_{k=1}^{n} (R_k)^{w_k},$$

is defined as the CPFFWG operator with $w = (w_k, w_k, \ldots, w_k)^t$ *as the weight vector of* $R_k (k = 1, 2, \ldots, n)$, $w_k \in [0, 1]$ *and* $\sum_{k=1}^{n} w_k = 1$.

As a result, the following consequential theorem is obtained.

Theorem 3.19. *Let* $\bar{R}_k = \left(\mu_{\bar{R}_k} e^{i2\pi\mu_{\bar{R}_k}(x)}, \eta_{\bar{R}_k} e^{i2\pi\eta_{\bar{R}_k}(x)}, \nu_{\bar{R}_k} e^{i2\pi\nu_{\bar{R}_k}(x)}\right)(k = 1, 2, \ldots, n)$ *be a collection of CPFNs, then by using CPFFWG operator, their aggregated value is also a CPFN, and*

$$CPFFWG\,(R_1, R_2, \ldots, R_n) = \bigotimes_{k=1}^{n} (R_k)^{w_k}$$

$$= \begin{pmatrix} \log_r \left(1 + \prod_{k=1}^{n} \left(r^{\mu_{R_k} e^{i2\pi\mu_{\bar{R}_k}(x)}} - 1\right)^{w_k}\right), \\ 1 - \log_r \left(1 + \prod_{k=1}^{n} \left(r^{1-\eta_{R_k} e^{i2\pi\eta_{\bar{R}_k}(x)}} - 1\right)^{w_k}\right), \\ 1 - \log_r \left(1 + \prod_{k=1}^{n} \left(r^{1-\nu_{R_k} e^{i2\pi\nu_{\bar{R}_k}(x)}} - 1\right)^{w_k}\right) \end{pmatrix}.$$

Proof. This theorem can be proved by using the method of proof of Theorem 3.5. □

Example 3.20. For $\bar{R}_1 = \left(0.5e^{i2\pi(0.5)}, 0.3e^{i2\pi(0.1)}, 0.1e^{i2\pi(0.4)}\right), \bar{R}_2 = \left(0.1e^{i2\pi(0.4)}, 0.5e^{i2\pi(0.4)}, 0.4e^{i2\pi(0.2)}\right), \bar{R}_3 = \left(0.1e^{i2\pi(0.1)}, 0.1e^{i2\pi(0.2)}, 0.7e^{i2\pi(0.5)}\right), \bar{R}_4 = \left(0.6e^{i2\pi(0.2)}, 0.1e^{i2\pi(0.2)}, 0.3e^{i2\pi(0.3)}\right)$ with weights $= (0.2, 0.3, 0.1, 0.4)$ and $r = 2$, step by step working of the CPFFWG operator is given as follows:

$$CPFFWG\left(\bar{R}_1, \bar{R}_2, \ldots, \bar{R}_n\right) = \begin{pmatrix} \log_r \left(1 + \prod_{k=1}^{n} \left(r^{\mu_{R_k} e^{i2\pi\mu_{\bar{R}_k}(x)}} - 1\right)^{w_k}\right), \\ 1 - \log_r \left(1 + \prod_{k=1}^{n} \left(r^{1-\eta_{R_k} e^{i2\pi\eta_{\bar{R}_k}(x)}} - 1\right)^{w_k}\right) \\ 1 - \log_r \left(1 + \prod_{k=1}^{n} \left(r^{1-\upsilon_{R_k} e^{i2\pi\upsilon_{\bar{R}_k}(x)}} - 1\right)^{w_k} \end{pmatrix}$$

$$\begin{pmatrix} \log_2\begin{pmatrix} 1+\begin{pmatrix} 2^{(0.5)e^{i2\pi(0.5)}} \\ -1 \end{pmatrix}^{0.2} + \begin{pmatrix} 2^{(0.1)e^{i2\pi(0.4)}} \\ -1 \end{pmatrix}^{0.3} \\ \begin{pmatrix} 2^{(0.1)e^{i2\pi(0.1)}} \\ -1 \end{pmatrix}^{0.1} + \begin{pmatrix} 2^{(0.6)e^{i2\pi(0.2)}} \\ -1 \end{pmatrix}^{0.4} \end{pmatrix} \\ 1-\log_2\begin{pmatrix} 1+\begin{pmatrix} 2^{(0.3)e^{i2\pi(0.1)}} \\ -1 \end{pmatrix}^{0.2} + \begin{pmatrix} 2^{(0.5)e^{i2\pi(0.4)}} \\ -1 \end{pmatrix}^{0.3} \\ \begin{pmatrix} 2^{(0.1)e^{i2\pi(0.2)}} \\ -1 \end{pmatrix}^{0.1} + \begin{pmatrix} 2^{(0.1)e^{i2\pi(0.2)}} \\ -1 \end{pmatrix}^{0.4} \end{pmatrix} \\ 1-\log_2\begin{pmatrix} 1+\begin{pmatrix} 2^{(0.1)e^{i2\pi(0.4)}} \\ -1 \end{pmatrix}^{0.2} + \begin{pmatrix} 2^{(0.4)e^{i2\pi(0.2)}} \\ -1 \end{pmatrix}^{0.3} \\ \begin{pmatrix} 2^{(0.7)e^{i2\pi(0.5)}} \\ -1 \end{pmatrix}^{0.1} + \begin{pmatrix} 2^{(0.3)e^{i2\pi(0.3)}} \\ -1 \end{pmatrix}^{0.4} \end{pmatrix} \end{pmatrix}$$

$$= \left(0.239e^{i2\pi(0.263)}, 0.272e^{i2\pi(0.221)}, 0.425e^{i2\pi(0.357)}\right).$$

The CPFFWG operator makes it simple to prove the following properties:

Theorem 3.21. *(Idempotent). Let* $\bar{R}_k = \left(\mu_{\bar{R}_k} e^{i2\pi\mu_{\bar{R}_k}(x)}, \eta_{\bar{R}_k} e^{i2\pi\eta_{\bar{R}_k}(x)}, \nu_{\bar{R}_k} e^{i2\pi\nu_{\bar{R}_k}(x)}\right) (k = 1, 2, \ldots, n)$ *be identical CPFNs' collection, i.e., for every* $k, R_k = R$. *So,* $CPFFWG\,(R_1, R_2, \ldots, R_n) = R$.

Theorem 3.22. *(Boundedness). Let* $\bar{R}_k = \left(\mu_{\bar{R}_k} e^{i2\pi\mu_{\bar{R}_k}(x)}, \eta_{\bar{R}_k} e^{i2\pi\eta_{\bar{R}_k}(x)}, \nu_{\bar{R}_k} e^{i2\pi\nu_{\bar{R}_k}(x)}\right) (k = 1, 2, \ldots, n)$ *be CPFNs' collection. Take* $R^- = \min\{R_1, R_2, \ldots, R_n\}$ *an* $R^+ = \max\{R_1, R_2, \ldots, R_n\}$. *Then* $R^- \leq CPFFWG(R_1, R_2, \ldots, R_n) \leq R^+$.

Theorem 3.23. *(Monotonicity Property). Let the sets* R_i *and* $R'_k(k = 1, 2, \ldots, n)$ *of CPFNs, if for every* $k, R_i \leq R'_k$, *we have* $CPFFWG\,(R_1, R_2, \ldots, R_n) \leq CPFFWG\,(R'_1, R'_2, \ldots, R'_n)$.

At this point, CPFFOWG operator has been introduced.

Definition 3.24. *Let* $\bar{R}_k = \left(\mu_{\bar{R}_k} e^{i2\pi\mu_{\bar{R}_k}(x)}, \eta_{\bar{R}_k} e^{i2\pi\eta_{\bar{R}_k}(x)}, \nu_{\bar{R}_k} e^{i2\pi\nu_{\bar{R}_k}(x)}\right) (k = 1, 2, \ldots, n)$ *be CPFNs' collection. The n-dimensional CPFFOWG operator takes the form of a function* $R^n \to R$ *such that,*

$$CPFFOWG\,(R_1, R_2, \ldots R_n) = \bigotimes_{k=1}^{n} \left(R_{\rho(k)}\right)^{w_k},$$

where $w = (w_1, w_2, \ldots, w_n)^t$ *is a representation of weight vector for* $R_k(k = 1, 2, \ldots, n), w_k \in [0, 1]$; $\sum_{k=1}^{n} w_k = 1$. *Moreover,* $(\rho(1), \rho(2), \ldots, \rho(n))$ *appear to be representation of permutation of* $(k = 1, 2, \ldots, n)$, *such that for every* $k = 1, 2, \ldots, n$, $R_{\rho(k-1)} \geq R_{\rho(k)}$.

On the basis of Frank product operation on CPFN utilizing CPFFOWG operators, the following theorem is constructed.

Theorem 3.25. *Let* $\bar{R}_k = \left(\mu_{\bar{R}_k} e^{i2\pi\mu_{\bar{R}_k}(x)}, \eta_{\bar{R}_k} e^{i2\pi\eta_{\bar{R}_k}(x)}, \nu_{\bar{R}_k} e^{i2\pi\nu_{\bar{R}_k}(x)}\right) (k = 1, 2, \ldots, n)$ *is CPFNs' collection. The n-dimensional CPFFOWG operator takes the form of a function* $R^n \to R$. *So,*

$$CPFFOWG\,(R_1, R_2, \ldots R_n) = \otimes_{k=1}^{n} \left(R_{\rho(k)}\right)^{w_k}$$

$$= \begin{pmatrix} \log_r \left(1+\prod_{k=1}^{n}\left(r^{\mu_{R_{\rho(k)}} e^{i2\pi\mu_{R_k}(x)}}-1\right)^{w_k}\right), \\ 1-\log_r\left(1+\prod_{k=1}^{n}\left(r^{1-\eta_{R_{\rho(k)}}}-1\right)^{w_k}\right), \\ 1-\log_r\left(1+\prod_{k=1}^{n}\left(r^{1-\nu_{R_{\rho(k)}}}-1\right)^{w_k}\right) \end{pmatrix}.$$

Here $w=(w_1,w_2,\ldots,w_n)^t$ is weight vector satisfying $w_k \in [0,1]; \sum_{k=1}^{n} w_k = 1$. Moreover, $(\rho(1),\rho(2),\ldots,\rho(n))$ appear to be representation of permutation of $(k=1,2,\ldots,n)$, such that for every $k=1,2,\ldots,n, R_{\rho(k-1)} \geq R_{\rho(k)}$.

The CPFFOWG operator can be used to investigate the properties provided below.

Theorem 3.26. *(Idempotent). Let $\bar{R}_k = \left(\mu_{\bar{R}_k} e^{i2\pi\mu_{\bar{R}_k}(x)}, \eta_{\bar{R}_k} e^{i2\pi\eta_{\bar{R}_k}(x)}, \nu_{\bar{R}_k} e^{i2\pi\nu_{\bar{R}_k}(x)}\right)(k=1,2,\ldots,n)$ be a collection of identical CPFNs, i.e., $\bar{R}_k = \bar{R}$ for all k. Then, $CPFFOWG\left(\bar{R}_1,\bar{R}_2,\ldots,\bar{R}_n\right)=\bar{R}$.*

Theorem 3.27. *(Boundedness). Let $\bar{R}_k = \left(\mu_{\bar{R}_k} e^{i2\pi\mu_{\bar{R}_k}(x)}, \eta_{\bar{R}_k} e^{i2\pi\eta_{\bar{R}_k}(x)}, \nu_{\bar{R}_k} e^{i2\pi\nu_{\bar{R}_k}(x)}\right)(k=1,2,\ldots,n)$ be a collection of CPFN. Let $\bar{R}^- = \min\{\bar{R}_1,\bar{R}_2,\ldots,\bar{R}_n\}$ an $\bar{R}^+ = \max\{\bar{R}_1,\bar{R}_2,\ldots,\bar{R}_n\}$. Then $\bar{R}^- \leq CPFFOWG(\bar{R}_1,\bar{R}_2,\ldots,\bar{R}_n) \leq \bar{R}^+$.*

Theorem 3.28. *(Monotonicity). Let the sets $\bar{R}_k$ and $\bar{R}'_k(k=1,2,\ldots,n)$ of CPFNs, if for every $k, \bar{R}_k \leq \bar{R}'_k$, then $CFFFOWG\left(\bar{R}_1,\bar{R}_2,\ldots,\bar{R}_n\right) \leq CPFFOWAG\left(\bar{R}'_1,\bar{R}'_2,\ldots,\bar{R}'_n\right)$.*

Theorem 3.29. *(Commutativity). Let the sets $\bar{R}_k$ and $\bar{R}'_k(k=1,2,\ldots,n)$ of CPFNs, then CPFFOWG $\left(\bar{R}_1,\bar{R}_2,\ldots,\bar{R}_n\right) = CPFFOWG\left(\bar{R}'_1,\bar{R}'_2,\ldots,\bar{R}'_n\right)$, where $\bar{R}'_k$ denotes the permutation of $\bar{R}_k(k=1,2,\ldots,n)$.*

The weights associated with the CPFFWG operator in Definition 3.18 are in the most basic form of a PF value, but the weights associated with CPFFOWG operator in Definition 3.24 are in the real form of the ordered locations of CPF values. The weights given in CPFFWG and CPFFOWG operators convey different viewpoints that are conflicting with one another in this way. However, both viewpoints are intended to be similar in a broad sense. Only to overcome such a shortcoming, we are now introducing CPFFHG operator.

Definition 3.30. *Let $\bar{R}_k = \left(\mu_{\bar{R}_k} e^{i2\pi\mu_{\bar{R}_k}(x)}, \eta_{\bar{R}_k} e^{i2\pi\eta_{\bar{R}_k}(x)}, \nu_{\bar{R}_k} e^{i2\pi\nu_{\bar{R}_k}(x)}\right)$ (where k varies from 1 to n) be the CPFNs' collection. Then, a function $\bar{R}^n \to \bar{R}$ such that,*

$$CPFFHG\left(\bar{R}_1,\bar{R}_2,\ldots,\bar{R}_n\right) = \bigotimes_{k=1}^{n}\left(\bar{R}_{\rho(k)}\right)^{\bar{w}_k}$$

$$\begin{pmatrix} \log_r\left(1+\prod_{k=1}^{n}\left(r^{1-\dot{\mu}_{\bar{R}_{\rho(k)}} e^{i2\pi\mu_{\rho_k}(x)}}-1\right)^{\bar{w}_k}\right), \\ 1-\log_r\left(1+\prod_{k=1}^{n}\left(r^{1-\dot{\eta}_{\bar{R}_{\rho(k)}} e^{i2\pi\eta_{\rho_k}(x)}}-1\right)^{\bar{w}_k}\right), \\ 1-\log_r\left(1+\prod_{k=1}^{n}\left(r^{1-\nu_{\bar{R}_{\rho(k)}}} e^{i2\pi\nu_{\rho_k}(x)}-1\right)^{\bar{w}_k}\right) \end{pmatrix},$$

is defined as the CPFFHG operator of dimension n with $\bar{w} = (\bar{w}_1,\bar{w}_2,\ldots,\bar{w}_n)^t$ as aggregation associated weight vector, $\sum_{k=1}^{n}\bar{w}_k = 1, w=(w_1,w_2,\ldots,w_n)^t$ as weight vector of $\bar{R}_k(k=1,2,\ldots,n), w_k \in [0,1]$ and $\sum_{k=1}^{n} w_k = 1$. $\bar{R}_{\bar{R}(k)}$ represents k^{th} weighted greatest CPF value for $\dot{p}_k\,(\dot{p}_k = nw_kp_k, k=1,2,\ldots,n)$, and n being the balancing coefficient.

Remark 3.31. *When* $w = \left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)^t$*, then* $\bar{R}_k = n \times \frac{1}{n} \times \bar{R}_k = \bar{R}_k$ *for* $k = 1, 2, \ldots, n$*. When this happens, the CPFFHG operator becomes the CPFFOWG operator. The CPFFHG operator becomes the CPFFWG operator, if* $\bar{w} = \left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)^t$*. As a result, the CPFFWG and the CPFFOWG operators are particular varieties of the CPFFHG operators. Therefore, the CPFFHG operator, which indicates the magnitude of the stated disagreements and their structured situations, appears to be a generalization of both the CPFFWG and the CPFFOWG operators.*

4 Model for the MADM Using Complex Picture Fuzzy Data

The purpose of this part is to discuss an effective strategy for solving the MADM process, as well as a method that may be used to identify the attribute weights that are necessary.

4.1 An Overview of the DM Issue

An innovative method to MADM problems has been proposed in this part, in which we will use CPF information along with manipulation of the CPFFWA and CPFFWG operators. For this purpose, let $P = \{P_1, P_2, \ldots, P_m\}$ represent a discrete collection of m alternatives to be chosen and $Q = \{Q_1, Q_2, \ldots, Q_n\}$ represent an order of attributes to be evaluated. Also, the weight vector is $w = \{w_1, w_2, \ldots, w_n\}$ related to attributes $H_j (j = 1, 2, \ldots, n)$ where $w_k (k = 1, 2, 3, \ldots, n) \in \mathrm{R}$ such that $w_k > 0$; $\sum_{k=1}^{n} w_k = 1$. We let $P = (\pi_{ij})_{m \times n} = ((\mu_{ij}, \eta_{ij}, v_{ij}))_{m \times n}$ as the CPF decision matrix, where π_{ij} is the possible value for which the alternative F_i satisfies the attribute H_j with the condition $\mu_{ij} + \eta_{ij} + v_{ij} \leq 1$ and $\mu_{ij}, \eta_{ij}, v_{ij} \in [0, 1]$. The illustration of this algorithm has been given following:

4.2 Determination of the Attribute Weights

During the decision-making process, DM challenges inevitably include numerous attributes. They don't have to give each other the same amount of weight. Take, for example, a decision where, in one case, the product's price is more important than its functionality; in another, the product's functionality may be more important than its price, reliability, or other considerations. This means that while solving a problem, various attributes play a role, each with its relevance. For DM to be very effective, selecting the appropriate attribute weights is crucial. Following is the method that can be taken into account for computing the attributes' weights accurately.

For a CPFN, $\bar{R} = \left(\mu_{\bar{R}}(x) e^{i2\pi\mu_{\bar{R}}(x)}, \eta_{\bar{R}} e^{i2\pi\eta_{\bar{R}}(x)}, v_{\bar{R}} e^{i2\pi v_{\bar{R}}(x)}\right)$, its hesitation degree has been given as:

$$\tau(p) = 2 - \left(\mu_{\bar{R}} e^{i2\pi\mu_{\bar{R}}(x)} + \nu_{\bar{R}} e^{i2\pi\nu_{\bar{R}}(x)}\right).$$

Cases involving DM always require an algorithm which has a lesser hesitancy degree due to the fact that attribute plays an important role during making process. Simply, we can say that an algorithm with lesser hesitancy degree would be more accurate than the one having a greater hesitancy degree. As a result the object would be more important when the hesitancy degree is lesser as compared to when hesitancy is greater. Keeping this in consideration, the following hesitancy matrix $\mathcal{R}$ has been constructed for the given alternatives

$$\mathcal{R} = \begin{pmatrix} \tau_{\bar{R}_{11}} & \tau_{\bar{R}_{12}} & \cdots & \tau_{\bar{R}_{1n}} \\ \tau_{\bar{R}_{21}} & \tau_{\bar{R}_{22}} & \cdots & \tau_{\bar{R}_{2n}} \\ \vdots & \vdots & \ddots & \vdots \\ \tau_{\bar{R}_{m1}} & \tau_{\bar{R}_{m2}} & \cdots & \tau_{\bar{R}_{mn}} \end{pmatrix}.$$

Each $\tau_{\bar{R}_{pq}}$ has been calculated by using hesitation function. Hence, the weight vector $\hat{W}_j$ is determined as

$$\hat{\mathrm{W}}_j = \frac{2-\left(\frac{1}{m}\sum_{i=1}^{m}\tau_{\bar{R}_{ij}}\right)}{\sum_{j=1}^{n}\left(2-\left(\frac{1}{m}\sum_{i=1}^{m}\tau_{\bar{R}_{ij}}\right)\right)}.$$

Algorithm

Following is a presentation of the proposed MADM problem using CPF data related to the proposed CPFFWA and CPFFWG operators:

Step I: Construction of the CPF decision matrix $P=(\pi_{ij})_{m\times n}=((\mu_{ij},\eta_{ij},v_{ij}))_{m\times n}$.

Step II: Transformation of the matrix $P=(\pi_{ij})_{m\times n}=((\mu_{ij},\eta_{ij},v_{ij}))_{m\times n}$ into a normalize PF matrix $P'=\left(\pi'_{ij}\right)_{m\times n}=\left(\left(\mu'_{ij},\eta'_{ij},v'_{ij}\right)\right)_{m\times n}$ by Equation (1).

$$\pi'_{ij}=\begin{cases}(\mu_{ij},\eta_{ij},v_{ij}), & \text{if } H_j \text{ a benefit attribute}\\ (v_{ij},\eta_{ij},\mu_{ij}), & \text{if } H_j \text{ a cost attribute}\end{cases} \tag{1}$$

Step III: Determination of attribute weights of alternatives by using hesitation function, and the following

$$\hat{\mathrm{W}}_j = \frac{2-\left(\frac{1}{m}\sum_{i=1}^{m}\tau_{\bar{R}_{ij}}\right)}{\sum_{j=1}^{n}\left(2-\left(\frac{1}{m}\sum_{i=1}^{m}\tau_{\bar{R}_{ij}}\right)\right)}.$$

Step IV: Calculation of the information σ_k, which is collective, for the alternative A_k with the aid of following equation:

$$\sigma_f = CPFFWA\left(\pi'_{f'}{}', \pi'_{f2_f}, \ldots, J'_{fn}\right)=\bigoplus_{k=1}^{n}\left(w_k\pi_{fk}\right)$$

$$=\left(\begin{array}{c}1-\log_r\left(1+\prod_{k=1}^{n}\left(r^{1-\mu' p_{fk}}-1\right)^{w_k}\right),\log_r\left(1+\prod_{k=1}^{n}\left(r^{\eta' p_{fk}}-1\right)^{w_k}\right),\\ \log_r\left(1+\prod_{k=1}^{n}\left(r^{v' p_{fk}}-1\right)^{w_k}\right)\end{array}\right). \tag{2}$$

And

$$\sigma_f = \mathrm{CPFFWG}\left(\pi_{f1}{}', \pi_{f2}{}', \ldots, \pi'_{fn}\right)=\bigotimes_{k=1}^{n}\left(\gamma_{fk}\right)^{w_k}$$

$$=\left(\begin{array}{c}\log_r\left(1+\prod_{k=1}^{n}\left(r^{\mu' p_{fk}}-1\right)^{w_k}\right),1-\log_r\left(1+\prod_{k=1}^{n}\left(r^{1-\eta' p_{fk}}-1\right)^{w_k}\right),\\ 1-\log_r\left(1+\prod_{k=1}^{n}\left(r^{1-v' p_{fk}}-1\right)^{w_k}\right.\end{array}\right). \tag{3}$$

Step V: Usage of definition 2.3 to calculate the score value for each alternative.

Step VI: The optimal decision is to select F_k if $\Delta\left(\sigma_f\right)=\max_l\left\{\Delta\left(\sigma_l\right)\right\}$.

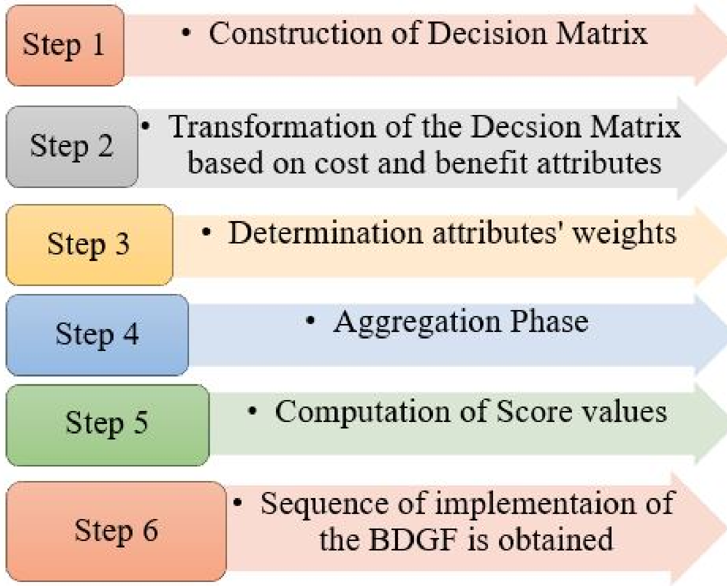

Figure 2: Sequential representation of the algorithm.

5 Numerical Illustration

In order to show the potential assessment of commercialization with the aid of PF data, we are prepared to draw a numerical problem in this part.

A BDGF is of great importance for companies since it regulates the rules under which data flows through different streams and appropriate access is granted to the users. Most of the companies work very much on improving their data assets but fail to understand that a robust data governance framework is needed in which segregation of users' access to sensitive data, access to the data within the organization among stakeholders in hierarchical order is of primary importance, and the responsibilities of the employees are well organized. Failing to have this type of robust framework can lead to uncertain results for the company, hence making the company's survival vulnerable.

Suppose a renowned international organization has come up with the idea of utilizing a handsome amount from its net annual profit in order to improve the company's reputation. Due to increasing risks and malfunctioning in their data assets, the company decides to use the amount in an optimized implementation of its BDGF.

BDGF covers more than one area to be focused on, and also cost attributes and benefit attributes spread uncertainty among decision makers. This brings them to think: In what order the BDGF should be implemented? What area of the BDGF should be focused on very first, followed by the next and so on. Hence an MADM problem arises to be solved for true implementation of the framework so that the company could get the most benefit and make its data assets more profitable. Following are the alternative choices/focus areas of the BDGF which have gained managing board's attention:

1. A_1 : Policy and Standards

2. A_2 : Data Quality

3. A_3 : Data Privacy and Security

4. A_4 : Architecture

Table 2: Decision matrix containing information about alternatives and attributes.

	B_1		$\boldsymbol{B}_2$
A_1	$\left(0.5e^{i2\pi(0.5)}, 0.3e^{i2\pi(0.1)}0.2e^{i2\pi(0.4)}\right)$	A_1	$\left(0.1e^{i2\pi(0.4)}, 0.5e^{i2\pi(0.4)}0.4e^{i2\pi(0.2)}\right)$
A_2	$\left(0.6e^{i2\pi(0.2)}, 0.2e^{i2\pi(0.3)}0.1e^{i2\pi(0.4)}\right)$	A_2	$\left(0.2e^{i2\pi(0.6)}, 0.4e^{i2\pi(0.2)}0.3e^{i2\pi(0.1)}\right)$
A_3	$\left(0.5e^{i2\pi(0.5)}, 0.3e^{i2\pi(0.1)}0.2e^{i2\pi(0.4)}\right)$	A_3	$\left(0.3e^{i2\pi(0.3)}, 0.3e^{i2\pi(0.4)}0.2e^{i2\pi(0.1)}\right)$
A_4	$\left(0.3e^{i2\pi(0.6)}, 0.1e^{i2\pi(0.1)}0.3e^{i2\pi(0.2)}\right)$	A_4	$\left(0.4e^{i2\pi(0.2)}, 0.2e^{i2\pi(0.4)}0.3e^{i2\pi(0.4)}\right)$
A_5	$\left(0.2e^{i2\pi(0.7)}, 0.4e^{i2\pi(0.1)}0.4e^{i2\pi(0.1)}\right)$	A_5	$\left(0.5e^{i2\pi(0.3)}, 0.1e^{i2\pi(0.1)}0.1e^{i2\pi(0.6)}\right)$
	B_3		B_4
A_1	$\left(0.7e^{i2\pi(0.5)}, 0.1e^{i2\pi(0.2)}0.1e^{i2\pi(0.1)}\right)$	A_1	$\left(0.6e^{i2\pi 0.2}, 0.1e^{i2\pi(0.2)}0.3e^{i2\pi(0.3)}\right)$
A_2	$\left(0.4e^{i2\pi(0.1)}, 0.1e^{i2\pi(0.1)}0.1e^{i2\pi(0.7)}\right)$	A_2	$\left(0.2e^{i2\pi 0.3}, 0.6e^{i2\pi(0.3)}0.1e^{i2\pi(0.3)}\right)$
A_3	$\left(0.6e^{i2\pi(0.4)}, 0.1e^{i2\pi(0.5)}0.1e^{i2\pi(0.1)}\right)$	A_3	$0.5e^{i2\pi(0.1)}, 0.2e^{i2\pi 0.1} \left(0.2e^{i2\pi(0.1)}\right)$
A_4	$\left(0.5e^{i2\pi(0.5)}, 0.2e^{i2\pi(0.1)}0.2e^{i2\pi(0.2)}\right)$	A_4	$\left(0.4e^{i2\pi(0.4)}, 0.3e^{i2\pi(0.4)}0.2e^{i2\pi(0.1)}\right)$
A_5	$\left(0.8e^{i2\pi(0.2)}, 0.1e^{i2\pi(0.2)}0.1e^{i2\pi(0.2)}\right)$	A_5	$\left(0.5e^{i2\pi(0.2)}, 0.4e^{i2\pi(0.2)}0.1e^{i2\pi(0.1)}\right)$

5. A_5 : Data Warehouse and Business Intelligence (BI)

As it is difficult to choose amongst the options because they each meet distinct criteria, the problem of making a decision arises. Keeping this in view, the governing board has therefore established the following noteworthy attributes:

1. B_1 : Profit enhancement.

2. B_2 : Benefits of the clients.

3. B_3 : Maintenance cost.

4. B_4 : Mangement support.

Given that each alternative claims to maximize a distinct attribute, making a decision in this situation is challenging. A_1, A_2, A_3, A_4 and A_5 are focus areas of the BDGF.Moreover, B_1, B_2, B_4 are related to benefit attributes, and B_3 is related to cost attributes. Let $P = (\pi_{ij})_{m\times n} = ((\mu_{ij}, \eta_{ij}, v_{ij}))_{m\times n}$, a CPF matrix, be the representation of the alternative A_i with respect to the attributes B_i. Table 2 shows the assessment of the alternatives.

We use the CPFFWA and the CPFFWG operators to create an MADM theory with CPF data for the sake of choosing the best alternative $A_i (i = 1, 2, 3, 4, 5)$ by the following way:

Step I: The CPF decision matrix $P = (\pi_{ij})_{m\times n} = ((\mu_{ij}, \eta_{ij}, v_{ij}))_{m\times n}$ has been created as follows:

Step II: By a careful exploitation of Equation (1), the CPF matrix of table 1 has been normalized as $P' = \left({\pi'_{ij}}'\right)_{m\times n} = \left(\left(\mu'_{ij}, \eta'_{ij}, v'_{ij}\right)\right)_{m\times n}$ given as follows:

Step III: Now we determine the weights of attributes by using the hesitancy function and

$$\hat{\mathrm{W}}_j = \frac{2 - \left(\frac{1}{m}\sum_{i=1}^{m} \tau_{\bar{R}_{ij}}\right)}{\sum_{j=1}^{n}\left(2 - \left(\frac{1}{m}\sum_{i=1}^{m} \tau_{\bar{R}_{ij}}\right)\right)}.$$

The resultant attributes weights are obtained:

$$\hat{W} = (0.268012, 0.259366, 0.242075, 0.230548).$$

Table 3: Transformed decision matrix.

	B_1		B_1
A_1	$\left(0.5e^{i2\pi(0.5)}, 0.3e^{i2\pi(0.1)}, 0.2e^{i2\pi(0.4)}\right)$	A_1	$\left(0.1e^{i2\pi(0.4)}, 0.5e^{i2\pi(0.4)}, 0.4e^{i2\pi(0.2)}\right)$
A_2	$\left(0.6e^{i2\pi(0.2)}, 0.2e^{i2\pi(0.3)}, 0.1e^{i2\pi(0.4)}\right)$	A_2	$\left(0.2e^{i2\pi(0.6)}, 0.4e^{i2\pi(0.2)}, 0.3e^{i2\pi(0.1)}\right)$
A_3	$\left(0.5e^{i2\pi(0.5)}, 0.3e^{i2\pi(0.1)}, 0.2e^{i2\pi(0.4)}\right)$	A_3	$\left(0.3e^{i2\pi(0.3)}, 0.3e^{i2\pi(0.4)}, 0.2e^{i2\pi(0.1)}\right)$
A_4	$\left(0.3e^{i2\pi(0.6)}, 0.1e^{i2\pi(0.1)}, 0.3e^{i2\pi(0.2)}\right)$	A_4	$\left(0.4e^{i2\pi(0.2)}, 0.2e^{i2\pi(0.4)}, 0.3e^{i2\pi(0.4)}\right)$
A_5	$\left(0.2e^{i2\pi(0.7)}, 0.4e^{i2\pi(0.1)}, 0.4e^{i2\pi(0.1)}\right)$	A_5	$\left(0.5e^{i2\pi(0.3)}, 0.1e^{i2\pi(0.1)}, 0.1e^{i2\pi(0.6)}\right)$
	B_3		B_4
A_1	$\left(0.1e^{i2\pi(0.1)}, 0.1e^{i2\pi(0.2)}, 0.7e^{i2\pi(0.5)}\right)$	A_1	$\left(0.6e^{i2\pi(0.2)}, 0.1e^{i2\pi(0.2)}, 0.3e^{i2\pi(0.3)}\right)$
A_2	$\left(0.1e^{i2\pi(0.7)}, 0.1e^{i2\pi(0.1)}, 0.4e^{i2\pi(0.1)}\right)$	A_2	$\left(0.2e^{i2\pi(0.3)}, 0.6e^{i2\pi(0.3)}, 0.1e^{i2\pi(0.3)}\right)$
A_3	$\left(0.1e^{i2\pi(0.1)}, 0.2e^{i2\pi(0.5)}, 0.6e^{i2\pi(0.4)}\right)$	A_3	$\left(0.5e^{i2\pi(0.1)}, 0.2e^{i2\pi(0.1)}, 0.2e^{i2\pi(0.1)}\right)$
A_4	$\left(0.2e^{i2\pi(0.2)}, 0.2e^{i2\pi(0.1)}, 0.5e^{i2\pi(0.5)}\right)$	A_4	$\left(0.4e^{i2\pi(0.4)}, 0.3e^{i2\pi(0.4)}, 0.2e^{i2\pi(0.1)}\right)$
A_5	$\left(0.1e^{i2\pi(0.2)}, 0.1e^{i2\pi(0.2)}, 0.8e^{i2\pi(0.2)}\right)$	A_5	$\left(0.5e^{i2\pi(0.2)}, 0.4e^{i2\pi(0.2)}, 0.1e^{i2\pi(0.1)}\right)$

Table 4: Aggregated vales.

	CPFFWA		CPFFWG
σ_1	$\left(0.35e^{i2\pi(0.32)}, 0.20e^{i2\pi(0.20)}, 0.36e^{i2\pi(0.33)}\right)$	σ_1	$\left(0.23e^{i2\pi(0.26)}, 0.27e^{i2\pi(0.23)}, 0.42e^{i2\pi(0.35)}\right)$
σ_2	$\left(0.30e^{i2\pi(0.48)}, 0.26e^{i2\pi(0.20)}, 0.18e^{i2\pi(0.18)}\right)$	σ_2	$\left(0.23e^{i2\pi(0.40)}, 0.34e^{i2\pi(0.22)}, 0.23e^{i2\pi(0.23)}\right)$
σ_3	$\left(0.36e^{i2\pi(0.27)}, 0.24e^{i2\pi(0.21)}, 0.26e^{i2\pi(0.20)}\right)$	σ_3	$\left(0.30e^{i2\pi(0.20)}, 0.25e^{i2\pi(0.29)}, 0.31e^{i2\pi(0.26)}\right)$
σ_4	$\left(0.32e^{i2\pi(0.37)}, 0.18e^{i2\pi(0.20)}, 0.31e^{i2\pi(0.25)}\right)$	σ_4	$\left(0.31e^{i2\pi(0.31)}, 0.19e^{i2\pi(0.25)}, 0.33e^{i2\pi(0.31)}\right)$
σ_5	$\left(0.34e^{i2\pi(0.39)}, 0.20e^{i2\pi(0.13)} 0.24e^{i2\pi(0.19)}\right)$	σ_5	$\left(0.27e^{i2\pi(0.31)}, 0.26e^{i2\pi(0.14)} 0.42e^{i2\pi(0.28)}\right)$

It is worth noting that $\sum_{j=1}^{n} \hat{W}_j = 1$

Step IV: By taking $r = 2$, and using the CPFFWA and the CPFFWG operators, the collective values $\sigma_f (f = 1, 2, 3, 4, 5)$ alternatives A_i 's have been obtained as follows:

Step V: The definition 2.3 has been used to compute score values $\Delta(\sigma_i)\,(i = 1, 2, 3, 4, 5)$ of the overall CPFN $\sigma_i(1, 2, 3, 4, 5)$ as follows:

Note 3: There may occur many instances, although the probability is somewhat very low, that the score values of two or more alternatives becomes equal. In that case the order of the alternatives is decided by their accuracy values.

Step VI: Finally, we can say that the company should select A_2 as the most preferable alternative in both the cases. So the enterprise should: first consider focusing on Data Quality, then on Data Warehouse and Business Intelligence (BI), followed by Data Privacy and Security, Architecture, and Policy and Standards

5.1 Analysis of Changing the Parameter r on the Outcome of Decision-making:

Various values, for the sake of ranking our considered alternatives, of operational parameter r could be applied in our proposed method. Keeping this in consideration, we set numerous r values for classifying the innovative

Table 5: Order of alternatives for implementation.

Operators	$\Delta(\sigma_1)$	$\Delta(\sigma_2)$	$\Delta(\sigma_3)$	$\Delta(\sigma_4)$	$\Delta(\sigma_5)$	Alternatives' Ranking/Order
CPFFWA	0.5096	0.6018	0.5540	0.5445	0.5851	$A_2 > A_5 > A_3 > A_4 > A_1$
CPFFWG	0.4300	0.5413	0.4820	0.4963	0.4701	$A_2 > A_5 > A_3 > A_4 > A_1$

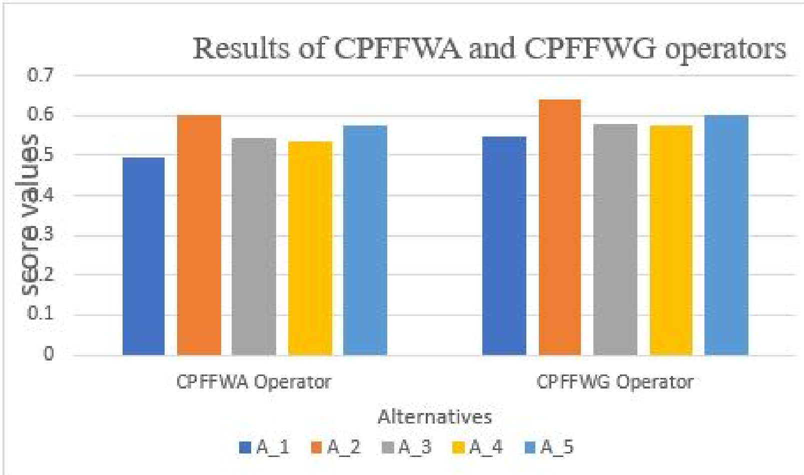

Figure 3: Score values obtained by CPFFWA and CPFFWG operators.

Table 6: Behavior of alternative with changing values of r in the CPFFWA operator.

r	$\Delta(\sigma_1)$	$\Delta(\sigma_2)$	$\Delta(\sigma_3)$	$\Delta(\sigma_4)$	$\Delta(\sigma_5)$	Order of ranking	Optimal alternative
2	0.4958	0.6033	0.5428	0.5330	0.5741	$A_2 > A_5 > A_3 > A_4 > A_1$	A_2
3	06819	0.7497	0.7115	0.7053	0.7313	$A_2 > A_5 > A_3 > A_4 > A_1$	A_2
4	0.7495	0.8016	0.7714	0.7665	0.7870	$A_2 > A_5 > A_3 > A_4 > A_1$	A_2
10	0.8482	0.8805	0.8623	0.8594	0.8718	$A_2 > A_5 > A_3 > A_4 > A_1$	A_2
30	0.8972	0.9191	0.9068	0.9048	0.9132	$A_2 > A_5 > A_3 > A_4 > A_1$	A_2

numerical MADM example in order to investigate the adaptability and sensitivity of the parameter r.

From table 6 and figure 4, it is evident that if $2 \leq r \leq 30$ although the obtained aggregated outcomes of the alternatives $A_i(i = 1, 2, 3, 4, 5)$ are different yet the order of ranking does not change. The order of ranking of the alternatives is in this case is depicted to be $A_2 > A_5 > A_3 > A_4 > A_1$. Moreover, the figure 3 depicts that the value of alternatives keeps on becoming refined as the value of parameter r increases. For example, the value of alterative 2 starts from 0.6033 and reaches 0.9191 as the value of r reaches 30, hence showing a refined behavior. The similar trend can also be observed for the remaining alternative from the figure 3.

From table 7 and figure 5, it is evident that if $\leq r \leq 30$, altought the obtained aggregated outcomes of the alternatives $A_i(i = 1, 2, 3, 4, 5)$ are different yet the order of ranking does not change. The order of raking

Table 7: Behavior of alternative with changing values of r in the CPFFWG operator.

r	$\Delta(\sigma_1)$	$\Delta(\sigma_2)$	$\Delta(\sigma_3)$	$\Delta(\sigma_4)$	$\Delta(\sigma_5)$	Order of ranking	Optimal alternative
2	0.4300	0.5413	0.4820	0.4963	0.4701	$A_2 > A_5 > A_3 > A_4 > A_1$	A_2
3	0.2713	0.3415	0.3041	0.3131	0.3257	$A_2 > A_5 > A_3 > A_4 > A_1$	A_2
4	0.2150	0.2706	0.2410	0.2481	0.2350	$A_2 > A_5 > A_3 > A_4 > A_1$	A_2
10	0.1294	0.1629	0.1451	0.1494	0.1415	$A_2 > A_5 > A_3 > A_4 > A_1$	A_2
30	0.0876	0.1103	0.0982	0.1011	0.0958	$A_2 > A_5 > A_3 > A_4 > A_1$	A_2

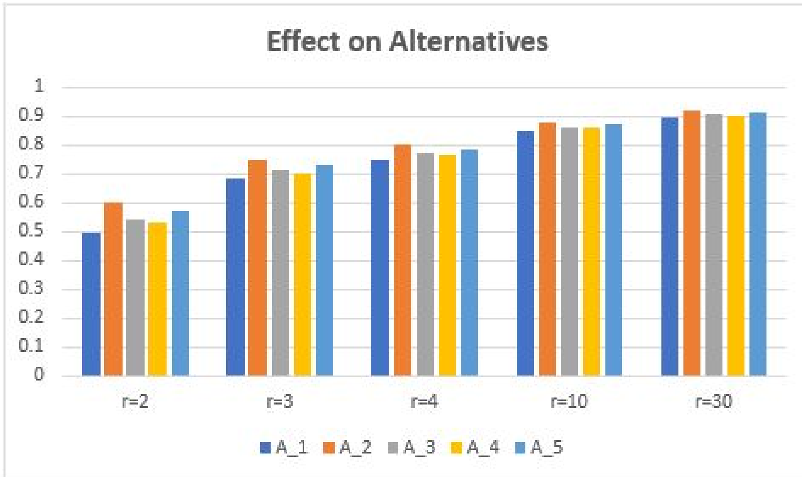

Figure 4: Departure of value of alternatives from their initial values to the finest value as r increases.

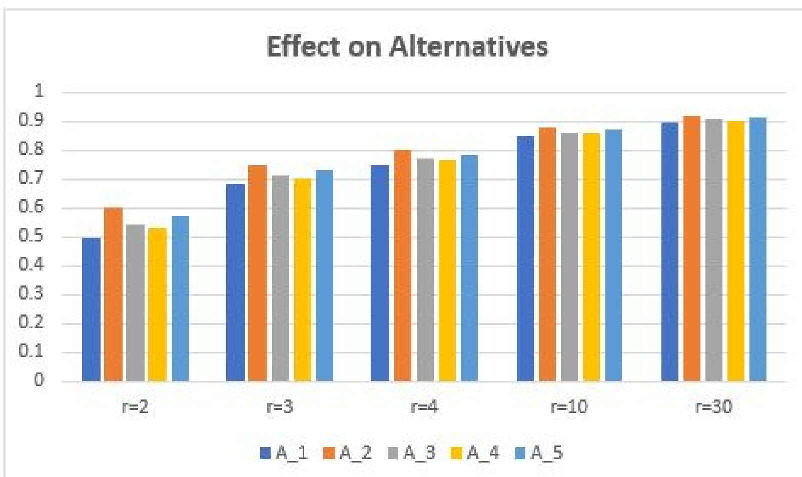

Figure 5: The value of alternatives keeps on becoming refined as the value of parameter r increases.

of the alternatives is in this case is depicted to be $A_4 > A_3 > A_5 > A_1 > A_2$. It is to be noted that although order of ranking of the alternatives are different, yet the best alternative is the same i.e., A_2. Moreover, the figure 5 depicts that the value of alternatives keeps on becoming refined as the value of parameter r increases. For example, the value of alterative 2 starts from 0.5413 and reaches 0.1103 - depicting the maximum refining- as the value of r reaches 50, hence showing a refined behavior. The similar trend can also be observed for the remaining alternative from the figure 5.

Generally, we can say that our proposed method has enough flexibility and accessibility which clearly allows decision makers to take the value of parameters based upon their choice.

It should be noted that CPFFWG operator has shown more flexible behavior than CPFFWA operator in our proposed MADM method by giving more refinement in the score values of the alternatives. Contrary to CPFFWG, the CPFFWA does not show that much flexibility. Therefore, it can be concluded that CPFFWG operator has responded more to variation in values of r than CPFFWA in this MADM problem and becomes more important for smoothly solving this type of MADM problem while keeping variations in values of r according to the decision maker's choice.

Table 8: Comparison of our suggested operators.

Aggregation Operators	Score values					Ranking/Order
	A_1	A_2	A_3	A_4	A_5	
Current work	0.4958	0.6033	0.5428	05330	0.5741	$A_2 > A_5 > A_3 > A_4 > A_1$
Current work	0.5494	0.6396	0.5791	0.5744	0.6018	$A_2 > A_5 > A_3 > A_4 > A_1$
CPFHWA[64]	-0.4151	-0.4934	-0.3868	-0.3376	-04312	$A_2 > A_5 > A_3 > A_4 > A_1$
CPFHWG[64]	-0.4969	-0.3737	-0.3545	-0.3174	-0.479	$A_2 > A_5 > A_3 > A_4 > A_1$
PFHWA[62]	0.5029	0.5640	0.5546	0.5085	0.5519	$A_2 > A_5 > A_3 > A_4 > A_1$
PFHWG[62]	0.4002	0.4956	0.4865	0.4892	0.4192	$A_2 > A_5 > A_3 > A_4 > A_1$
CIFWA[15]	0.36284	0.4516	0.4371	0.4216	0.4501	$A_2 > A_5 > A_3 > A_4 > A_1$

6 Comparative Studies

We contrast our suggested Frank aggregation operators with other current, well-known aggregation operators in the CPF context to ensure their usefulness and to explore their merits. Table 3 presents the comparison outcomes.

6.1 Comparison with Picture Fuzzy (PF) Operators:

We contrast our suggested approach with PFSs operators. When compared to the PFHWG and PFHWA operators given by Wei [62], we can see that in the presence of parameter r they appear to be mere particular incidences of our suggested operators. Moreover, these operators also suffer from the absence of a periodicity function due to which they just become very particular cases of our suggested operators when periodicity functions are taken to be zero. Furthermore owing to the fact the order of alternatives (sequence of implementation of the BDGF) remains unaltered, our approach becomes more consistent. These observations incline us to state that our newly established techniques are therefore more broadly applicable.

6.2 Comparison with CIF Operators:

We compare the operators of CIFSs with our proposed method. In contrast to the CIFWA operators [15], we can observe that they seem to be just specific instances of our proposed operators when parameter r is present. Moreover, these operators lack a hesitation function, which makes them special instances of our proposed operators when the hesitancy functions are assumed to be zero. Furthermore, our method becomes more consistent because the alternatives' order (sequence of implementation of the BDGF doesn't change. These findings lead us to the following conclusion: When compared to CIF operators, our recently developed methods are therefore more widely applicable.

6.3 Comparison with CPF Operators:

We contrast our suggested approach with CPFSs operators. When compared to the CPFHWA and CPFHWG operators [64] we can see that the ranking of the alternatives does not change. The score values obtained by these operators are compared with our suggested operators and are shown in Table 8, the ranking order (sequence of implementation of the BDGF has been shown in Table 9.

Additionally, our observation is also that some AOs [65] and [56] are not capable of dealing with our decision matrix. Our suggested operators, which are based on the t-norm and t-conorm of Frank, are more sophisticated and may take into account the link between different arguments. Our suggested operators also

Table 9: Comparison of our suggested operators.

Aggregation Operators	Framework	Ranking/Order
CPFFWA (our proposed)	CPFSs	$A_2 > A_5 > A_3 > A_4 > A_1$
CPFFWG (our proposed)	CPFSs	$A_2 > A_5 > A_3 > A_4 > A_1$
CPFHWA[64]	CPFSs	$A_2 > A_5 > A_3 > A_4 > A_1$
CPFHWF[64]	CPFSs	$A_2 > A_5 > A_3 > A_4 > A_1$
CIFWA[15]	CIFSS	$A_2 > A_5 > A_3 > A_4 > A_1$
CPyFWA[65]	CPyFs	Failed
CIVFWA[56]	CIVFSs	Failed
PFHWA[62]	PFSs	$A_2 > A_5 > A_3 > A_4 > A_1$
PFHWG[62]	PFSs	$A_2 > A_5 > A_3 > A_4 > A_1$

display Lukasiewicz sum and product as the parameter gets closer to infinity. In light of the various values of r, we have come to the conclusion that practically all of the arithmetic and geometric aggregation operations related to CPFNs are, in fact, a part of CPF Frank aggregating operator.

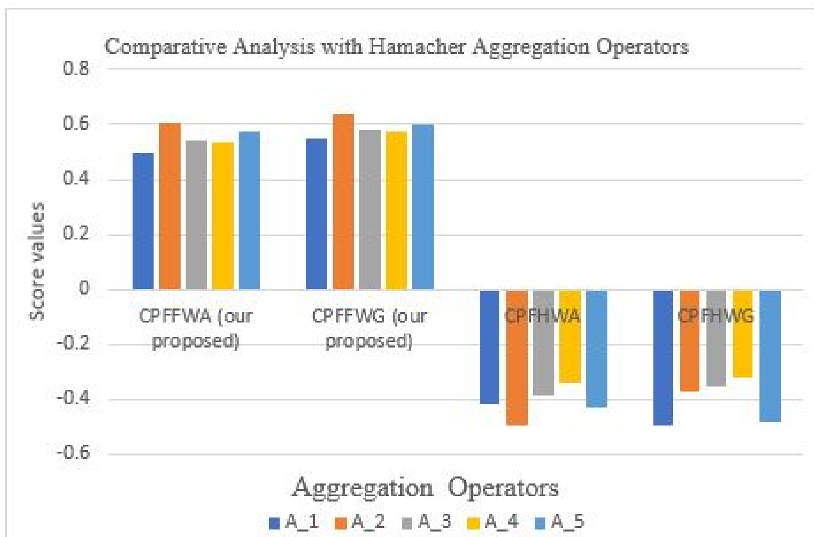

Figure 6: comparison with CPFHWA and CPHWG operators.

7 Conclusion

In conclusion, our study has developed a novel approach, CPFSs, which extends standard PFSs and IFSs. The study concentrated on the MADM problems, demonstrating the versatility and usefulness of CPFSs through the use of multiple aggregation approaches. Our research revealed compelling findings, demonstrating the stability and superiority of complicated CPFFA operators. Specifically, the use of the CPFFWA, CPFFWG, CPFFHA, and CPFFOWA operators has enhanced the applicability of CPFSs for MADM problems. In aggregating complex judgment criteria, Frank weighted geometric performed admirably. The methodological approach, which included Frank techniques for aggregation, gave a methodical framework for addressing MADM problems more rigorously than in the frameworks of PFSs, IFSs, CIFS etc. This research article has enabled decision makers to make more sound and appealing decisions for their maximum benefit. Despite these encouraging outcomes, it is critical to recognize some limits. The study concentrated on a specific set of aggregation approaches and may not cover the complete range of options. The use of CPFSs in real-world applications necessitates additional validation and testing. To address the aforementioned limitations and extend the impact of our findings, future research directions should explore the following: Investigate extensions of CPFSs, such as complex q-rung picture fuzzy sets, to enhance the scope and applicability of the proposed model. Extend the application of CPFSs to various decision-making techniques beyond MADM, including but not limited to COPRAS [66], and VIKOR[67]. Explore and develop additional complex picture fuzzy aggregation operators utilizing different t-norms and t-conorms to enhance the flexibility and robustness of the proposed model. Conduct extensive empirical studies to validate the proposed CPFS-based model in diverse real-world decision-making scenarios, ensuring their practical relevance and reliability. Our method also finds its applications in some notable problems such as plastic waste management [68], electric vehicle charging station site selection problems[69], and bio-medical waste management[70] etc. By focusing on the results obtained and providing a detailed methodological overview, we aim to offer a comprehensive understanding of the capabilities and limitations of CPFSs, facilitating their practical adoption in decision-making processes.

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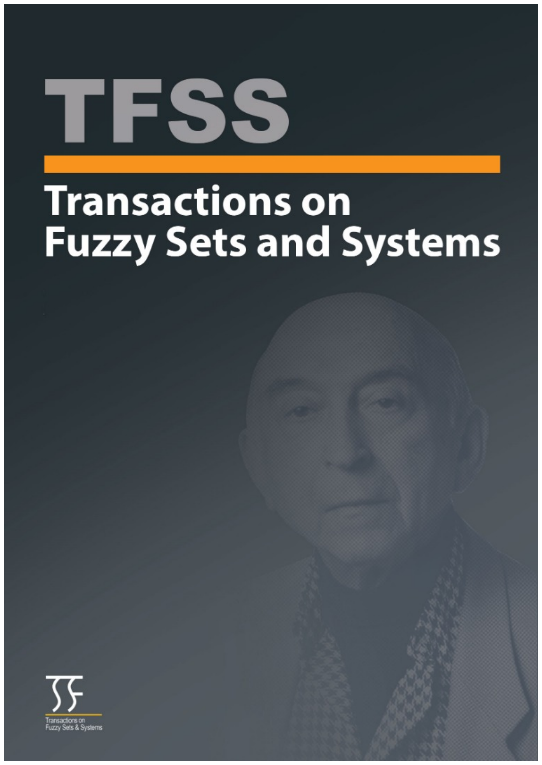

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New Elements in Hilbert Algebras

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New Elements in Hilbert Algebras

Ardavan Najafi*

Abstract. In this paper, the notion of the commutator of elements of a Hilbert algebra are introduced and some properties are given. The notions of involution element and Engel element in Hilbert algebras are introduced. Many different characterizations of them are given. Then, left (right) k-Engel elements as a natural generalization of commutators are introduced, and we discuss Engel elements, which are defined by left and right commutators. Finally, we will also study the relationships between these elements.

AMS Subject Classification 2020: MSC 06A11; MSC 03G25; MSC 03B22
Keywords and Phrases: Hilbert algebras, Commutator, Engel (involution) elements, Left (right) k-Engel element.

1 Introduction

Hilbert algebras are important tools for certain investigations in algebraic logic, since they can be considered as fragments of any propositional logic containing a logical connective implication and the constant 1 which is considered as the logical value true. Following the introduction of Hilbert algebras by L. Henkin in the early 50-ties and A. Diego [1], the algebra and related concepts were developed by D. Busneag [2]. Y.B. Jun gave characterizations of deductive systems in Hilbert algebras [3], introduced the notion of commutative Hilbert algebras and gave some characterizations of a commutative Hilbert algebra.
A. Diego [1] proved that Hilbert algebras form a variety that is locally finite. They were studied from various points of view. Concerning congruence properties it is shown in [4] that Hilbert algebras form a congruence distributive variety the congruences which are in a one to one correspondence with ideals [5].
The present author introduced the commutator of two elements in a BCI-algebra, and used this notion to define a solvable BCI-algebra and considered solvable BCI-algebras using commutators. Then we gave a new definition for solvability, nilpotency, centralizer and pseudo-center in a BCI-algebra and considered their properties (see [6]).
In this paper, we present a definition for the notion of Engel elements in Hilbert algebras based on commutators. We define also the notions of left k-Engel elements and right k-Engel elements as a natural generalization of commutators in Hilbert algebra, give several characterizations of them and prove that a Hilbert algebra is commutative if and only if 1 is only commutator of it. So, the class of commutative Hilbert algebras and 1-Engel BCI-algebras are equal. One of the most important concepts in the study of groups is the notion of nilpotency. Finally, we present a definition for the involution elements in Hilbert algebras. We give several characterizations of them and we illustrate also these notions with some examples.

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2 Preliminaries

We include some elementary aspects of Hilbert algebras that are necessary for this paper, and for more details we refer to [1–5, 7].
By a Hilbert algebra, we main an algebra $(H, \to, 1)$ of type $(2, 0)$, where H is a non-empty set, $\to$ is a binary operation on H, $1 \in H$ is an element such that the following three axioms are satisfied for every $x, y, z \in H$:
$(H1)$ $x \to (y \to x) = 1$,
$(H2)$ $(x \to (y \to z)) \to ((x \to y) \to (x \to z)) = 1$,
(H3) if $x \to y = y \to x = 1$, then $x = y$.
In a Hilbert algebra H, the following properties hold:
$(P1)$ $x \to 1 = 1, 1 \to x = x$ and $x \to x = 1$,
$(P2)$ $x \to (y \to z) = (x \to y) \to (x \to z)$,
$(P3)$ $x \to (y \to z) = y \to (x \to z)$
If H is a Hilbert algebra, then the relation $x \leq y$ iff $x \to y = 1$ is a partial order on H, called the natural ordering on H. With respect to this ordering 1 is the greatest element of H and the following property is satisfied.
$(P4)$ $x \leq y$ implies $z \to x \leq z \to y$ and $y \to z \leq x \to z$.
For any x and y in a Hilbert algebra H, define $x \vee y$ as $(y \to x) \to x$. A Hilbert algebra H is said to be commutative, if for all $x, y \in H$,

$$(y \to x) \to x = (x \to y) \to y \quad i.e., \quad x \vee y = y \vee x.$$

Note that $x \vee y$ is the least upper bound of x and y, hence each commutative Hilbert algebra H is a semilattice with respect to $\vee$ (see [2]) and hence $\vee$ is commutative and associative. A non empety subset S of a Hilbert algebra H is called a subalgebra of H, if $x \to y \in S$, whenever $x, y \in S$. A bounded Hilbert algebra is a Hilbert algebra H with an element $0 \in H$ such that $0 \to x = 1$, for every $x \in H$. In a bounded Hilbert algebra H we define a unary operation $*$ as $x^* = x \to 0$, for each $x \in H$.
A Hilbert algebra is prelinear if $(x \to y) \vee (y \to x) = 1$, for all $x, y \in H$. We say that an element x of H is minimal if $y \leq x$ (i.e., $y \to x = 1$) implies $x = y$, for any $y \in H$.

Example 2.1. [1] It is of great importance that every partially ordered set $(P, \to, 1)$ with the greatest element 1 can be regarded as a Hilbert algebra, namely, if for any $x, y \in P$ we define:

$$x \to y = \begin{cases} 1 & \text{if x} \leq \text{y} \\ y & \text{otherwise.} \end{cases}$$

then (P, $\to$, 1) is a Hilbert algebra the natural ordering on which coincides with the relation $\leq$.

Lemma 2.2. *[8] If H is a bounded Hilbert algebra and $x, y \in H$, then*
i) $1^* = 0$ *and* $0^* = 1$,
ii) $x \leq x^{**}$,
iii) $x^{***} = x^*$,
iv) $x \to y^* = y \to x^*$,
v) if $x \leq y$, *then* $y^* \leq x^*$.

From now on, $(H, \to, 1)$ or simply H is a Hilbert algebra, unless otherwise specified.

3 Commutators of Two Elements

In this section, we introduce commutators of two elements of a Hilbert algebra and investigate some of their properties.

Definition 3.1. For elements x and y of a Hilbert algebra H, we define the commutator of x and y by $(y \vee x) \to (x \vee y)$ denoted by $[x, y]$, Namely,

$$[x, y] = (y \vee x) \to (x \vee y) = ((x \to y) \to y) \to ((y \to x) \to x)$$

For every $x \in H$, we obtain $[x, x] = (x \vee x) \to (x \vee x) = x \to x = 1$. Also, since 1 is the greatest element of H, $[1, x] = (x \vee 1) \to (1 \vee x) = 1 \to 1 = 1$. Similarity $[x, 1] = 1$. The set of all the commutators of elements of H is denoted by $Com(H)$ or $K(H)$. Obviously, $1 \in K(H)$.

Example 3.2. Let $H = \{0, a, b, c, 1\}$, with $0 < a, b < c < 1$, and a, b are incompatible, be a Hilbert algebra in which $\to$ operation is defined by the left table and the commutators of elements of H are calculation in the right table

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	b	1	1
b	a	a	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

$[,]$	0	a	b	c	1
0	1	1	1	1	1
a	1	1	1	1	1
b	1	1	1	1	1
c	c	c	c	1	1
1	1	1	1	1	1

Lemma 3.3. *For any $x, y, z \in H$, $z \to [x, y] = [z \to x, z \to y]$.*

Proof.

$$\begin{aligned}
[z \to x, z \to y] =& (((z \to x) \to (z \to y)) \to (z \to y)) \\
& \to (((z \to y) \to (z \to x)) \to (z \to x)) \\
=& ((z \to (x \to y)) \to (z \to y)) \to ((z \to (y \to x)) \to (z \to x)) \\
=& (z \to ((x \to y) \to y)) \to (z \to ((y \to x) \to x)) \\
=& z \to (((x \to y) \to y) \to ((y \to x) \to x)) \\
=& z \to [x, y]
\end{aligned}$$

In this proof we use (P_2) in Lines 3, 4 , 5. □

In the following theorem, we will show that $[x, y]$ is an upper bound for x and y, but in Example 4.5 we show that it is not a supremum of x, y, in general.

Theorem 3.4. *For all $x, y \in H$, we have*
i) $x \leq [x, y]$ and $y \leq [x, y]$,
ii) if $x \leq y$, then $[x, y] = 1$,
iii) $[x \to y, x] = [x, y \to x] = 1$,
iv) $x \to y \leq [x, y]$,
v) $[x, y] = y$ if and only if $y = 1$.

Proof. i) As $x \to [x, y] = [x \to x, x \to y] = [1, x \to y] = 1$, then $x \leq [x, y]$. Also $y \to [x, y] = [y \to x, y \to y] = [y \to x, 1] = 1$, then $y \leq [x, y]$.
ii) Let $x \leq y$, then $x \to y = 1$. By use definition of $[x, y]$ and property P_3, we obtain

$$\begin{aligned}
[x, y] =& ((x \to y) \to y) \to ((y \to x) \to x) \\
=& (1 \to y) \to ((y \to x) \to x) \\
=& y \to ((y \to x) \to x) \\
=& (y \to x) \to (y \to x) = 1.
\end{aligned}$$

iii)

$$\begin{aligned}[x \to y, x] =&(((x \to y) \to x) \to x) \to ((x \to (x \to y)) \to (x \to y))\\ =&(((x \to y) \to x) \to x) \to (((x \to x) \to (x \to y)) \to (x \to y))\\ =&(((x \to y) \to x) \to x) \to (((1 \to (x \to y)) \to (x \to y))\\ =&(((x \to y) \to x) \to x) \to ((x \to y) \to (x \to y))\\ =&(((x \to y) \to x) \to x) \to 1 = 1.\end{aligned}$$

Since $x \leq y \to x$, by (ii), we have $[x, y \to x] = 1$.
iv)

$$\begin{aligned}(x \to y) \to [x, y] =&(x \to y) \to (((x \to y) \to y) \to ((y \to x) \to x))\\ =&((x \to y) \to y) \to ((x \to y) \to ((y \to x) \to x))\\ =&(x \to y) \to (y \to ((y \to x) \to x))\\ =&(x \to y) \to ((y \to x) \to (y \to x))\\ =&(x \to y) \to 1 = 1.\end{aligned}$$

v) Let $[x, y] = y$. Then $y = (y \vee x) \to (x \vee y) \geq (x \vee y) \geq y$. Hence $x \vee y = y$. Therefore $x \to y = x \to (x \vee y) = 1$, because $x \leq (x \vee y)$. So $x \to y = 1$ and hence $x \leq y$ and by (ii), we obtain $[x, y] = 1$. Thus $y = 1$.
Conversely, if $y = 1$, then $[x, y] = [x, 1] = 1$. □
By Example 3.2 we see that $[x, y] \neq [y, x]$, in general. Also, in this example we see $[c, b] = c$, so $[x, y] = x$ it cannot be said that $x = 1$, in general.

Theorem 3.5. *If H is a bounded Hilbert algebra and $x, y \in H$, then*
i) $[x, 0] = x^{**} \to x$,
ii) $[0, x] = 1$,
iii) $[x^*, x] = 1$.

Proof. i)

$$\begin{aligned}[x, 0] =&((x \to 0) \to 0) \to ((0 \to x) \to x)\\ =&(x^* \to 0) \to (1 \to x)\\ =&x^{**} \to x.\end{aligned}$$

ii) Since $0 \leq x$ for every $x \in H$, by Theorem 3.4 part (ii), we have $[0, x] = 1$.
iii)

$$\begin{aligned}[x^*, x] =&((x^* \to x) \to x) \to ((x \to x^*) \to x^*)\\ =&((x^* \to x) \to x) \to ((x \to (x \to 0)) \to x^*)\\ =&((x^* \to x) \to x) \to (((x \to x) \to (x \to 0)) \to x^*)\\ =&((x^* \to x) \to x) \to ((1 \to x^*) \to x^*)\\ =&((x^* \to x) \to x) \to (x^* \to x^*)\\ =&((x^* \to x) \to x) \to 1\\ =&1.\end{aligned}$$

□

Theorem 3.6. *H is commutative if and only if $[x,y]=1$, for every $x,y\in H$.*

Proof. Let H be a commutative Hilbert algebra. Then for every $x,y\in H$, $y\vee x=x\vee y$. Hence $[x,y]=(y\vee x)\to(x\vee y)=1$.
Conversely, let for every $x,y\in H$, we have $[x,y]=[y,x]=1$. Form $[x,y]=(y\vee x)\to(x\vee y)=1$, we deduce $y\vee x\le x\vee y$. But $[y,x]=(x\vee y)\to(y\vee x)=1$, so $x\vee y\le y\vee x$. Therefore $x\vee y=y\vee x$, for every $x,y\in H$. Hence H is a commutative Hilbert algebra. □

Theorem 3.7. *Let H be a finite Hilbert algebra of order n with $n\ge 2$ and $x,y\in H$. Then $[x,y]$ is not a minimal element of H.*

Proof. Suppose that there exist x and y in H such that $[x,y]$ is a minimal element of H. Then $[x,y]=y$, as $y\le[x,y]$. Hence by Theorem 3.4 (v) we deduce $y=1$. Whence $[x,y]=[x,1]=1$. This is a contradiction. Because, if 1 is a minimal element of H, then from $x\le 1$ we deduce $H=\{1\}$. □

4 Engel Elements in Hilbert Algebras

In this section, we introduce the notion of the Engel element of n elements of a Hilbert algebra.
Let $x_1,...,x_n$ be elements of H. For all positive integer n we define inductively $[x_1,...,x_n]$ as follows: $[x_1]=x_1$ and

$$[x_1,...,x_n]=([x_1,...,x_{n-1}]\vee x_n)\to(x_n\vee[x_1,...,x_{n-1}])$$

If $x_2=x_3=...=x_n$, then we denote $[x_1,...,x_n]$ by $[x_1,{}_nx_2]$. Note that $[x_1]=[x_1,{}_0x_2]=x_1$.

Definition 4.1. Suppose that $x,y\in H$. For a non-negative integer n we define inductively the *n-Engel left commutator* $[x,{}_ny]$ as follows:

$$[x,{}_0y]=x,...,[x,{}_ny]=[[x,{}_{n-1}y],y]$$

Also the *n-Engel right commutator* $[{}_nx,y]$ of the pair (x,y) is defined by induction as follows:

$$[{}_0x,y]=y,...,[{}_nx,y]=[x,[{}_{n-1}x,y]].$$

Especially, $[x,{}_1y]=[{}_1x,y]=[x,y]=(y\vee x)\to(x\vee y)$.

For a positive integer k, an element x of H is called a *right k-Engel element* of H whenever $[x,{}_ky]=1$, for all $y\in H$. An element x of H is called a *right Engel element* if it is right k-Engel for some non-negative integer k.
We denote by $R(H)$ and $R_k(H)$ the set of right Engel elements and right k-Engel elements, respectively. So

$$R_k(H)=\{x\in H:[x,{}_ky]=1,\forall y\in H\}.$$

and

$$R(H)=\bigcup_{k\in N}R_k(H).$$

Notice that the variable element y appears on the right of bracket and if n can be chosen independently of y, then x is a right n-Engel element of H. Left Engel elements are defined in a similar way.
For a positive integer k, an element x of H is called a *left k-Engel element* of H whenever $[y,{}_kx]=1$ for all

$y \in H$. Also x is said to be a *left Engel element* of H if it is left k-Engel for some non-negative integer k. We denote by $L(H)$ and $L_k(H)$ the set of left Engel elements and left k-Engel elements, respectively.

$$L_k(H) = \{x \in H : [y, {}_k x] = 1, \forall y \in H\}.$$

and

$$L(H) = \bigcup_{k \in N} L_k(H).$$

Where the variable y is on the left of bracket. Also, since $[x, 1] = [1, x] = 1$, for every $x \in H$, $1 \in R(H) \cap L(H)$. An element x of H that is both the left and right Engel element is said to be an *Engel element*.
The set of all Engel elements of H is denoted by $En(H)$. Obviously, 1 is an Engel element in any Hilbert algebra. Since $x, y \leq [x, y]$, for every $x, y \in H$,

$$[x, {}_0 y] = x \leq [x, {}_1 y] = [x, y] \leq [x, {}_2 y] = [[x, y], y] \leq [x, {}_3 y] = [[x, {}_2 y], y] \leq ...$$

Also

$$[{}_0 x, y] = y \leq [{}_1 x, y] = [x, y] \leq [{}_2 x, y] = [x, [x, y]] \leq [{}_3 x, y] = [x, [{}_2 x, y]] \leq ...$$

According to the above inequalities, we immediately have the following theorems.

Theorem 4.2. *Let $x, y \in H$ and m, n be non-negative integers. If $m \leq n$, then $[x,_m y] \leq [x,_n y]$ and $[_m x, y] \leq [_n x, y]$.*

Theorem 4.3.

$$R_1(H) \subseteq R_2(H) \subseteq R_3(H) \subseteq .... \subseteq R(H).$$
$$L_1(H) \subseteq L_2(H) \subseteq L_3(H) \subseteq .... \subseteq L(H).$$

Example 4.4. By simple calculation, for Hilbert algebra H in Example 3.2 we see that
$[c, {}_0 a] = c, [c, {}_1 a] = [c, a] = c, [c, {}_2 a] = [[c, a], a] = [c, a] = c$. So $[c, {}_n a] = c$ for $n \geq 2$, also $[{}_0 c, a] = a, [{}_1 c, a] = c, [{}_2 c, a] = [c, [c, a]] = [c, c] = 1$, so $[{}_n c, a] = 1$, for any $n \geq 2$.

$$R_1(H) = \{x \in H : [x, y] = 1, \forall y \in H\} = \{0, a, b, 1\}.$$

$$R_2(H) = \{x \in H : [x, {}_2 y] = 1, \forall y \in H\} = \{0, a, b, 1\}.$$

Also, for every $n \geq 2$ we have

$$R_n(H) = \{x \in H : [x, {}_n y] = 1, \forall y \in H\} = \{0, a, b, 1\}.$$

and

$$R(H) = \bigcup_{k \in N} R_k(H) = \{0, a, b, 1\}.$$

Also

$$L_1(H) = \{x \in H : [y, x] = 1, \forall y \in H\} = \{c, 1\}.$$

and for any $n \geq 2$ we obtain

$$L_n(H) = \{x \in H : [y, {}_n x] = 1, \forall y \in H\} = \{c, 1\}.$$

Therefore

$$L(H) = \bigcup_{k \in N} L_k(H) = \{c, 1\}.$$

Then $En(H) = R(H) \cap L(H) = \{1\}$.

Example 4.5. Let $H = \{a, b, c, 1\}$ be a Hilbert algebra in which $\rightarrow$ operation is defined by the left table and the commutators of elements of H are calculation in the right table

$\rightarrow$	a	b	c	1
a	1	a	a	1
b	1	1	a	1
c	1	1	1	1
1	a	b	c	1

$[,]$	a	b	c	1
a	1	1	1	1
b	1	1	a	1
c	1	1	1	1
1	1	1	1	1

By simple calculation, we see that $[a, {}_0 b] = a, [a, {}_1 b] = 1, [a, {}_2 b] = 1$ and so $[a, {}_n b] = 1$, also $[b, {}_0 c] = b, [b, {}_1 c] = a, [b, {}_2 c] = [[b, c], c] = [a, c] = 1$ and so $[b, {}_n c] = 1$, for any $n \geq 2$.
$[{}_0 c, a] = a, [{}_1 c, a] = 1, [{}_2 c, a] = [c, [c, a]] = [c, 1] = 1$, so $[{}_n c, a] = 1$, for any $n \geq 3$.

$$R_1(H) = \{a, c, 1\}.$$

for every $n \geq 2$

$$R_n(H) = \{a, b, c, 1\}.$$

$$L_1(H) = \{a, b, 1\}.$$

and for any $n \geq 2$ we obtain

$$L_n(H) = \{a, b, c, 1\}.$$

Then $En(H) = R(H) \cap L(H) = \{a, b, c, 1\}$.

In Example 4.5 we see that $c \leq b \leq a \leq 1$ and $[b, c] = a$. Hence $[b, c]$ is an upper bound for b and c, but it is not supremum.

Lemma 4.6. *Let $x, y \in H$. Then for each $n \in N$ the following assertions hold:*
i) $[{}_n x, y] = 1$, for every $n \geq 2$.
ii) if H is a Hilbert algebra with $|H| \geq 2$, then $[x, {}_n y]$ and $[{}_n x, y]$ are not minimal elements.

Proof. i) For any $x, y \in H$, since $x \leq [x, y]$, we obtain

$$[{}_2 x, y] = [x, [x, y]] = 1.$$

But

$$1 = [{}_2 x, y] \leq [{}_3 x, y] \leq [{}_4 x, y] \leq [{}_5 x, y] \leq ...$$

Hence $[{}_3 x, y] = [{}_4 x, y] = [{}_5 x, y] = ... = 1$.
ii) We proceed by induction on n. For $n = 1$, $[x, y]$ is not a minimal element of H, by Theorem 3.7.
Now assume that for $n \in N$, $[x, {}_n y]$ and $[{}_n x, y]$ are not minimal elements of H. Since $[x, {}_{n+1} y] = [[x, {}_n y], y] \geq [x, {}_n y]$, $[x, {}_{n+1} y]$ is not minimal. Also $[{}_{n+1} x, y] = [x, [{}_n x, y]] \geq [{}_n x, y]$. Since $[{}_n x, y]$ is not a minimal element of H, $[{}_{n+1} x, y]$ is not too. Hence the result holds for $n + 1$ in both cases. □

5 Involution Elements

In this section, at first, we define the involution element in a bounded Hilbert algebra, and then we investigate the relationships of these elements with commutators.

Definition 5.1. For a bounded Hilbert algebra H, if an element x satisfies $x^{**} = x$, then x is called an involution of H.

The set of all involution elements of a bounded Hilbert algebra H is denoted by $S(H)$. The smallest element 0 and the greatest element 1 are two involutions of H, because $0^{**} = 1^* = 0$ and $1^{**} = 0^* = 1$. Since the elements 0 and 1 are contained in $S(H)$. Hence $S(H)$ is not empty.

Example 5.2. It is easy to see that $(H, \to, 1)$ in Example 3.2 is a bounded Hilbert algebra with unit 1. We obtain $S(H) = \{0, a, b, 1\}$. In this example $c \notin S(H)$, because $c^{**} = 0^* = 1 \neq c$. We saw that $K(H) = \{c, 1\}$ and $En(H) = \{1\}$. Hence $En(H), S(H)$ and $K(H)$ are the separate sets from each other, in general.
Also, if we choose $P = [0, 1]$ in Example 2.1, we obtain, $S(P) = \{0, 1\}$ and for any $0 < x \leq 1$, we get $[x, 0] = [x,_2 0] = [x,_3 0] = ... = x$. So $En(P) = K(P) = (0, 1]$.

Theorem 5.3. *In a bounded Hilbert algebra H, we have*

$$x^* \to y = y^* \to x$$

for all x and y in $S(H)$.

Proof. Let $x, y \in S(H)$. Then $x^* \to y = x^* \to y^{**} = y^* \to x^{**} = y^* \to x$. □

Theorem 5.4. *For any bounded Hilbert algebra H, $S(H)$ is a bounded subalgebra of H.*

Proof. Let $x, y \in S(H)$. Then by Lemma 2.2 and Theorem 5.3,

$$\begin{aligned}
(x \to y)^{**} \to (x \to y) =& (x \to y)^{**} \to (x \to y^{**}) \\
=& (x \to y)^{**} \to (y^* \to x^*) \\
=& y^* \to ((x \to y)^{**} \to x^*) \\
=& y^* \to (x^{**} \to (x \to y)^{***}) \\
=& y^* \to (x \to (x \to y)^*) \\
=& y^* \to ((x \to y) \to x^*) \\
=& (x \to y) \to (y^* \to x^*) \\
=& (x \to y) \to (x \to y^{**}) \\
=& (x \to y) \to (x \to y) \\
=& 1.
\end{aligned}$$

Therefore $(x \to y)^{**} \leq (x \to y)$. But $(x \to y) \to (x \to y)^{**} = (x \to y) \to (((x \to y) \to 0) \to 0) = ((x \to y) \to 0) \to ((x \to y) \to 0) = 1$. Hence $x \to y \leq (x \to y)^{**}$. We deduce $(x \to y)^{**} = (x \to y)$, which says $x \to y \in S(H)$, and consequently $S(H)$ is closed with respect to the binary operation $\to$. Also $0, 1 \in S(H)$, namely, $S(H)$ is a bounded subalgebra of H. □

Theorem 5.5. *Let $x, y \in S(H)$. Then for all $x, y \in H$ we have*
i) $x \to y = y^* \to x^*$,
ii) $x \leq y^*$ implies $y \leq x^*$.

Proof. i) Since $x, y \in S(H)$, we have $x^{**} = x$ and $y^{**} = y$. Hence by Lemma 2.2 part (iv), $x \to y = x \to y^{**} = y^* \to x^*$.
ii) Let $x \leq y^*$, we get $x \to y^* = 1$. Hence by Lemma 2.2 part (iv), $1 = x \to y^* = y \to x^*$. So, $y \leq x^*$. □
In the following theorem, we express the relationship between an involution element and the commutators.

Theorem 5.6. *Let H be a bounded Hilbert algebra, then for $x \in H$, $[x, 0] = 1$ if and only if $x \in S(H)$.*

Proof. Let H be a bounded Hilbert algebra. Let $x \in H$ and suppose that $[x, 0] = 1$. Since $[x, 0] = x^{**} \to x$, we get $x^{**} \to x = 1$, so $x^{**} \leq x$. But $x \leq x^{**}$ and therefore $x^{**} = x$. Thus $x \in S(H)$.
Conversely, let $x \in S(H)$, for bounded Hilbert algebra H. Then $x^{**} = x$. So, we obtain $[x, 0] = x^{**} \to x = x \to x = 1$. □

6 Conclusion

In the present paper, we have introduced the concepts of Engel elements in Hilbert algebras and investigated some of their properties. To develop the theory of Hilbert algebras, one of the most encouraging ideas could be investigating the Engel degree of Hilbert algebras and finding a relation diagram between subclasses of Hilbert algebras. For instance, 1-Engel Hilbert algebras are strictly commutative Hilbert algebras. It is hoped that this work contributes to further studies. Therefore, we think that the results presented in this paper and the forthcoming works can pave the way for a bright future for the theory of the Hilbert algebras. The major goal of Engelś theory in Hilbert algebras can be stated as follows: to find conditions on H which will ensure that $L(H)$ and $R(H)$ are subalgebras or ideals, if possible.

Conflict of Interest: The author declares no conflict of interest.

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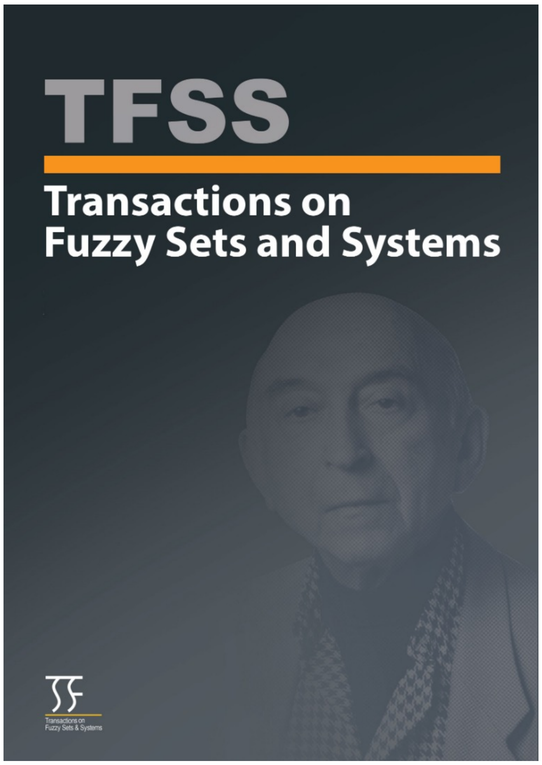
Transactions on Fuzzy Sets and Systems



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A Journey from Traditional to Fuzzy Methods of Decision-Making

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A Journey from Traditional to Fuzzy Methods of Decision-Making

Michael Gr. Voskoglou*

Abstract. Decision-Making (DM) is one of the most important components of human cognition. Starting with a review of the traditional criteria for DM, this work presents also a method for the verification of a decision, a step of the DM process which, due to its special interest, is usually examined separately from its other steps. Frequently in everyday life, however, the data of a DM problem are vague and characterized by uncertainty. In such cases the traditional techniques for DM, which are based on principles of the bivalent logic (yes-no), cannot help effectively in making the right decision. The first who introduced principles of the fuzzy sets theory in DM were Bellman and Zadeh in 1970 and an example is given here illustrating their fuzzy criterion for DM. Also, among the several fuzzy methods proposed later by other researchers for a more effective DM, a hybrid method is developed here for parametric multiple-criteria DM using soft sets and grey numbers (or intuitionistic fuzzy sets, or neutrosophic sets) as tools, which improves an earlier method proposed by Maji et al. in 2002. All the DM approaches presented in this paper are illustrated with everyday practical examples.

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Keywords and Phrases: Decision-making (DM), Fuzzy set (FS), Intuitionistic FS (IFS), Neutrosophic set (NS), Grey number (GN), Soft set (SS).

1 Introduction

Decision-making (DM), one of the most important components of human cognition, is the process of choosing a solution between two or more alternatives for the purpose of achieving the optimal result for a given problem. Obviously DM has sense if, and only if, there exists more than one feasible solution, together with one or more suitable criteria helping the decision maker to choose the best among these solutions. We recall that a solution is characterized as feasible, if it satisfies all the restrictions imposed onto the real system by the statement of the problem as well as all the natural restrictions imposed onto the problem by the real system; e.g. if x denotes the quantity of stock of a product, we must have $x \geq 0$. The choice of the suitable criterion, especially when the results of DM are affected by random events, depends upon the desired goals of the decision maker; e.g. optimistic or conservative criterion, etc.

The rapid technological progress, the impressive development of transportation means, the globalization of human society, the continuous changes appearing in the local and international economies, and other related reasons, led during the last 60-70 years to a continuously increasing complexity of the problems of our everyday life. As a result the DM process became in many cases a very difficult task, which is impossible to be based on the decision makers experience, intuition and skills only, as it usually happened in the past. Thus, from the beginning of 1950 a progressive development started of a systematic methodology for the DM

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process, termed *Statistical Decision Theory*, which is based on principles of Probability Theory, Statistics, Economics, Psychology and of other related scientific sectors [1].

The DM process involves the following steps:

• $\mathbf{d_1}$: *Analysis* of the decision problem, i.e. understanding, simplifying and reformulating the problem in a form permitting the application of the standard DM techniques it.

• $\mathbf{d_2}$: *Collection* and *interpretation* of all the necessary information related to the problem.

• $\mathbf{d_3}$: *Determination* of all the feasible solutions.

• $\mathbf{d_4}$: *Choice* of the best solution in terms of the suitable, according to the decision-makers goals, criterion (-ia).

One could add one more step to the DM process, the *verification* of the chosen decision according to the results obtained by applying it in practice. However, this step is extended to areas which, due to their depth and importance, have become autonomous. Therefore, it is usually examined separately from the other steps of the DM process.

Note that the first three steps of the DM process are continuous, in the sense that the completion of each one of them usually needs some time, during which the decision- maker's reasoning is characterized by transitions between hierarchically neighbouring steps. In other words, the DM process, the flow diagram of which is represented in Figure 1, cannot be characterized as a linear process.

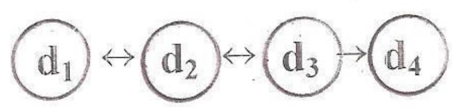

Figure 1: The flow diagram of the DM process

For facilitating the DM process, at the step of analysis a decision problem is usually represented by a *decision matrix*, otherwise termed as the *matrix of the pay-offs*. Each row of this matrix corresponds to an event and each column of it corresponds to a decision. The events are all the possible outcomes of the corresponding DM problem, whereas the entries of the matrix correspond to the results of each decision (pay-offs). Mathematically speaking, in a DM problem with n events and m possible decisions the decision matrix is an n x m matrix of the form $[a_{ij}]$, where a_{ij} denotes the pay-off corresponding to the event E_i and the decision D_j. Table 1, for example, represents the decision matrix of the classical DM problem of the judge.

Table 1: Decision matrix of the DM problem of the judge

Events	**Decisions of the judge**	
	INNOCENT	GUILTY
INNOCENT	An innocent is decided to be innocent	An innocent is decided to be guilty
GUILTY	A guilty is decided to be innocent	A guilty is decided to be guilty

An alternative way to represent a DM problem is the use of a *decision tree*, which has the form of a logical diagram. The decision tree of the DM problem of the judge, for example, is shown in Figure 2.

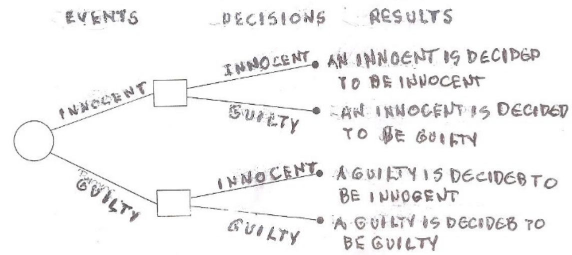

Figure 2: The decision tree of the DM problem of the judge

The use of a decision tree is usually preferred in the case of composite and complicated DM problems.

In this review paper, starting from the traditional criteria for DM, based on principles of the bivalent logic [2, 3], we also present a method for studying the verification of a decision, based on the calculation of the GPA index. Next the criterion of Bellman and Zadeh is presented for DM under fuzzy conditions [4] and a parametric method for multiple criteria DM is developed [5–8], which improves an earlier DM of Maji et al. [9] using soft sets as tools.

2 Traditional Criteria for Decision-Making

According to the existing information, a decision is made under conditions of *certainty, risk, uncertainty* or *complete ignorance*. In the first case the DM is obviously an easy task, whereas the complete lack of information is something that happens very seldom. Uncertainty in the field of Management is understood to be a situation in which all the possible outcomes of future action are known, but not the probabilities of the appearance of each outcome. On the contrary, in a situation of risk both the outcomes of an action and the probabilities of them to happen are known. The turn of a coin, for example, is a situation of risk, whereas the color of the first car that will pass in front of an observer is a situation of uncertainty.

As already mentioned in the previous section, a necessary condition for the DM is the existence of at least one suitable criterion helping the decision-maker to make the right decision. When the pay-offs are numerical quantities, the most commonly used decision criteria among those reported in the literature [2, 3], are the following:

- ***Maximization of the minimal pay-offs (maxi min pay-offs)***

Using this criterion, the decision-maker considers the minimal pay-offs corresponding to each possible decision and chooses the maximal among them. This criterion, otherwise known as the *criterion of Wald*, is based on the law of Murphy, according to which the worst that could happen will happen. It is, therefore, a conservative criterion, which is frequently used when the decision-maker knows that he/she has no chance to make a wrong estimation. On the other end, the *maximization of the maximal pay-offs (maxi max pay-offs)* is a super optimistic criterion, which is used very rarely, because it involves a great risk.

- ***Minimization of the maximal lost opportunities (mini max lost opportunities)***

The *lost opportunity* x_{ij} is defined to be the difference of the maximal pay-off corresponding to the event E_i, minus the pay-off a_{ij}, for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$. To apply this criterion, one forms the n x m *matrix of the lost opportunities* $\{x_{ij}\}$ and chooses the column (and therefore the decision) corresponding to the minimal among the maximal lost opportunities. This criterion, also known as the *regret criterion* because of the decision-makers disappointment with the lost opportunities, is more optimistic than the criterion of Wald.

- ***Maximization of the expected pay-offs***

Let p_i be the probability of appearance of the event E_i, $i = 1, 2, \ldots, n$, then the *expected pay-off* a_j corresponding to the possible decision j, $j = 1, 2, \ldots, m$, is defined by

$$a_j = \sum_{i=1}^{n} p_i a_{ij}. \tag{1}$$

According to this criterion, which obviously can be applied when the decision is made under conditions of risk, the right decision corresponds to the $\max(a_1, a_2, \ldots, a_m)$. For applying this criterion under conditions of uncertainty, i.e. when the probabilities p_i are not known, one may assume that all of them are equal to each other, a simplification which is not always true in practice. In this case equation (1) takes the form

$$a_j = \frac{1}{n} \sum_{i=1}^{n} a_{ij}. \tag{2}$$

In this form the criterion is known as the *criterion of Laplace*.

- ***Minimization of the expected lost opportunities***

Under conditions of risk, if p_i denotes the probability of realization of the event E_i, $i = 1, 2, \ldots, n$, the expected lost opportunity x_j corresponding to the possible decision D_j, $j = 1, 2, \ldots, m$, is defined by

$$x_j = \sum_{i=1}^{n} p_i x_{ij}. \tag{3}$$

According to this criterion, the right decision corresponds to the $\min(x_1, x_2, \ldots, x_m)$. In case of uncertainty one may set again $p_i = \frac{1}{n}$.

Remark 2.1. On the basis of the definition of the lost opportunities it becomes evident that the criteria of the maximization of the expected pay-offs and of the minimization of the expected lost opportunities are equivalent, leading always to the same decision, e.g. see below the case (iii) of Example 2.2.

- ***Criterion of optimism - pessimism***

In this criterion an *optimism* index q_j is assigned to the maximal pay-off, say t_j, of each decision D_j, $j = 1, 2, \ldots, m$. Also the *pessimism index* $1q_j$ is assigned to the minimal pay-off, say s_j, of the same decision. The index q_j either depends on the personal goals of the decision-maker, or it is determined with the help of existing statistical data. Then the expected pay-off a_j of the possible decision j, $j = 1, \ldots, m$, is calculated by the formula

$$a_j = q_j t_j + (1 - q_j) s_j, \tag{4}$$

and the right decision corresponds to the $\max(a_1, a_2, \ldots, a_m)$. This criterion is also referred to as the criterion of Hurwicz.

Example 2.2. The management of an industry must choose the optimal among three methods, say A_1, A_2, A_3, for the production of a good, which will be put on sale at a price of 100 euros per unit. The application of A_1 requires an initial capital of one million euros for buying and setting the necessary equipment, plus 50 euros per unit for the production expenses. The corresponding amounts of money are 1.6 million, 40 euros for A_2 and 3 million, 30 euros for A_3 respectively. The markets research has shown that the probability for a low demand of the good (25000 units) is 10%, for a mediocre demand (100000 units) is 70% and for a high demand (150000 units) is 20%. Further, the optimistic indices for each method of production were estimated to be $q_1 = q_2 = 0.6$ and $q_3 = 0.8$ respectively.

Find which the optimal choice for the industry is by applying the criteria:

i) Maxi min pay-offs,
ii) Mini max lost opportunities,
iii) Maximization of the expected pay-offs or minimization of the expected lost opportunities.
iv) Optimism pessimism.

Solution: Denote by E_1, E_2 and E_3 the events of low, mediocre and high demand of the good respectively.

i) The pay-offs a_{ij} are equal to the revenue from the sale of the good minus the initial capital and the expenses for the production of the good. In case of the event E_1 and the method A_2, for example, one finds that $a_{12} = 25000.100-(1600000+25000.40) = -100000$ euros. The matrix of pay-offs (in thousands of euros) is the following:

$$\begin{array}{c} \\ E_1 \\ E_2 \\ E_3 \end{array} \begin{array}{c} \begin{array}{ccc} A_1 & A_2 & A_3 \end{array} \\ \begin{bmatrix} 250 & -100 & -1250 \\ 4000 & 4400 & 4000 \\ 6500 & 7400 & 7500 \end{bmatrix} \end{array}$$

The minimal pay-offs corresponding to each method of production are 250, -100 and -1250 respectively and the maximal pay-off among them is 250. Therefore, the industry must choose the method A_1.

ii) With the help of the matrix of pay-offs one calculates the lost opportunities; for example, $x_{32} = 7.500-7.400 = 100$, $x_{33} = 7.500-7.500 = 0$, The matrix of the lost opportunities is, therefore, the following:

$$\begin{array}{c} \\ E_1 \\ E_2 \\ E_3 \end{array} \begin{array}{c} \begin{array}{ccc} A_1 & A_2 & A_3 \end{array} \\ \begin{bmatrix} 0 & 350 & 1500 \\ 400 & 0 & 400 \\ 1000 & 100 & 0 \end{bmatrix} \end{array}$$

The maximal lost opportunities for each method of production are 1000, 350 and 1500, with $\min(1.000, 350, 1500) = 350$. Therefore, the industry must choose the method A_2. This decision is more optimistic than the decision made with the help of the previous criterion, since it corresponds to a maximum possible pay-off of 7.400.000 euros, in comparison to the 6.500.000 euros corresponding to the previous decision.

iii) From the problems data it turns out that $p_1 = 0.1$, $p_2 = 0.7$ and $p_3 = 0.2$. Therefore, equation (1) gives that the expected payoffs for each decision are $a_1 = (0.1).250 + (0.7).4000 + (0.2).6500 = 4125$ and similarly $a_2 = 4450$, $a_3 = 4175$. Therefore, since $\max(4125, 4450, 4175) = 4450$, the industry must choose the method A_2.

Also, with the help of equation (3) one finds that the expected lost opportunities for each method of production are $x_1 = 0.(0.1) + 400.(0.7) + 1000.(0.2) = 480$ and similarly $x_2 = 55$ and $x_3 = 430$. Therefore, since $\min(x_1, x_2, x_3) = 55$, the industry must choose again the method A_2 (see Remark 2.1).

iv) The maximal pay-off of the method A_1 is 6500 and the minimal is 250. Therefore, equation (4) gives that $a_1 = (0.6).6500 + (1-0.6).250 = 4000$ and similarly $a_2 = 4400$, $a_3 = 3500$. Therefore, since $\max(a_1, a_2, a_3) = 4400$, the industry must choose the method A_2.

3 Verification of a Decision

As it was already mentioned, the *verification* of a decision is a step of the DM process, which is usually examined separately from its other steps. A method will be presented here for investigating this important step of the DM process by using the *Grade Point Average (GPA)* index.

It is recalled that the GPA index is a weighted mean which is frequently used for assessing a groups quality performance (since greater coefficients are assigned to the higher grades) during a certain activity. For this, consider the qualitative grades $A =$ excellent, $B =$ very good, $C =$ good, $D =$ satisfactory and $F =$ unsatisfactory (failed). Then the GPA index is calculated by the formula

$$\text{GPA} = \frac{0n_F + n_D + 2n_C + 3n_B + 4n_A}{n}. \tag{5}$$

In formula (5) n denotes the total number of the groups members and nA, nB, nC, nD and nF denote the numbers of the groups members that demonstrated excellent, very good, good, satisfactory and unsatisfactory performance respectively [10, Chapter 6]. In case of the worst performance ($nF = n$), formula (5) gives that GPA $= 0$, whereas in case of the ideal performance ($nA = n$) it gives that GPA $= 4$. Therefore, we have in general that $0 \leq$ GPA ≤ 4.

Our method is illustrated with the help of the following example:

Example 3.1. The car industry circulates a new car in the market in two different types, the luxury (L) Class and the regular (R) Class. Six months after the purchase with their cars, the customers were asked to complete a written questionnaire concerning the degree of their satisfaction for their cars. Their answers were divided by the industrys marketing department into the following five categories: $A =$ Fully satisfied customers, $B =$ Very well satisfied customers, $C =$ Satisfied customers, $D =$ Rather satisfied customers and $E =$ Unsatisfied customers. The data collected from the customers answers are depicted in Table 2. What is the general conclusion obtained by the car industry concerning the degree of satisfaction of its customers for their new cars?

Table 2: Questionnaires data

Customers Categories	L Class	R Class
A	60	60
B	30	90
C	30	45
D	30	45
E	20	15
Total	170	255

Solution: Replacing the data of Table 2 to formula (5) one finds that the GPA index concerning the degree of satisfaction of the owners of the L Class and the R Class is equal to $\frac{42}{17} \approx 2.47$ and $\frac{43}{17} \approx 2.529$ respectively. Taking into account that $0 \leq$ GPA ≤ 4, this means that the owners were satisfied with their cars at a percentage of $\frac{2.47x100}{4} \approx 61.75\%$ for the L Class and $\frac{2.529x100}{4} \approx 63.22\%$ for the R Class.

4 Criterion of Bellman and Zadeh for Decision-Making under Fuzzy Conditions

Frequently in everyday life the data of a DM problem are fuzzy; e.g. when a company wants to employ as a sales manager a well-experienced person whose residence is not very far from the companys place. In such cases the traditional techniques of DM, which are based on principles of bivalent logic (yes-no), cannot help effectively in making the right decision. On the contrary, *fuzzy sets (FSs)* and their extensions, due to their nature of including multiple values, offer a rich field of resources for this purpose; e.g. see [4, 11–18], etc.

It is recalled that Zadeh in 1965 extended the concept of the crisp set to that of a FS by replacing the characteristic with the membership function as follows [19]:

Definition 4.1. A FS, say A, in the universal set of the discourse U is of the form $A = \{(x, m(x)) : x \in U\}$, where $m : U \to [0, 1]$ is its membership function. The value $m(x)$ is called the membership degree of x in A, for all x in U. The closer $m(x)$ to 1, the better x satisfies the characteristic property of A.

For example, if A is the FS of the high mountains and $m(x) = 0.7$, then x is a rather high mountain, if $m(x) = 0.4$, then x is a rather low mountain, etc.
Bellman and Zadeh were the first, in 1970, who applied principles of FS theory to DM, their method being known as *Criterion of Bellman and Zadeh for DM* [4].

A DM problem under fuzzy conditions is characterized by its *fuzzy goal* (G) and by the *fuzzy constraints* C_i, $i = 1, 2, \ldots, n$, where n is a positive integer. The steps of the method of Bellman and Zadeh are the following:

- *Choice of the universal set of the discourse U*
- *Fuzzification of the decision problems data*

In this step the fuzzy goal G and the fuzzy constraints C_i are expressed as fuzzy sets (FSs) in U by defining properly the corresponding membership functions m_G and m_{C_i}.

- *Evaluation of the fuzzy data*

The *fuzzy decision F*, expressed as a fuzzy set in U, is equal to the intersection of the FSs G and C_i of U Therefore, the membership function m_F of F is defined by

$$m_F(x) = m_G \cap m_{C_1} \cap \ldots \cap m_{C_i}(x) = \min\{m_G(x), m_{C_1}(x), \ldots, m_{C_2}(x)\}, \quad (6)$$

for all x in U.

- *Defuzzification*

The solution of the problem corresponds to the element x of U having the highest membership degree in F.

The following example illustrates the DM model of Bellman and Zadeh in practice:

Example 4.2. A company is willing to employ as a sales manager the candidate with the best qualifications (G), provided that his/her salary demand is not very high (C_1) and that his/her residence is in a close distance from the companys central offices (C_2). There are four candidates for this position, say , , C and D, with annual salary demands of 29050, 25000, 14050, and 6250 euros respectively. Who of them is the best choice for the company under the fuzzy constraints C_1 and C_2?

Solution: In this problem the universal set of the discourse is the set $U = \{A, B, C, D\}$ of the four candidates. In order to express the fuzzy goal and the fuzzy constraints as FSs in U, one must properly define the corresponding membership functions.

For example, having in mind that there is not any general criterion available for the definition the membership functions, the membership function $m_{C_1} : U \to [0, 1]$ of the fuzzy constraint C_1, may be defined by $m_{C_1} = 1$ for $s(x) < 6000$, $m_{C_1}(x) = 1 - 2x10^{-5}xs(x)$ for $6000 \leq s(x) \leq 30000$ and $m_{C_1}(x) = 0$ for $s(x) > 30000$, where $s(x)$ denotes the salary of the candidate x, for all x in U. Then $m_{C_1}(A) = 1 - 2x0.2905 = 0.419$ and similary $m_{C_1}(B) = 0.5$, $m_{C_1}(C) = 0.719$ and $m_{C_1}(D) = 0.875$. Consequently, the constraint C_1 can be written as a FS in U in the form $C_1 = \{(A, 0.419), (B, 0.5), (C, 0.719), (D, 0.875)\}$.

Assume further that in an analogous way the fuzzy goal G and the fuzzy constraint C_2 were expressed as fuzzy sets in U in the form $G = \{(A, 0.9), (B, 0.6), (C, 0.8), (D, 0.6)\}$ and $C_2 = \{(A, 0.1), (B, 0.9), (C, 0.7), (D, 1)\}$ respectively. Therefore, with the help of equation (6) it is straightforward to check that F can be written as a FS in U in the form $F = \{(A, 0.1), (B, 0.5), (C, 0.7), (D, 0.6)\}$. The highest membership degree in F is 0.7 and corresponds to C. Therefore the candidate C is the best choice for the company.

The fuzzy model of Bellman and Zadeh can be suitably modified to accommodate the relative importance that could exist for the goal and constraints by using *weighting coefficients*, whose sum is always equal to 1. The following example illustrates this case:

Example 4.3. Revisit Example 4.2 and assume that the management of the company, taking into account the existing companys budget and the results of the oral interviews of the four candidates, decided to attach weights $w = 0.5$, $w_2 = 0.2$ and $w_3 = 0.3$ to the goal G and to the constraints C_1 and C_2 respectively. Which will be the best companys choice under these new conditions?

Solution: In this case the membership function of the fuzzy decision F is defined as a linear combination of the weighted goal and constraints of the form

$$m_F(x) = w_1 x m_G(x) + w_2 x m_{C_1}(x) + w_3 x m_{C_2}(x), \tag{7}$$

where $m_G(x)$, $m_{C_1}(x)$, $m_{C_2}(x)$ are the membership degrees in G, C_1 and C_2 respectively of each x in U (see Example 4.2) and the coefficients w_1, w_2 and w_3 are the weights attached to the fuzzy goal and to the fuzzy constraints C_1 and C_2 respectively. Therefore, the membership degree of candidate A in the fuzzy decision F is equal to $m_F(A) = 0.5x0.9 + 0.2x0.419 + 0.3x0.1 = 0.638$. In the same way one finds that $m_F(B) = 0.67$, $m_F(C) = 0.7538$ and $m_F(D) = 0.775$. Therefore, candidate D is the companys best choice in this case.

5 MultipleCriteria Parametric Decision-Making

Following the criterion of Bellman and Zadeh, several other methods were proposed by other researchers for DM in fuzzy environments; e.g. [11–18], etc. Here we will present a hybrid, parametric, multiple-criteria DM method using soft sets, grey numbers and intuitionistic fuzzy sets as tools.

5.1 Decision-Making with Soft Sets

Molodstov introduced in 1999 the concept of *soft set (SS)* for tackling the uncertainty in a parametric manner, not needing, therefore, the definition of a membership function. Namely, a SS is defined as follows [20]:

Definition 5.1. Let E be a set of parameters, let A be a subset of E, and let f be a map from A into the power set $P(U)$ of the universe U. Then the SS (f, A) in U is defined as the set of the ordered pairs $(f, A) = \{(e, f(e)) : e \in A\}$. In other words, an SS is a parametric family of subsets of U. The term "soft" was introduced due to the fact that the form of (f, A) depends on the parameters of A. A FS in U with membership function $y = m(x)$ is a SS in U of the form $(f, [0, 1])$, where $f(a) = \{x \in U : m(x) \geq a\}$ is the corresponding a-cut of the FS, for each a in $[0, 1]$. Consequently the concept of SS is a generalization of the concept of FS. Most notions and operations defined on FSs are extended in a natural way to SSs.

Maji et al. [9] utilized the tabular form of a SS as a tool for DM in a parametric manner. Here this method is illustrated with the following example:

Example 5.2. Let $V = \{H_!, H_2, H_3, H_4, H_5, H_6\}$ be a set of houses and let $Q = \{e_1, e_2, e_3, e_4\}$ be the set of the parameters e_1 = beautiful, e_2 = wooden, e_3 = in the country and e_4 = cheap. Assume that H_1, H_2, H_6 are beautiful, H_2, H_3, H_5, H_6 are wooden, H_3, H_5 are the houses in the country and H_4 is the unique cheap house. Assume further that one is interested in buying a beautiful, wooden and cheap house in the country choosing among the previous six houses. Which is the best choice for the candidate buyer?

Solution: Consider the map $g : Q \to P(V)$ defined by $g(e_1) = \{H_1, H_2, H_6\}$, $g(e_2) = \{H_2, H_3, H_5, H_6\}$, $g(e_3) = \{H_3, H_5\}$ and $g(e_4) = \{H_4\}$ and the

$$\text{SS } (g, Q) = \{(e_1, \{H_1, H_2, H_6\}), (e_2, \{H_2, H_3, H_5, H_6\}), (e_3, \{H_3, H_5\}), (e_4, \{H_4\})\}.$$

The tabular representation of the SS (g, Q), which is shown in Table 3, is formed by assigning the binary elements 1, 0 to each of the houses having (not having) the property described by the corresponding parameter

The *choice value* of each house is calculated by adding the binary elements of the corresponding row of the tabular matrix containing it. The houses H_1 and H_4, therefore, have choice value 1 and all the other houses have choice value 2. Consequently, the buyer must choose one of the houses H_2, H_3, H_5 or H_6.

Table 3: Tabular form of the soft set (g, Q)

	e_1	e_2	e_3	e_4
H_1	1	0	0	0
H_2	1	1	0	0
H_3	0	1	1	0
H_4	0	0	0	1
H_5	0	1	1	0
H_6	1	1	0	0

5.2 Decision-Making Using Grey Numbers in the Decision Matrix

The decision of Example 5.2 was not very helpful for the candidate buyer, since it excluded only two among the six available for sale houses. This gave us the hint to modify the DM method of Maji et al. by using *grey numbers (GNs)* in the tabular form of the corresponding *SS* [6].

Definition 5.3. A GN is understood to be a real number with known boundaries whose exact value is unknown. A GN, say G, is represented with the help of a closed real interval. Namely, we write $G \in [a, b]$, with a, b in the set $\mathbf{R}$ of real numbers. Frequently, however, G is accompanied by a *whitenization function* $g : [a, b] \to [0, 1]$, such that the closer $g(x)$ to 1, the more x approximates the exact value of G, for all x in $[a, b]$.

It is recalled that GNs are used as tools for performing all the necessary calculations in the theory of *grey systems* introduced by Deng in 1982 [21] as an alternative to Zadehs FSs for tackling the existing in real world uncertainty. The known arithmetic of the real intervals [22] is used for performing the arithmetic operations between GNs. Here we will make use of the *addition* of GNs and the *scalar multiplication* of a positive number with a GN, which are defined as follows:

Definition 5.4. Let $G_1 \in [a_1, b_1]$, $G_2 \in [a_2, b_2]$ be two given GNs and let k be a positive number. Then the sum $G_1 + G_2 \in [a_1 + a_2, b_1 + b_2]$ and the scalar product $kG_1 \in [ka_1, kb_1]$. When no whitenization function is assigned to $G \in [a, b]$, then the real number

$$W(G) = \frac{a + b}{2}, \tag{8}$$

is used for approximating the unknown exact value of G.

Revisiting now Example 5.2 one observes that the parameters e_1 and e_4 do have not a bivalent texture. In fact, how beautiful a house is depends on the subjective criteria of each observer, whereas its low or high price depends on the financial ability of the candidate buyer. For this reason, the characterization of the parameters e_1 and e_4 in Table 3 by using the binary elements 0, 1 is not the suitable one. One way to tackle this problem, is to replace the binary elements 0, 1 corresponding to the parameters e_1 and e_5 with GNs. This is illustrated with the following example.

Example 5.5. Revisit Example 5.2 and assume that the candidate buyer, after studying more carefully the existing information about the six available houses, decided to use Table 4 instead of Table 3 for making the final decision, where $G_1 \in [0.85, 1]$, $G_2 \in [0.6, 0.74]$, $G_3 \in [0.5, 0.59]$ and $G_4 \in [0, 0.49]$ are the GNs replacing the binary elements 0, 1 in the columns of e_1 and e_3. Which will be the optimal decision in this case?

Table 4: Revised tabular form of the soft set (g, Q) using grey numbers

	e_1	e_2	e_3	e_5
H_1	G_1	0	0	G_2
H_2	G_1	1	0	G_4
H_3	G_2	1	1	G_2
H_4	G_3	0	0	G_1
H_5	G_4	1	1	G_3
H_6	G_1	1	0	G_4

Solution: In Table 4 one calculates the choice value V_i of the house H_i, $i = 1, 2, 3, 4, 5, 6$ with the help of Definition 5.3 and equation (8) as follows: $V_1 = W(G_1 + G_2) = W([1.45, 1.74]) = \frac{1.45 + 1.74}{2} = 1.595$ and similarly $V_2 = 1 + W(G_1 + G_4) = 2.17$, $V_3 = 2 + W(G_2 + G_2) = 3.34$, $V_4 = W(G_3 + G_1) = 1.47$, $V_5 = 2 + W(G_4 + G_3) = 3.215$, $V_6 = 1 + W(G_1 + G_4) = 2.47$. Therefore, the optimal decision is to buy the house H_3. A second way for tackling this problem is to use *triangular fuzzy numbers (TFNs)* instead of GNs [5]. These two methods are equivalent, providing always the same outcomes.

5.3 Decision-Making Using Intuitionistic Fuzzy Pairs in the Decision Matrix

As we have seen in the previous example, the use of the GNs instead of the binary elements 0, 1 for characterizing the fuzzy parameters that exist in the tabular decision matrix, helps the decision-maker to make a better decision. DM situations, however, appear frequently in everyday life, in which the decision-maker is not sure about the accuracy of these characterizations. In such cases, one way to perform the DM process is to use intuitionistic fuzzy pairs instead of GNs in the tabular matrix of the corresponding soft set [8].

It is recalled that Atanassov in 1986, in order to tackle more effectively the existing in real life uncertainty, added to Zadehs membership degree the degree of non-membership and extended the concept of FS to the concept of *intuitionistic FS (IFS)* as follows [23]:

Definition 5.6. An IFS, say A, in the universe U is of the form $A = \{(x, m(x), n(x)) \;:\; x \in U,\ 0 \leq m(x) + n(x) \leq 1\}$, where $m : U \to [0, 1]$ and $n : U \to [0, 1]$ are its membership and non-membership functions of A respectively.

For example, if A is the set of the high mountains and $m(x) = 0.6$, $n(x) = 0.2$, then there is a 60% belief that x is a high mountain, but at the same time there is a 20% belief that I is not a high mountain. For brevity an IFS is denoted here by $A = \langle n, m\rangle$ and its elements are written in the form of *intuitionistic fuzzy pairs (IFPs)* (m, n), with $m + n \leq 1$. For the needs of the present work we define the addition of $IFPs$ and the scalar multiplication of a positive number with an IFP in the same way as for the ordinary ordered pairs, i.e. as follows:

Definition 5.7. Let $A = \langle m, n\rangle$ be an IFS, let (m_1, n_1), (m_2, n_2) be elements of A and let k be a positive number. Then:

i) The sum $(m_1, n_1) + (m_2, n_2) = (m_1 + m_2, n_1 + n_2)$

ii) ii) The scalar product $k(m_1, n_1) = (km_1, kn_1)$.

It becomes evident that the above defined sum and the scalar product are not closed operations in A, since it can be either $(m_1 + m_2) + (n_1 + n_2) > 1$ or (and) $km_1 + kn_1 > 1$.

We also define the *mean value* of a finite number of IFPS of A in the following way:

Definition 5.8. Let $A = \langle m, n \rangle$ be an IFS and let $(m_1, n_1), (m_2, n_2), \ldots, (m_k, n_k)$ be a finite number of elements of A. Then the mean value of these elements is defined to be $(m, n) = \frac{1}{k}[(m_1, n_1) + (m_2, n_2) + \ldots + (m_k, n_k)]$. It becomes evident that (m, n) is always an element of A.

The use of IFPs in the decision matrix will be illustrated with the following example:

Example 5.9. A company wants to employ a person among six candidates, say A_1, A_2, A_3, A_4, A_5 and A_6. The ideal qualifications for the new employee are to have satisfactory previous experience (p_1), to hold a university degree (p_2), to have a driving license (p_3) and to be young (p_4). Assume that A_2, A_3, A_5, A_6 are the holders of a university degree and that A_3, A_5 are the holders of a driving license. Assume further that the company has difficulty assigning accurate characterizations to the six candidates with respect to the fuzzy parameters p_1 and p_4. It was decided, therefore, to use $IFPs$ instead of the binary elements 0, 1 in the tabular decision matrix. For this, the analysts of the company considered the $IFSs$ of the candidates with satisfactory previous experience and of the young candidates, as well as the trivial IFSs of the holders of a university degree and of a driving license and represented their elements in the form of IFPs. As a result the tabular decision matrix took the form of Table 5.

Table 5: Tabular representation of the DM process using IFPs

	p_1	p_2	p_3	p_4
A_1	$(1, 0)$	$(0, 1)$	$(0, 1)$	$(0.6, 0.1)$
A_2	$(1, 0)$	$(1, 0)$	$(0, 1)$	$(0.2, 0.6)$
A_3	$(0.5, 0.1)$	$(1, 0)$	$(1, 0)$	$(0.6, 0.2)$
A_4	$(0.5, 0.3)$	$(0, 1)$	$(0, 1)$	$(1, 0)$
A_5	$(0.5, 0.4)$	$(1, 0)$	$(1, 0)$	$(0.6, 0.1)$
A_6	$(1, 0)$	$(1, 0)$	$(0, 1)$	$(0.4, 0.2)$

Which will be the best choice for the company?

Solution: In this case the choice value of each candidate A_i, $i = 1, 2, 3, 4, 5, 6$, is equal the *mean value* of the IFPs contained in the row of A_i. With the help of Definitions 5.7 and 5.8, therefore, one finds that the choice value of A_1 is equal to

$$\frac{1}{4}[(1, 0) + 2(0, 1) + (0.6, 0.1)] = \frac{1}{4}(1.6, 2.1) = (0.4, 0.525).$$

In the same way the choice values of A_2, A_3, A_4, A_5 and A_6 can be find to be equal to $(0.55, 0.4)$, $(0.775, 0.075)$, $(0.375, 0.575)$, $(0.775, 0.125)$ and $(0.6, 0.3)$ respectively. The company now may use either an optimistic criterion by choosing the candidate with the greatest membership degree, or a conservative criterion by choosing the candidate with the lower non-membership degree, i.e. one of the candidates A_3 and A_5 in the first case, or the candidate A_3 in the second case. A combination of the two criteria leads finally to the choice of the candidate A_3.

5.4 Decision-Making Using Neutrosophic Triplets in the Decision Matrix

An alternative way for tackling the previous DM problem is to use *neutrosophic sets (NSs)* instead of $IFSs$ writing their elements in the form of *neutrosophic triplets (NTs)* in the tabular decision matrix [7]. In fact, Smarandache in 1995, inspired by the frequently appearing in the everyday life neutralities, like

$\langle tall, medium, short\rangle$, $\langle friend, neutral, enemy\rangle$,$\langle win, draw, defeat\rangle$, etc., introduced the degree of indeterminacy or neutrality and extended the notion of IFS to the notion of NS [24]. The simplest form a NS, known as a *single-valued NS (SVNS)* is defined in the following way [25]:

Definition 5.10. A SVNS, say A, in the universe U has the form

$$A = \{(x, m(x), i(x), n(x)) \ : \ x \in U, \ m(x), i(x), n(x) \in [0, 1], \ 0 \leq m(x) + i(x) + n(x) \leq 3\}.$$

In the SVNS A $m(x)$ is the degree of membership (or truth), $i(x)$ is the degree of indeterminacy (or neutrality) and $n(x)$ is the degree of non-membership (or falsity) of x in A, for all x in U. When $0 \leq m(x) + i(x) + n(x) \leq 1$, then the data about x in A are characterized by incomplete information, when $m(x)+i(x)+n(x) = 1$ by complete and when $m(x)+i(x)+n(x) > 1$ by inconsistent (contradiction relevant) information. A NS may contain simultaneously elements characterized by all these types of information. For brevity we write $A = \langle m, i, n\rangle$ and the elements of A as NTs in the form (m, i, n), with $0 \leq m + i + n \leq 3$. For example, if A is the NS of the high mountains and $(0.6, 0.3, 0.2) \in A$, then there exists a 60% belief that x is a high mountain, but at the same time a 30% belief that x is neither a high nor a low mountain and a 20% belief that it is a low mountain.
The *sum* of NTSs, the *scalar product* of a positive number cross a NT and the mean value of a finite number of NTs of a NS are defined similarly with the corresponding operations for $IFPs$ (see Definitions 5.7 and 5.8). The advantage of using NTs instead of IFPs in the decision matrix is that they enable one to handle data connected incomplete and/or inconsistent information. The following example illustrates this situation.

Example 5.11. Revisiting Example 5.9 assume that the company, due to the existence of incomplete and inconsistent information for some candidates, decided to use NSs instead of IFSs for the formation of the decision matrix. Thus, considering the NSs of the candidates with satisfactory previous experience and of the young candidates, as well as the trivial NSs of the holders of a university degree and of the holders of a driving license and representing their elements in the form of NTs formed the decision matrix shown in Table 6. Which is the best choice for the company in this case?

Table 6: Tabular representation of the DM process using NTs

	p_1	p_2	p_3	p_4
A_1	$(1, 0, 0)$	$(0, 0, 1)$	$(0, 0, 1)$	$(0.6, 0.3, 0.1)$
A_2	$(1, 0, 0)$	$(1, 0, 0)$	$(0, 0, 1)$	$(0.2, 0.2, 0.7)$
A_3	$(0.5, 0.4, 0.2)$	$(1, 0, 0)$	$(1, 0, 0)$	$(0.6, 0.2, 0.1)$
A_4	$(0.5, 0.2, 0.2)$	$(0, 0, 1)$	$(0, 0, 1)$	$(1, 0, 0)$
A_5	$(0.5, 0.1, 0.4)$	$(1, 0, 0)$	$(1, 0, 0)$	$(0.6, 0.3, 0.1)$
A_6	$(1, 0, 0)$	$(1, 0, 0)$	$(0, 0, 1)$	$(0.4, 0.3, 0.2)$

Solution: In this case the choice value of the candidate A_1 is equal to $\frac{1}{4}[(1, 0, 0)+2(0, 0, 1)+(0.6, 0.3, 0.1)] = \frac{1}{4}(1.6, 0.3, 2.1) = (0.4, 0.075, 0.525)$ and in the same way the choice values of A_2, A_3, A_4, A_5 and A_6 are approximately equal to $(0.55, 0.07, 0.425)$, $(0.775, 0.15, 0.075)$, $(0.375, 0.05, 0.55)$, $(0.775, 0.13, 0.125)$ and $(0.6, 0.075, 0.3)$ respectively. Consequently, applying the optimistic criterion the company must choose one of the candidates A_3 or A_5, whereas applying the conservative criterion it must choose the candidate A_3. The final choice of the company, therefore, must be again the candidate A_3, although the indeterminacy degree of candidate A_5 is slightly smaller $(0.13 < 0.15)$.

5.5 Weighted Parametric Decision-Making

Cases appear frequently in DM in which the decision-makers goals are not equally important. In such cases, weight coefficients, whose sum is equal to 1, are assigned to each parameter. Assume, for instance, that the weight coefficients 0.4, 0.3, 0.2 and 0.1 have been assigned to the parameters p_1, p_2, p_3 and p_4 respectively of Example 5.9. Then the weighted choice value of the candidate A_1 is equal to

$$\frac{1}{4}[0.4(1,0) + 0.3(0,1) + 0.2(0,1) + 0.1(0.6,0.1)] = \frac{1}{4}(0.46, 0.51) = (0.115, 0.1275).$$

In the same way one finds that the choice values of the candidates A_2, A_3, A_4, A_5 and A_6 are $(0.18, 0.065)$, $(0.19, 0.015)$, $(0.075, 0.115)$, $(0.19, 0.0425)$ and $(0.185, 0.055)$ respectively. The combination of the two criteria, therefore, shows again that the best decision for the company is to employ the candidate A_3.

Remark 5.12. (i) The parametric DM method presented in this work is of a general character, therefore it can be applied to all the analogous cases of multiple-criteria DM. Other examples that have been already presented in earlier works of the present author are related to decisions for buying a car [5], choosing a new player for a football team [7], etc.
(ii) There is no objective criterion for defining the membership function of a FS, its definition depends on the personal criteria of each observer. The same problem exists for all the extensions of FSs involving membership functions and in particular for the membership, non-membership and indeterminacy functions of the IFSs and of the NSs. As a result, the characterization of the fuzzy parameters p_1 and p_4 in Examples 5.9 and 5.11 using IFPs and NTs respectively was purely based on the companys analysts personal criteria. An analogous problem appears when using GNs (or TFNs) for characterizing the fuzzy parameters in the decision matrix (see Section 5.2), although no whitenization function was used for the corresponding GNs. This is, therefore, a general limitation of the parametric DM method presented in this work.

6 Conclusion

Frequently in everyday life the goal and/or the constraints of a DM problem are expressed in a vague way, characterized by uncertainty. The first who studied DM problems under fuzzy conditions were Bellman and Zadeh in 1970. Since then, several DM methods have been proposed by other researchers using FSs or their extensions as tools. In this work, starting from the traditional DM criteria of bivalent logic and continuing with the fuzzy criterion of Bellman and Zadeh, we also presented a hybrid model for multiple-criteria parametric DM in fuzzy environments. This model improves a DM method of Maji et al. using SSs as tools, by replacing the binary elements 0, 1 in the tabular matrix of the corresponding SS either with GNs (or TFNs), or by IFPs, or by NTs, depending on the form of the corresponding DM problem. In addition, a method was presented, based on the calculation of the GPA index, for the verification of a decision, a step of the DM process, which, due to its special importance, is usually examined separately from its other steps. All the DM methods presented in this work are illustrated by suitable examples, connected to everyday life situations. It seems that suitable combinations of two or more theories related to FS (e.g. SSs with GNs or with IFSs or with NSs in this work) provide better results than each one of these theories alone does. This is, therefore, a promising area for further research.

Conflict of Interest: The author declares no conflict of interest.

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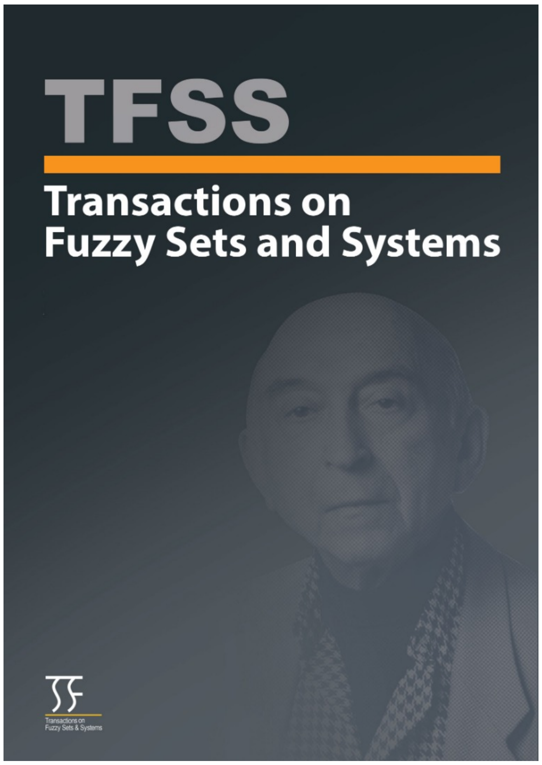

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Stable topology on ideals for residuated lattices

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Stable Topology on Ideals for Residuated Lattices

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Abstract. Residuated lattices are the major algebraic counterpart of logics without contraction rule, as they are more generalized logic systems including important classes of algebras such as Boolean algebras, MV-algebras, BL-algebras, Stonean residuated lattices, MTL-algebras and De Morgan residuated lattices among others, on which filters and ideals are sets of provable formulas. This paper presents a meaningful exploration of the topological properties of prime ideals of residuated lattices. Our primary objective is to endow the set of prime ideals with the stable topology, a topological framework that proves to be more refined than the well-known Zariski topology. To achieve this, we introduce and investigate the concept of pure ideals in the general framework of residuated lattices. These pure ideals are intimately connected to the notion of annihilator in residuated lattices, representing precisely the pure elements of quantales. In addition, we establish a relation between pure ideals and pure filters within a residuated lattice, even though these concepts are not dual notions. Furthermore, thanks to the concept of pure ideals, we provide a rigorous description of the open sets within the stable topology. We introduce the i-local residuated lattices along with their properties, demonstrating that they coincide with local residuated lattices. The findings presented in this study represent an extension beyond previous work conducted in the framework of lattices, and classes of residuated lattices.

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1 Introduction

It is well known that non-classical logic is a formal and useful technique for computer science to deal with fuzzy and uncertain information in classification problems, artificial intelligence, data organization, and formal concept analysis. In this way, several algebraic structures such as MV-algebras, BL-algebras, Gödel algebras, MTL-algebras, De Morgan residuated lattices, and residuated lattices (see [1–4]) have been introduced, and provide an algebraic framework to fuzzy logic and fuzzy reasoning. Among these structures, Pavelka showed in [5] that residuated lattices are more generalized logic systems on which filters and ideals are sets of provable formulas. The study of their algebraic properties is therefore deciphered through the notions of ideal and filter.

In the framework of residuated lattices, previous works, such as [3, 6–9], were more focused on filters. In [9], Busneag et al. endowed the set of prime filters with the spectral topology and used the concepts of co-annihilator as well as pure filter to study the stable topology. On the other hand, the notion of ideal was recently introduced in residuated lattices by Busneag et al. in [10], generalizing the one in BL-algebras. A

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year later, Luo ([11]) pursued this by bringing in another definition for ideals of residuated lattices, with which he introduced a congruence relation associated with ideals. That congruence relation was later on revised by Liu et al. ([12]), who set forth the concept of fuzzy ideals. In [2], Holdon established the equivalence between Luo's and Busneag's definition of an ideal of a residuated lattice, providing additional properties on ideals of residuated lattices. Motivated by the fact that the Zariski topology allows tools from topology to be used to interpret algebraic varieties, Dana Piciu ([13]) introduced the Zariski topology on ideals of residuated lattices. One can observe that in that topology, clopen sets are *stable*, that is, they are simultaneously stable under ascent and descent. The question that arises: does it exist stable sets other than clopen sets? If so, how can we describe them?

To answer this question, we introduce and study pure ideals in residuated lattices, based on the notion of annihilator in residuated lattices which generalize the one done in De Morgan residuated lattices [14] and MV-algebras ([15, 16]).

The paper is organized as follows: in Section 2, we recall basic notions of residuated lattices and describe some properties that will be needed in the sequel. In Section 3, we introduce the concept of pure ideal in residuated lattices and provide some of its properties. Moreover, we discuss the relationship between pure ideals and pure filters of a residuated lattice. Section 4 is devoted to the characterization of open stable sets by the means of pure ideals of a residuated lattice, setting up the stable topology.

2 Preliminaries

A *residuated lattice* ([4, 9, 11, 12]) is an algebraic structure $(L; \vee, \wedge, \odot, \rightarrow, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$, where:

(L1) $(L; \vee, \wedge, 0, 1)$ is a bounded lattice;

(L2) $(L; \odot, 1)$ is a commutative ordered monoid;

(L3) For every $x, y, z \in L, x \leq y \rightarrow z$ iff $x \odot y \leq z$.

In what follows, unless otherwise specified, by $\mathcal{L}$ we denote a residuated lattice $(L;\ \vee,\ \wedge,\ \odot,\ \rightarrow,\ 0,\ 1)$. A subset X of L is *proper* if $X \neq L$. Every residuated lattice $\mathcal{L}$ has the *negation* operation defined by $x' := x \rightarrow 0$, for all $x \in L$.

We will use the notations

$$x^n := \underbrace{x \odot \cdots \odot x}_{n\ \ times}, \text{ for any } x \in L \text{ and } n \geq 1;$$

$$X' := \{x' : x \in X\}, \text{ for any } X \subseteq L.$$

Recall from [2, 8] that a residuated lattice $\mathcal{L}$ is called:

(i) a *De Morgan* residuated lattice if the De Morgan law $(x \wedge y)' = x' \vee y'$, for all $x, y \in L$ holds;

(ii) an *MTL-algebra* if it satisfies $(x \rightarrow y) \vee (y \rightarrow x) = 1$, for all $x, y \in L$ *(prelinearity)*;

(iii) a *BL-algebra* if it is an MTL-algebra where $x \wedge y = x \odot (x \rightarrow y)$, for all $x, y \in L$ (*divisibility*);

(iv) an *MV-algebra* if it is a BL-algebra that verifies $x'' = x$, for all $x \in L$ (*double negation*).
When $x'' = x$ for all x in L, we say that $\mathcal{L}$ is *regular*.

The following rules of calculus in residuated lattices shall be needed in the sequel.

Proposition 2.1. *[2, 4, 8–10] Let L be a residuated lattice. Then, for all $x, y, z \in L$, we have:*

(P1) $x \leq y$ *iff* $x \to y = 1,\ x \odot y \leq x \wedge y,\ x \odot y \leq x \to y$;

(P2) $x \to (y \to z) = (x \odot y) \to z = y \to (x \to z)$;

(P3) $x \to y \leq (x \odot y')',\ \ x \odot (y \to z) \leq y \to (x \odot z) \leq (x \odot y) \to (x \odot z)$;

(P4) *If* $x \leq y$, *then* $y \to z \leq x \to z\ ,\ z \to x \leq z \to y,\ x \odot z \leq y \odot z,\ y' \leq x'$;

(P5) $x \odot (x \to y) \leq y, \ \ x \leq (x \to y) \to y$ *and* $((x \to y) \to y) \to y = x \to y$;

(P6) $1 \to x = x, \ \ x \to x = 1, \ \ x \to 1 = 1, \ \ x \leq y \to x,$
$x \to y \leq y' \to x', \ \ x \leq x'', \ \ x''' = x'$;

(P7) $x \odot x' = 0,\ x \odot y = 0$ *iff* $x \leq y'$;

(P8) $(x \odot y)' = x \to y' = y \to x' = x'' \to y', \ \ (x \wedge y)' \geq x' \vee y',$
$(x \vee y)' = x' \wedge y', \ \ 0' = 1$ *and* $1' = 0$;

(P9) $x \odot (y \vee z) = (x \odot y) \vee (x \odot z), x \odot (y \wedge z) \leq (x \odot y) \wedge (x \odot z),\ x \vee (y \odot z) \geq (x \vee y) \odot (x \vee z)$ *and hence* $(x \vee y)^{mn} \leq x^m \vee y^n$, *for every* $n, m \geq 1$;

(P10) $x \to (y \wedge z) = (x \to y) \wedge (x \to z),\ (x \vee y) \to z = (x \to z) \wedge (y \to z),$
$(x \wedge y) \to z \geq (x \to z) \vee (y \to z),\ x \to (y \vee z) \geq (x \to y) \vee (x \to z)$;

(P11) $x' \odot y' \leq (x \odot y)',\ x'' \odot y'' \leq (x \odot y)'',\ x' \odot y' \leq (x' \to y)'$ *and* $x, y \leq (x' \odot y')'$;

(P12) $x \vee y = 1$ *implies* $x \odot y = x \wedge y$ *and* $x^n \vee y^n = 1$, *for every* $n \geq 1$.

The operation $\oplus$ defined on L by $x \oplus y = (x' \odot y')' = x' \to y''$, for all $x, y \in L$ is commutative, associative, and compatible with the order [10].

For any $x \in L$, $nx := \underbrace{x \oplus \cdots \oplus x}_{\text{n times}}$, $n \geq 1$.

Recall from [13, 17] that

(P13) For every $m, n \geq 2$,

$$[(x')^n]' = nx, x \wedge (ny) \leq n(x'' \wedge y'') \text{ and } (mx) \wedge (ny) = (mn)(x'' \wedge y'').$$

The operation $\oslash$ defined for every $x, y \in L$ by $x \oslash y := x' \to y$ is neither associative nor commutative and is called the pseudo-addition (see [11]). We shall have in mind that it is compatible with the order.

Remark 2.2. We easily see that the operation $\oslash$ verifies $x \oslash (y \wedge z) = (x \oslash y) \wedge (x \oslash z)$ and $x \oslash (y \oslash z) = y \oslash (x \oslash z)$, for every $x, y, z \in L$.

We recall that a nonempty subset F of $\mathcal{L}$ is called a *filter* ([9]) if it verifies:
(F1) For every $x, y \in L$, if $x \leq y$ and $x \in F$, then $y \in F$;
(F2) For every $x, y \in F$, $x \odot y \in F$.
A filter F of $\mathcal{L}$ is *proper* if $F \neq L$ (i.e., $0 \notin F$).

A *deductive system* of a residuated lattice $\mathcal{L}$ is a nonempty subset F of $\mathcal{L}$ containing 1 such that for all $x, y \in L$, $x \to y \in F$ and $x \in F$ imply $y \in F$.

It is known that in a residuated lattice, filters and deductive systems coincide.

A filter M of $\mathcal{L}$ is called a *maximal filter* if it is a maximal element of the set of all proper filters of $\mathcal{L}$. A residuated lattice $\mathcal{L}$ is called *local* if it has a unique maximal filter ([9]).
From [11], a nonempty subset I of a residuated lattice $\mathcal{L}$ is said to be an *ideal* of $\mathcal{L}$ if the following properties hold:

(I1) For every $x, y \in I$, $x \oslash y \in I$;

(I2) For every $x, y \in L$, if $x \leq y$ and $y \in I$, then $x \in I$.

An ideal I of $\mathcal{L}$ is *proper* if $I \neq L$ (i.e., $1 \notin I$). I is a *maximal ideal* of $\mathcal{L}$ if it is not contained in any other proper ideal of $\mathcal{L}$ ([13]). A residuated lattice is called *i-local* if it has a unique maximal ideal.
Holdon proved the following proposition.

Proposition 2.3. *[2] A nonempty subset I of a residuated lattice $\mathcal{L}$ is an ideal of $\mathcal{L}$ if and only if:*

(I'1) For every $x, y \in I$, $x \oplus y \in I$;

(I2) For every $x, y \in L$, if $x \leq y$ and $y \in I$, then $x \in I$.

We denote by $\mathcal{I}(\mathcal{L})$ the set of ideals of $\mathcal{L}$. We shall notice that $\{0\}$ and L are trivial ideals of $\mathcal{L}$, and each ideal of $\mathcal{L}$ contains 0.

Proposition 2.4. *[2] Let L be a residuated lattice, and I an ideal of $\mathcal{L}$. Then, $x \in I$ iff $x'' \in I$, for every $x \in L$.*

We recall that an algebraic structure $(L; \wedge, \vee, \odot, 0, 1)$ is a *quantale* if $(L; \wedge, \vee, 0, 1)$ is a complete lattice and $(L; \odot)$ a semigroup such that the operator $\odot$ verifies the infinite distributive laws: $a \odot \bigvee X = \bigvee\{a \odot x : x \in X\}$, for all $a \in L$ and $X \subseteq L$ ([18, 19]). An element $a \in L$ is said to be *compact* if for every $X \subseteq L$ such that $a \leq \bigvee X$, there is a finite subset $X_1 \subseteq X$ such that $a \leq \bigvee X_1$.

Recall also from [20] that a *Heyting algebra* is a lattice $(L; \wedge, \vee)$ with 0 in which for every $x, y \in L$, there is an element $x \to y := \bigvee\{a : a \wedge x \leq y\} \in L$, called the *pseudocomplement* of x with respect to y. We say that L is *pseudocomplemented* if every element of L is pseudocomplemented with respect to 0. For any $x \in L$, we will denote by x^* the pseudocomplement of x with respect to 0. Note that a complete Heyting algebra is a quantale in which the operators $\odot$ and $\wedge$ coincide [21]. It is also known that a residuated lattice $\mathcal{L}$ is a Heyting algebra iff $x \odot y = x \wedge y$, for every $x, y \in L$ (see [2]).

An element x of a quantale L is called *pure* if for every compact element a of L, $a \leq x$ implies $x \vee a^* = 1$ (see [18, 22]).

Given a nonempty subset X of L, the least ideal of $\mathcal{L}$ containing X (called the *ideal generated* by X) will be denoted $\langle X \rangle$, and for all $x \in L$, $\langle \{x\} \rangle$ will be denoted $\langle x \rangle$.

Proposition 2.5. *[13] Let $\mathcal{L}$ be a residuated lattice and $x \in L$. Then,*

(i) $\langle X \rangle := \{a \in L : a \leq x_1 \oplus ... \oplus x_n$, *for some* $n \geq 1$ *and* $x_1, x_2, ..., x_n \in X\}$. *Particularly,* $\langle x \rangle = \{a \in L : a \leq nx$, *for some* $n \geq 1\}$.

(ii) For any $I \in \mathcal{I}(L)$, if $x \notin I$, then
$\langle I \cup \{x\} \rangle = \{a \in L : a \leq i \oplus nx$, *for some* $i \in I$ *and* $n \geq 1\}$.

(iii) $(\mathcal{I}(L), \wedge, \vee, \to)$ *is a complete Heyting algebra*
where $I \wedge J := I \cap J$,
$I \vee J = \langle I \cup J \rangle := \{x \in L : x \leq i \oplus j, i \in I, j \in J\}$ *and*
$I \to J := \{x \in L : \langle x \rangle \cap I \subseteq J\}$, *for* $I, J \in \mathcal{I}(L)$.

Definition 2.6. [12, 13] Let P be a proper ideal of a residuated lattice $\mathcal{L}$. Then,

(i) P is called a *prime ideal* of $\mathcal{L}$ if P is a prime element of $(\mathcal{I}(\mathcal{L}), \wedge, \vee, \to)$, that is, if I, J are ideals of $\mathcal{L}$ and $I \cap J \subseteq P$, then $I \subseteq P$ or $J \subseteq P$.

(ii) P is a *prime ideal of the second kind* of $\mathcal{L}$ if for every $x, y \in A$, $x \wedge y \in P$ implies $x \in I$ or $y \in P$.

(iii) We say that P is a *prime ideal of third kind* of $\mathcal{L}$ if for all $x, y \in L$, $(x \to y)' \in P$ or $(y \to x)' \in P$.

(iv) A prime ideal P which is minimal in the poset of prime ideals containing an ideal I is called a *minimal prime ideal belonging to I*. A minimal prime ideal belonging to $\{0\}$ is called *minimal prime ideal*. In other words, P is a *minimal prime ideal* if P is prime, and for every prime ideal Q, if $Q \subseteq P$, then $P = Q$.

We denote by $Max_{Id}(\mathcal{L})$, $Spec_{Id}(\mathcal{L})$, and by $Min_{Id}(\mathcal{L})$ the set of maximal ideals of $\mathcal{L}$, the set of all prime ideals of $\mathcal{L}$, and the set of minimal prime ideals of $\mathcal{L}$, respectively. Note that $Max_{Id}(\mathcal{L}) \subseteq Spec_{Id}(\mathcal{L})$, and $Min_{Id}(\mathcal{L}) \subseteq Spec_{Id}(\mathcal{L})$ (see [13]).

Proposition 2.7. *[2, 13] Let M be a proper ideal of $\mathcal{L}$. Then, the following are equivalent:*

(i) $M \in Max_{Id}(\mathcal{L})$

(ii) For any $x \in L$, $x \notin M$ iff $(nx)' \in M$, for some natural number $n \geq 1$

(iii) For all $x \notin M$ there is $y \in M$ and $n \geq 1$ such that $y \oplus (nx) = 1$

It follows from Zorn's lemma that every proper ideal of a residuated lattice is contained in a maximal ideal.

The next proposition characterizes prime ideals of residuated lattices.

Proposition 2.8. *[13] Let P be a proper ideal of $\mathcal{L}$. Then, the following are equivalent:*

(i) P is prime.

(ii) $x'' \wedge y'' \in P$ implies $x \in P$ or $y \in P$, for all $x, y \in A$

(iii) If $I, J \in \mathcal{I}(\mathcal{L})$ and $I \cap J = P$, then $I = P$ or $J = P$.

Obviously, every prime ideal of third kind of $\mathcal{L}$ is a prime ideal of second kind of $\mathcal{L}$. The converse, however, is not true, see [12].

Moreover, every prime ideal of second kind of $\mathcal{L}$ is also a prime ideal of $\mathcal{L}$. But the converse is not always guaranteed. Nevertheless, these three types of prime ideals coincide in a De Morgan residuated lattice (see [13]).

Consequently, the results presented in this study, using prime ideals of residuated lattices, constitute an extension of what was done with prime ideals of the second kind in [2, 23].

Before stating the prime ideal theorem, recall that a nonempty subset F of L is a *lattice filter* (or *ℓ-filter*) of $\mathcal{L}$ if:

(i) $\forall x, y \in F, x \wedge y \in F$;

(ii) $\forall x \in F, \forall y \in L, x \leq y \Rightarrow y \in F$.

If F is a filter of $\mathcal{L}$, then F is also a lattice filter of $\mathcal{L}$. But the converse is not always true ([9]).

Theorem 2.9. *(Prime ideal theorem) [17] Let $\mathcal{L}$ be a residuated lattice. If I is an ideal and F is a lattice filter of $\mathcal{L}$ such that $I \cap F = \emptyset$, then there exists a prime ideal P of $\mathcal{L}$ such that $I \subseteq P$ and $P \cap F = \emptyset$.*

As a direct consequence of the prime ideal theorem, for any proper ideal I of $\mathcal{L}$, we have $I = \cap\{P \in spec_{Id}(\mathcal{L}) : I \subseteq P\}$.

Proposition 2.10. *[13] For every ideal I of $\mathcal{L}$ and $x \in A \setminus I$, there is a minimal prime ideal P such that $I \subseteq P$ and $x \notin P$. Singularly, for every $x \in L$, there exists a minimal prime ideal P such that $x \notin P$, whenever $x \neq 0$.*

For any nonempty subset X of L, the ideal

$$X^{\perp} := \{a \in L : x'' \wedge a'' = 0, \text{ for all } x \in X\}$$

is called the *annihilator* of X in $\mathcal{L}$ (see moi, moi2). For all $x \in L$, $\{x\}^{\perp}$ will simply be denoted $x^{\perp}$.

Recall that for any ideal I of $\mathcal{L}$, $I^{\perp}$ is the pseudocomplement of I in $(\mathcal{I}(\mathcal{L}); \wedge, \vee, \rightarrow)$, that is $I^{\perp} = I^{*}$. Recall also that the set $\{a \in L : x \wedge a = 0, \text{ for all } x \in X\}$ is not always an ideal of $\mathcal{L}$ as shown in [24]. However, the above definition of annihilator in residuated lattices has the benefit of generalizing the existing one in subclasses of residuated lattices such as De Morgan residuated lattices, MTL-algebras, BL-algebras, MV-algebras.

Below are some properties of annihilators in residuated lattices.

Lemma 2.11. *[24] Let $x, a, b \in L$. Then, we have:*

(i) $1^{\perp} = \{0\}$, $0^{\perp} = L$;

(ii) *If* $a \leq b$, *then* $b^{\perp} \subseteq a^{\perp}$;

(iii) $a^{\perp} \cap b^{\perp} = (a \vee b)^{\perp}$;

(iv) $a^{\perp} \cup b^{\perp} \subseteq (a \wedge b)^{\perp}$;

(v) *If* $x \in a^{\perp}$, *then* $a \leq x'$ *and* $x \leq a'$.

In order to make the paper self-contained, we recall the following result.

Theorem 2.12. *[24] Let X, Y be nonempty subsets of L. Then,*

(i) $X \subseteq Y$ *implies* $Y^{\perp} \subseteq X^{\perp}$;

(ii) $X^{\perp}$ *is an ideal. In addition if* $X \neq \{0\}$, *then* $X^{\perp}$ *is a proper ideal;*

(ii) $L^{\perp} = \{0\}$;

(iii) $X \subseteq X^{\perp\perp}$;

(iv) $X^{\perp} = X^{\perp\perp\perp}$;

(v) $X \cap X^{\perp} \subseteq \{0\}$;

(vi) $X^{\perp} \cup Y^{\perp} \subseteq (X \cap Y)^{\perp}$;

(vii) $(X \cup Y)^{\perp} = X^{\perp} \cap Y^{\perp}$;

(viii) $\langle X \rangle^{\perp} = X^{\perp}$. *Particularly,* $\emptyset^{\perp} = L$;

(ix) *If* $X \subseteq L$, *then* $\langle X \rangle \cap X^{\perp} = \{0\}$;

(x) $X^{\perp} = \bigcap_{x \in X} x^{\perp}$.

Let $Ann(\mathcal{L}) = \{X^{\perp}, X \subseteq L\}$ be the set of annihilators of $\mathcal{L}$. Since $X^{\perp} = \langle X \rangle^{\perp}$, we have $Ann(\mathcal{L}) = \{I^{\perp}, I \in \mathcal{I}(\mathcal{L})\}$. Then, $(Ann(\mathcal{L}), \wedge, \vee_{Ann(\mathcal{L})}, ^{\perp}, \{0\}, L)$ is a complete boolean algebra where $I \wedge J := I \cap J$ and $I \vee_{Ann(\mathcal{L})} J := (I \cup J)^{\perp\perp}$, for all $I, J \in Ann(\mathcal{L})$ (see [25]).

3 Pure Ideals of Residuated Lattices

The notion of pure ideal has been studied in rings by De Marco ([26]), in distributive lattices by Georgescu and Voiculescu ([27]), as well as in MV-algebras by Cavaccini et al. ([15]), and in De Morgan residuated lattices by Holdon ([23]) and Mihaela ([14]). In this section, we introduce the notion of pure ideal in residuated lattices using the concept of annihilator and explore some of its properties.

For any ideal I of a residuated lattice $\mathcal{L} = (L; \vee, \wedge, \odot, \rightarrow, 0, 1)$, we define

$$\sigma(I) := \{x \in L : \text{ there are } a \in I \text{ and } b \in x^{\perp} \text{ such that } a \oplus b = 1\}.$$

Proposition 3.1. *Let I be an ideal of $\mathcal{L}$, $\sigma(I)$ is an ideal of $\mathcal{L}$ and $\sigma(I) \subseteq I$.*

Proof. Since $0 \in I, 1 \in 0^{\perp}$ and $0 \oplus 1 = 1$, we obtain $0 \in \sigma(I)$. Thus $\sigma(I) \neq \emptyset$.
Let $x_1, x_2 \in L$, such that $x_1 \leq x_2$ and $x_2 \in \sigma(I)$. Then, there are $a_2 \in I$, $b_2 \in x_2^{\perp}$ such that $a_2 \oplus b_2 = 1$. From Lemma 2.11 (ii), $x_1 \leq x_2$ implies that $x_2^{\perp} \subseteq x_1^{\perp}$. Then, $b_2 \in x_1^{\perp}$ and $x_1 \in \sigma(I)$.

In addition, if $x_1, x_2 \in \sigma(I)$, then, there are $a_1, a_2 \in I$, $b_1 \in x_1^{\perp}, b_2 \in x_2^{\perp}$ such that $a_1 \oplus b_1 = 1 = a_2 \oplus b_2$.

Consider $a = a_1 \oplus a_2$ and $b = b_1 \wedge b_2$. Then, $a \in I$ from Proposition 2.3 (I'1).
Let us show that $b \in (x_1 \oslash x_2)^{\perp}$ and $a \oplus b = 1$.

$$\begin{aligned}
(x_1 \oslash x_2)'' \wedge b'' &= [(x_1' \rightarrow x_2)' \vee b']', && \text{from } (P8)\\
&= [(x_1' \rightarrow x_2)' \vee (b_1 \wedge b_2)']', \\
&\leq [(x_1' \rightarrow x_2)' \vee (b_1' \vee b_2')]', && \text{from } (P4) \text{ and } (P8)\\
&\leq [(x_1' \odot x_2') \vee (b_1' \vee b_2')]', && \text{from } (P4) \text{ and } (P11)\\
&\leq [((b_1' \vee b_2') \vee x_1') \odot ((b_1' \vee b_2') \vee x_2')]', && \text{from } (P4) \text{ and } (P9)\\
&= ((b_1' \vee b_2') \vee x_1') \rightarrow ((b_1' \vee b_2') \vee x_2')', && \text{from } (P8)\\
&= ((b_1' \vee b_2') \vee x_1') \rightarrow ((b_1' \vee b_2')' \wedge x_2''), && \text{from } (P8)\\
&= ((b_1' \vee b_2') \vee x_1') \rightarrow ((b_1'' \wedge b_2'') \wedge x_2''), && \text{from } (P8)\\
&= ((b_1' \vee b_2') \vee x_1') \rightarrow ((x_2'' \wedge b_2'') \wedge b_1''), \\
&= ((b_1' \vee b_2') \vee x_1') \rightarrow 0, && \text{since } b_2 \in x_2^{\perp}\\
&= ((b_1' \vee b_2') \vee x_1')', \\
&= (b_1' \vee b_2')' \wedge x_1'', && \text{from } (P8)\\
&= (b_1'' \wedge b_2'') \wedge x_1'', && \text{from } (P8)\\
&= (b_1'' \wedge x_1'') \wedge b_2'', \\
&= 0.
\end{aligned}$$

Therefore, $(x_1 \oslash x_2)'' \wedge b'' = 0$, which means that $b \in (x_1 \oslash x_2)^{\perp}$.

Also,

$$\begin{aligned}
a \oplus b &= (a_1 \oplus a_2) \oplus (b_1 \wedge b_2), \\
&= a_1 \oplus (a_2 \oplus (b_1 \wedge b_2)), && \text{since } \oplus \text{ is associative} \\
&= a_1 \oplus (a_2' \odot (b_1 \wedge b_2)')', \\
&= a_1 \oplus (a_2' \to (b_1 \wedge b_2)''), && \text{from } (P8) \\
&\geq a_1 \oplus (a_2' \to (b_1 \odot b_2)''), && \text{from } (P1) \text{ and } (P4) \\
&\geq a_1 \oplus (a_2' \to (b_1'' \odot b_2'')), && \text{from } (P11) \text{ and } (P4) \\
&\geq a_1 \oplus (b_1'' \odot (a_2' \to b_2'')), && \text{from } (P3) \\
&= a_1 \oplus (b_1'' \odot (a_2 \oplus b_2)), \\
&= a_1 \oplus (b_1'' \odot 1), && \text{from hypothesis} \\
&= a_1 \oplus b_1''
\end{aligned}$$

and we obtain $a \oplus b \geq a_1 \oplus b_1 = 1$ because $\oplus$ is compatible with the lattice order and from the hypothesis. Thus, $a \oplus b = 1$, and it follows that $(x_1 \oslash x_2) \in \sigma(I)$. Hence, $\sigma(I)$ is an ideal of $\mathcal{L}$.

Now, let us show that $\sigma(I) \subseteq I$.
Let $x \in \sigma(I)$. Then, there are $a \in I$, $b \in x^{\perp}$ (i.e., $x'' \wedge b'' = 0$) such that $a \oplus b = 1$. We have: $x'' = x'' \wedge 1 = x'' \wedge (a \oplus b) = x'' \wedge (a' \odot b')' \overset{(P8)}{=} [x' \vee (a' \odot b')]' \overset{(P4),\,(P9)}{\leq} [(x' \vee a') \odot (x' \vee b')]' \overset{(P8)}{=} (x' \vee a') \to (x' \vee b')' \overset{(P8)}{=} (x' \vee a') \to (x'' \wedge b'') = (x' \vee a') \to 0 = (x' \vee a')' \overset{(P8)}{=} (x'' \wedge a'')$, i.e., $x'' \leq x'' \wedge a''$. Thus, $x'' = x'' \wedge a''$, which implies that $x'' \leq a''$. Since $a'' \in I$, then $x'' \in I$. Therefore, $x \in I$. $\square$

The following lemma highlights some properties of the ideal $\sigma(I)$.

Lemma 3.2. *Let L be a residuated lattice and $I,\ J$ two ideals of L. Then,*

(i) $I \subseteq J$ *implies* $\sigma(I) \subseteq \sigma(J)$;

(ii) $\sigma(I \cap J) = \sigma(I) \cap \sigma(J)$;

(iii) $\sigma(I) \vee \sigma(J) \subseteq \sigma(I \vee J)$;

(iv) $\sigma(\sigma(I)) = \sigma(I)$.

Proof.

(i) Straightforward.

(ii) From (i) we obtain $\sigma(I \cap J) \subseteq \sigma(I) \cap \sigma(J)$.
On the other hand, let $x \in \sigma(I) \cap \sigma(J)$. Then, there are $a_1 \in I$, $a_2 \in J$, and $b_1, b_2 \in x^{\perp}$ such that $a_1 \oplus b_1 = 1 = a_2 \oplus b_2$. Set $a = a_1 \wedge a_2$, and $b = b_1 \oplus b_2$. Since $x^{\perp}$, I and J are ideals, it follows that

$b \in x^{\perp}$, $a \in I \cap J$. Moreover, we have:

$$\begin{aligned}
a \oplus b &= b \oplus a, \text{ since } \oplus \text{ is commutative} \\
&= (b_1 \oplus b_2) \oplus (a_1 \wedge a_2), \\
&= b_1 \oplus (b_2 \oplus (a_1 \wedge a_2)), \text{ since } \oplus \text{ is associative} \\
&= b_1 \oplus (b_2' \odot (a_1 \wedge a_2)')', \\
&= b_1 \oplus (b_2' \to (a_1 \wedge a_2)''), \text{ from } (P8) \\
&\geq b_1 \oplus (b_2' \to (a_1 \odot a_2)''), \text{ from } (P1) \text{ and } (P4) \\
&\geq b_1 \oplus (b_2' \to (a_1'' \odot a_2'')), \text{ from } P(11) \text{ and } (P4) \\
&\geq b_1 \oplus (a_1'' \odot (b_2' \to a_2'')), \text{ from } (P3) \\
&= b_1 \oplus (a_1'' \odot (b_2 \oplus a_2)), \text{ from the definition of } \oplus \\
&= b_1 \oplus (a_1'' \odot 1), \text{ since } b_2 \oplus a_2 = 1 \\
&= b_1 \oplus a_1'', \\
&\geq b_1 \oplus a_1, \text{ since } \oplus \text{ is compatible with the lattice order} \\
&= 1, \text{ from hypothesis}
\end{aligned}$$

Then, $a \oplus b = 1$. Therefore, there are $b \in x^{\perp}$, $a \in I \cap J$ such that $a \oplus b = 1$. It follows that $x \in \sigma(I \cap J)$. Hence, $\sigma(I) \cap \sigma(J) \subseteq \sigma(I \cap J)$.

(iii) Straightforward from (i) and the fact that $\sigma(I) \vee \sigma(J) = \langle \sigma(I) \cup \sigma(J) \rangle$.

(iv) It follows from Proposition 3.1 that $\sigma(\sigma(I)) \subseteq \sigma(I)$.
Conversely, let $x \in \sigma(I)$. Then, there are $a \in I$ and $b \in x^{\perp}$ such that $a \oplus b = 1$. We have $x \oplus b = b' \to x'' \overset{(P4)}{\geq} b' \to (x \odot a)'' \overset{(P4),\,(P11)}{\geq} b' \to (x'' \odot a'') \overset{(P3)}{\geq} x'' \odot (b' \to a'') = x'' \odot (b \oplus a) = x'' \odot 1 = 1$.
Therefore, $x \oplus b = 1$ with $x \in \sigma(I)$ and $b \in x^{\perp}$. Hence, $x \in \sigma(\sigma(I))$.

□

In light of Proposition 3.1, we define the concept of pure ideal in residuated lattices.

Definition 3.3. Let $\mathcal{L}$ be a residuated lattice. Then, I is a *pure ideal* of $\mathcal{L}$ if I is an ideal of $\mathcal{L}$ such that $\sigma(I) = I$.

Remark 3.4. We observe that $\{0\}$ and L are trivial pure ideals of $\mathcal{L}$.

We denote by $\mathcal{I}_{\sigma}(\mathcal{L})$ the set of pure ideals of $\mathcal{L}$. As an illustration of Definition 3.3, we have this example.

Example 3.5. Consider the residuated lattice $\mathcal{L}_1 = (L; \vee, \wedge, \odot, \to, 0, 1)$ where the underlying poset is depicted in Figure 1, and the operations $\to$ and $\odot$ are given in Table 1 ([28])

The only proper ideals of $\mathcal{L}_1$ are $I = \{0, d\}$ and $J = \{0, a, b, c\}$. We have: $0^{\perp} = L$, $a^{\perp} = b^{\perp} = c^{\perp} = \{0, d\}$, $d^{\perp} = \{0, a, b, c\}$, $e^{\perp} = \{0\}$, $f^{\perp} = \{0\}$, $1^{\perp} = \{0\}$. Thus, $\sigma(I) = \{0, d\} = I$, and $\sigma(J) = \{0, a, b, c\} = J$. Hence I and J are pure ideals of $\mathcal{L}_1$.

The next example shows that not all ideals of residuated lattices are pure ideals.

Example 3.6. Let $\mathcal{L}_2 = (L; \vee, \wedge, \odot, \to, 0, 1)$ be the residuated lattice ([29]) whose associated Hasse diagram is depicted in Figure 2, and the operations $\to$ and $\odot$ given in Table 2.

For $I = \{0, d, e, f\}$, we have $\sigma(I) = \{0\} \neq I$, meaning that I is not a pure ideal of $\mathcal{L}_2$.

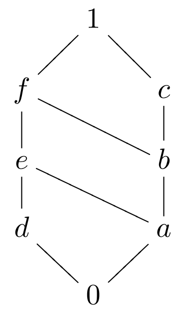

Figure 1: Hasse diagram of $\mathcal{L}_1$

Table 1: Operation tables of $\rightarrow$ and $\odot$ for $\mathcal{L}_1$ in Example 3.5

$\rightarrow$	0	a	b	c	d	e	f	1
0	1	1	1	1	1	1	1	1
a	d	1	1	1	d	1	1	1
b	d	f	1	1	d	f	1	1
c	d	e	f	1	d	e	f	1
d	c	c	c	c	1	1	1	1
e	0	c	c	c	d	1	1	1
f	0	b	c	c	d	f	1	1
1	0	a	b	c	d	e	f	1

$\odot$	0	a	b	c	d	e	f	1
0	0	0	0	0	0	0	0	0
a	0	a	a	a	0	a	a	a
b	0	a	a	b	0	a	a	b
c	0	a	b	c	0	a	b	c
d	0	0	0	0	d	d	d	d
e	0	a	a	a	d	e	e	e
f	0	a	a	b	d	e	e	f
1	0	a	b	c	d	e	f	1

The set of pure ideals is closed under the infimum and supremum as shown below.

Proposition 3.7. *If I and J are pure ideals of $\mathcal{L}$, then $I \cap J$ and $I \vee J$ are pure ideals of $\mathcal{L}$.*

Proof. Let I and J be pure ideals of $\mathcal{L}$. Then $\sigma(I) = I$ and $\sigma(J) = J$.
According to Proposition 3.1, we have $\sigma(I \cap J) \subseteq I \cap J$.
Now $I \subseteq \sigma(I)$ and $J \subseteq \sigma(J)$ imply that $I \cap J \subseteq \sigma(I) \cap \sigma(J)$. Moreover, $\sigma(I) \cap \sigma(J) = \sigma(I \cap J)$ from Lemma 3.2. It follows that $\sigma(I \cap J) = I \cap J$. Hence, $I \cap J$ is a pure ideal of $\mathcal{L}$.

In addition, applying Proposition 3.1 and Lemma 3.2, we have $\sigma(I) \vee \sigma(J) \subseteq \sigma(I \vee J) \subseteq I \vee J$. This implies that $I \vee J \subseteq \sigma(I \vee J) \subseteq I \vee J$. Therefore, $\sigma(I \vee J) = I \vee J$.
□

The next result is a characterization of pure ideals of residuated lattices.

Proposition 3.8. *An ideal I of $\mathcal{L}$ is pure if and only if $\langle I \cup x^{\perp} \rangle = L$, for all $x \in I$.*

Proof. Assume I is pure, that is $I = \sigma(I)$. Let $x \in I = \sigma(I)$. Then, there are $a \in I$ and $b \in x^{\perp}$ such that $a \oplus b = 1$. This means that $1 = a \oplus b \in \langle I \cup x^{\perp} \rangle$, and it follows that $\langle I \cup x^{\perp} \rangle = L$. Hence, for all $x \in I$, $\langle I \cup x^{\perp} \rangle = L$.
Conversely, suppose $\langle I \cup x^{\perp} \rangle = L$, for every $x \in I$. It suffices to show that $I \subseteq \sigma(I)$.
Let $x \in I$. From $\langle I \cup x^{\perp} \rangle = L$, we have $1 \in \langle I \cup x^{\perp} \rangle$; then, by Proposition 2.5, there are $i \in I$ and $j \in x^{\perp}$ such that $1 \leq i \oplus j$. This means that there are $i \in I$ and $j \in x^{\perp}$ such that $i \oplus j = 1$; i.e., $x \in \sigma(I)$. Then, $I \subseteq \sigma(I)$. □

The characterization obtained in 3.8 clearly shows that a pure ideal I of $\mathcal{L}$ is exactly a pure element of the quantale $\mathcal{I}(\mathcal{L})$.

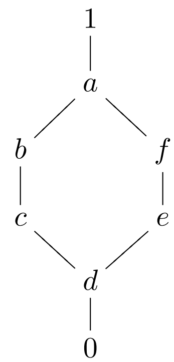

Figure 2: Hasse diagram of $\mathcal{L}_2$

Table 2: Operation tables of $\rightarrow$ and $\odot$ for $\mathcal{L}_2$ in Example 3.6

$\rightarrow$	0	a	b	c	d	e	f	1
0	1	1	1	1	1	1	1	1
a	d	1	a	a	f	f	f	1
b	e	1	1	a	f	f	f	1
c	f	1	1	1	f	f	f	1
d	a	1	1	1	1	1	1	1
e	b	1	a	a	a	1	1	1
f	c	1	a	a	a	a	1	1
1	0	a	b	c	d	e	f	1

$\odot$	0	a	b	c	d	e	f	1
0	0	0	0	0	0	0	0	0
a	0	c	c	c	0	d	d	a
b	0	c	c	c	0	0	d	b
c	0	c	c	c	0	0	0	c
d	0	0	0	0	0	0	0	d
e	0	d	0	0	0	d	d	e
f	0	d	d	0	0	d	d	f
1	0	a	b	c	d	e	f	1

Recall from [30] that a mapping $g : L \longrightarrow L$ on a bounded lattice that associates to any element a from L its image $g(a) \in L$ is an *interior operator* of L if it verifies the following properties for all $a, b \in L$:

(i) $a \leq b$ implies $g(a) \leq g(b)$;

(ii) $g(a) \leq a$;

(iii) $g^2(a) = g(a)$;

(iv) $g(1) = 1$.

The set $\mathcal{O} := \{a \in L : g(a) = a\}$ is the set of fixed elements of L by g.

As a direct consequence of Proposition 3.1, Lemma 3.2 (i), (iv), and Remark 3.4, we have the following proposition.

Proposition 3.9. *Let $\mathcal{L}$ be a residuated lattice. Then, the operator σ is an interior operator on $(\mathcal{I}(\mathcal{L}), \subseteq)$.*

We easily observe that $\mathcal{I}_\sigma(\mathcal{L})$ is the set of fixed elements of $\mathcal{I}(\mathcal{L})$ by σ.

Now, since the notions of ideal and filter in (non-regular) residuated lattices are not perfectly dual, we analyze the relation between pure ideals and pure filters studied in [9, 31]. We first of all recall some useful properties.

Let $(L; \wedge, \vee, \odot, \rightarrow, 0, 1)$ be a residuated lattice and X a nonempty subset of L. The set of elements of L having their negation in X is denoted and defined by:

$$N(X) := \{x \in L : x' \in X\}.$$

The following properties of the operator N shall be needed.

Remark 3.10. [29] Let F be a filter and I an ideal of $\mathcal{L}$. Then,

(i) $N(I)$ is a filter of $\mathcal{L}$, and $I = N(N(I))$;

(ii) $N(F)$ is an ideal of $\mathcal{L}$, and $F \subseteq N(N(F))$.

Proposition 3.11. *[32] Let $\mathcal{L}$ be a residuated lattice.*

(i) If I is a maximal ideal of $\mathcal{L}$, then $N(I)$ is a maximal filter of $\mathcal{L}$;

(ii) If F is a maximal filter of $\mathcal{L}$, then $N(F)$ is a maximal ideal of $\mathcal{L}$.

For all $x \in L$, the set

$$^{\perp}x := \{y \in L : x \vee y = 1\}$$

which is called the co-annihilator of x is a filter. Also, for any filter F of $\mathcal{L}$, the set

$$\delta(F) := \{x \in L : \text{ there are } f \in F \text{ and } z \in {}^{\perp}x \text{ such that } f \odot z = 0\}$$

is a filter of $\mathcal{L}$ and $\delta(F) \subseteq F$ (see [9]). Moreover, a filter F of $\mathcal{L}$ is called *pure filter* of $\mathcal{L}$ if $\delta(F) = F$ (i.e., $F \subseteq \delta(F)$).

Example 3.12. [9] Consider the residuated lattice $\mathcal{L}_3$ with the Hasse diagram of the underlying poset pictured in Figure 3, and the operations $\rightarrow$ and $\odot$ defined in Table 3. Then, $\{1\}$, $\{1, d\}$, $\{1, a, c\}$ and $\{0, a, b, c, d, 1\}$ are pure filters.

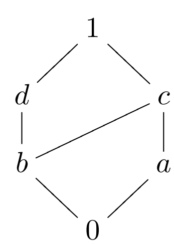

Figure 3: Hasse diagram of $\mathcal{L}_3$

Table 3: Operation tables of $\rightarrow$ and $\odot$ for $\mathcal{L}_3$ in Example 3.12

$\rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	d	1	d	1
b	c	c	1	1	1	1
c	b	c	d	1	d	1
d	a	a	c	c	1	1
1	0	a	b	c	d	1

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	0	a	0	a
b	0	0	0	0	b	b
c	0	a	0	a	b	c
d	0	0	b	b	d	d
1	0	a	b	c	d	1

The following proposition establishes a relation between pure ideals and pure filters of a residuated lattice.

Proposition 3.13. *Let $\mathcal{L}$ be a residuated lattice.*

(i) If F is a pure filter of $\mathcal{L}$, then $N(F)$ is a pure ideal of $\mathcal{L}$;

(ii) If I is a pure ideal of $\mathcal{L}$, then $N(I)$ is not necessarily a pure filter of $\mathcal{L}$.

Proof.

(i) Assume that F is a pure filter of $\mathcal{L}$; it suffices to show that $N(F) \subseteq \sigma(N(F))$.
For all $x \in N(F)$, we have $x' \in F = \delta(F)$, since F is pure. This implies that, there are $f \in F$ and $y \in {}^{\perp}(x')$ such that $f \odot y = 0$. We deduce that $f' \in F' \subseteq N(F)$.
In addition, since $x' \vee y = 1$, we obtain $1 = x' \vee y = x''' \vee y \leq x''' \vee y''$. Then, $x''' \vee y'' = 1$, which implies that $(x''' \vee y'')' = 0$. Thus, $x'''' \wedge y''' = 0$, and hence $y' \in (x'')^{\perp}$.
We also have $f \odot y = 0$, which implies that $f'' \odot y'' \overset{(P11)}{\leq} (f \odot y)'' = 0$. Therefore, $f'' \odot y'' = 0$, and then $(f'' \odot y'')' = 1$, i.e., $f' \oplus y' = 1$.
Thus, there are $f' \in N(F)$ and $y' \in (x'')^{\perp}$ such that $f' \oplus y' = 1$, which means that $x'' \in \sigma(N(F))$. Since $\sigma(N(F))$ is an ideal, it becomes clear that $x \in \sigma(N(F))$. Hence, $\sigma(N(F)) = N(F)$.

(ii) From Example 3.5, we have $I_1 = \{0, d\}$ is a pure ideal. But $N(I_1) = \{a, b, c, e, f, 1\}$ is not a pure filter, since $\delta(N(I_1)) = \{1, c\} \neq N(I_1)$.

□

Remark 3.14. If I is a pure ideal of a regular residuated lattice $\mathcal{L}$, then $N(I)$ is a pure filter of $\mathcal{L}$.

Indeed, assume that I is a pure ideal of a regular residuated lattice $\mathcal{L}$. It suffices to show that $N(I) \subseteq \delta(N(I))$.
For all $x \in N(I)$, $x' \in I = \sigma(I)$ since I is pure. Then, there are $a \in I$ and $b \in (x')^{\perp}$ such that $a \oplus b = 1$. We obtain that $a' \in I' \subseteq N(I)$. Since $x''' \wedge b'' = 0$, we have $(x''' \wedge b'')' = 1$, i.e., $(x'' \vee b')'' = 1$, from (P8). It follows from the regularity of $\mathcal{L}$ that $x'' \vee b' = 1$, and then $x \vee b' = 1$. Hence, $b' \in {}^{\perp}x$.

Moreover, since $a \oplus b = 1$, i.e., $(a' \odot b')' = 1$, we deduce that $a' \odot b' = 0$. Thus there are $a' \in N(I)$ and $b' \in {}^{\perp}x$ such that $a' \odot b' = 0$. Hence, $x \in \delta(N(I))$, as required.

The symbol $ord(x)$ which stands for the *order of nilpotence* or simply *order* of an element $x \in L$ is the smallest number $n \in \mathbb{N}^{\star}$ such that $x^n = 0$, i.e., $\underbrace{x \odot \cdots \odot x}_{n \ \ times} = 0$. If there is no such n, then the order of x is infinite, i.e., $ord(x) = \infty$. Obviously, we always have $ord(1) = \infty$. For every $x, y \in L$, if $x \leq y$ and $ord(y) < \infty$, then $ord(x) < \infty$. Similarly, if $x \leq y$ and $ord(x) = \infty$ then $ord(y) = \infty$.

Proposition 3.15. *[9] A residuated lattice $\mathcal{L}$ is local if and only if $ord(x) < \infty$ or $ord(x') < \infty$, for every $x \in L$.*

We say that a residuated lattice $\mathcal{L}$ is *locally finite* if $ord(x) < \infty$ for all $x \neq 1$ in L.

Proposition 3.16. *[14] For any $x \in L$,*

(i) There exists a proper ideal I of $\mathcal{L}$ such that $x \in I$ iff $ord(x') = \infty$;

(ii) $\langle x \rangle$ is proper iff $ord(x') = \infty$;

(iii) $ord(x') < \infty$ iff $x \notin P$ for every prime ideal P.

For any residuated lattice $\mathcal{L}$, we consider the set $\Re := \{x \in L : ord(x') = \infty\} = \{x \in L : (x')^n \neq 0, \text{ for all } n \geq 1\}$.

Proposition 3.17. *Let $\mathcal{L}$ be a residuated lattice. In case $\Re$ is an ideal of $\mathcal{L}$, if $x, y \in \Re$, then $(x')^n \oplus (y')^n \neq 0$, for all $n \geq 1$.*

Proof. Let $x^n, y^n \neq 0$ for all $n \geq 1$, as $x, y \in \Re$. Then, $x \oplus y \in \Re$, since $\Re$ is an ideal by hypothesis. This implies that $[(x \oplus y)']^n \neq 0$. But, $0 \neq [(x \oplus y)']^n = [(x' \odot y')'']^n \overset{P(11)}{\leq} [(x' \odot y')^n]'' = [(x')^n \odot (y')^n]'' \overset{P(11)}{\leq} [((x')^n)' \odot ((y')^n)']' = (x')^n \oplus (y')^n$. Hence, $(x')^n \oplus (y')^n \neq 0$, for all $n \geq 1$. □

A residuated lattice is called *i-local* if it has a unique maximal ideal.
We recall some characterizations of i-local residuated lattices.

Proposition 3.18. *[14] The following statements are equivalent:*

(i) $\Re$ is an ideal of $\mathcal{L}$.

(ii) $\langle \Re \rangle$ is a proper ideal of $\mathcal{L}$.

(iii) $\mathcal{L}$ is i-local.

(iv) $\Re$ is the only maximal ideal of $\mathcal{L}$.

The next proposition shows that the notions of local and i-local residuated lattices are equivalent.

Proposition 3.19. *A residuated lattice $\mathcal{L}$ is local if and only if it is i-local.*

Proof. Assume that $\mathcal{L}$ is local, that is, $\mathcal{L}$ has only one maximal filter F. From Proposition 3.11 (ii), $N(F)$ is a maximal ideal of $\mathcal{L}$.
Let I be a maximal ideal of $\mathcal{L}$. Then, from Proposition 3.11 (i), $N(I)$ is a maximal filter of $\mathcal{L}$. This implies that $N(I) = F$, by the uniqueness of the maximal filter. By applying Remark 3.10 (i), it yields that $I = N(N(I)) = N(F)$. Therefore, $N(F)$ is the unique maximal ideal of $\mathcal{L}$, that is, $\mathcal{L}$ is i-local.

Conversely, if $\mathcal{L}$ is i-local, then it has a unique maximal ideal I. Applying Proposition 3.11 (i), $N(I)$ is a maximal filter of $\mathcal{L}$.
Consider a maximal filter F of $\mathcal{L}$. We deduce from Proposition 3.11 (ii) that $N(F)$ is a maximal ideal of $\mathcal{L}$. Since $\mathcal{L}$ is i-local, then $N(F) = I$. Thus, from the maximality of F and applying Remark 3.10 (ii), yields we obtain $F = N(N(F)) = N(I)$. Hence, $\mathcal{L}$ has only one maximal filter $N(I)$, that is, $\mathcal{L}$ is local. □

Corollary 3.20. *A residuated lattice $\mathcal{L}$ is i-local if and only if $ord(x) < \infty$ or $ord(x') < \infty$, for every $x \in L$.*

Since the notion of ideal of residuated lattices is also defined from the commutative and associative operation $\oplus$, we now introduce the concept of *$\oplus$-order* of an element, from which we will provide a new characterization of i-local residuated lattices.

Definition 3.21. Let $\mathcal{L}$ be a residuated lattice, and $x \in L$. Then, the *$\oplus$-order* of x denoted $ord_{\oplus}(x)$ is the smallest number $n \in \mathbb{N}^{\star}$ such that $nx = 1$. When there is no such n, we say that the $\oplus$-order of x is infinite, that is, $ord_{\oplus}(x) = \infty$.

In the example below, we compute the $\oplus$-order of some elements of the residuated lattice $\mathcal{L}_2$ from Example 3.6.

Example 3.22. In the residuated lattice $\mathcal{L}_2$ of Example 3.6, we have:

- $ord_{\oplus}(a) = 1 < \infty = ord(a)$;
- $ord(d) = 1 < \infty = ord_{\oplus}(d)$;
- $ord(e) = 2 < \infty = ord_{\oplus}(e)$.

Proposition 3.23. *A residuated lattice $\mathcal{L}$ is i-local if and only if $ord_{\oplus}(x) < \infty$ or $ord_{\oplus}(x') < \infty$, for every $x \in L$.*

Proof. Assume that $\mathcal{L}$ is i-local, that is $\mathcal{L}$ has a unique maximal ideal I. Suppose by contrary that there is $x \in L$ such that $ord_{\oplus}(x) = \infty = ord_{\oplus}(x')$. Then, $\langle x \rangle$ is proper; otherwise, if $\langle x \rangle = L$, then from Proposition 2.5 (i) there exists $n \in \mathbb{N}^{\star}$ such that $nx = 1$, which is a contradiction.

Similarly, $\langle x' \rangle$ is proper. Since I is the unique maximal ideal of $\mathcal{L}$, we deduce that $\langle x \rangle$, $\langle x' \rangle \subseteq I$, which implies that $x, x' \in I$. Thus, $1 = x \oplus x' \in I$, a contradiction. Hence, $ord_{\oplus}(x) < \infty$ or $ord_{\oplus}(x') < \infty$.

Conversely, assume that $ord_{\oplus}(x) < \infty$ or $ord_{\oplus}(x') < \infty$, for every $x \in L$. Suppose by contrary that there are two distinct maximal ideals I and J of $\mathcal{L}$. Then, for any $y \in I \setminus J$, there is $n \in \mathbb{N}^{\star}$ such that $(ny)' \in J$, from Proposition 2.7 (ii). Set $a = ny$; then, $a' \in J$, which implies that $ma' \in J$ for all $m \in \mathbb{N}$. Thus, $ord_{\oplus}(a') = \infty$, implying from hypothesis that $ord_{\oplus}(a) < \infty$. This means that there is $k \in \mathbb{N}^{\star}$ such that $ka = 1$, that is, $kny = 1$. Since $y \in I$, we have $kny \in I$, that is, $1 = kny \in I$ which contradicts the fact that I is maximal. Therefore, $\mathcal{L}$ has only one maximal ideal, and hence is i-local. □

4 Stable Topology for Ideals of Residuated Lattices

Piciu in [13] endowed the set of prime ideals of a residuated lattice $\mathcal{L}$ with the Zariski topology. Let X be a nonempty subset of L and $D(X) := \{P \in Spec_{Id}(\mathcal{L}) : X \nsubseteq P\}$. The following proposition presents some properties of $D(X)$.

Proposition 4.1. *[13] Let $x, y \in L$, and $X, X_1, X_2, \{X_\gamma\}_{\gamma \in \Gamma} \subseteq L$. Then,*

(i) $X_1 \subseteq X_2$ *implies* $D(X_1) \subseteq D(X_2)$;

(ii) $D(X) = Spec_{Id}(\mathcal{L})$ *if and only if* $\langle X \rangle = L$. *Particularly,* $D(x) = Spec_{Id}(\mathcal{L})$ *if and only if* $\langle x \rangle = L$;

(iii) $D(X) = \emptyset$ *if and only if* $X = \{0\}$ *or* $X = \emptyset$. *In particular,* $D(x) = \emptyset$ *if and only if* $x = 0$;

(iv) $D(1) = D(L) = Spec_{Id}(\mathcal{L})$ *and* $D(\{0\}) = D(\emptyset) = \emptyset$;

(v) $\bigcup_{\gamma \in \Gamma} D(X_\gamma) = D(\bigcup_{\gamma \in \Gamma} X_\gamma)$;

(vi) $D(X) = D(\langle X \rangle)$;

(vii) $D(X_1) \cup D(X_2) = D(\langle X_2 \rangle \cup \langle X_1 \rangle)$ *and* $D(X_1) \cap D(X_2) = D(\langle X_2 \rangle \cap \langle X_1 \rangle)$;

(viii) $\langle X_1 \rangle = \langle X_2 \rangle$ *if and only if* $D(X_1) = D(X_2)$;

(ix) $D(x) \cup D(y) = D(x \vee y) = D(x \oplus y)$; $D(x) \cap D(y) = D(x'' \wedge y'')$.

The family $\{D(\{x\})\}_{x \in L}$ where $D(x) = \{P \in Spec_{Id}(\mathcal{L}) : x \notin P\}$, for all $x \in L$, is a basis for a topology $\tau_{\mathcal{L}} := \{D(X) : X \subseteq L\}$ on $Spec_{Id}(\mathcal{L})$. The topological space $(Spec_{Id}(\mathcal{L}), \tau_{\mathcal{L}})$ is called the *prime ideals space* of $\mathcal{L}$.

One can observe that for any ideal I of $\mathcal{L}$, $D(I)$ is an open set and $V(I) := \{P \in Spec_{Id}(\mathcal{L}) : I \subseteq P\}$ is a closed set for $(Spec_{Id}(\mathcal{L}), \tau_{\mathcal{L}})$. The set $D(I)$ is *stable under descent*, that is, if $P \in D(I)$, $Q \in Spec_{Id}(\mathcal{L})$ and $Q \subseteq P$, then $Q \in D(I)$. Moreover, $V(I)$ is *stable under ascent*, that is, if $P \in V(I)$, $Q \in Spec_{Id}(\mathcal{L})$ and $P \subseteq Q$, then $Q \in V(I)$. Therefore, the sets that are simultaneously open and closed (called clopen) are stable, that is, they are stable under ascent and descent.
The *stable topology* for $\mathcal{L}$ is the collection $\mathcal{S}_L$ of open stable subsets $D(I)$ of $Spec_{Id}(\mathcal{L})$ defined by $\mathcal{S}_L := \{D(I) : I \in \mathcal{I}(\mathcal{L}) \text{ and } D(I) \text{ is stable under ascent}\}$.

In what follows, we characterize open stable sets by means of pure ideals.

Theorem 4.2. *Let $\mathcal{L}$ be a residuated lattice and I an ideal of $\mathcal{L}$. Then, I is pure iff $D(I)$ is stable in $Spec_{Id}(\mathcal{L})$.*

Proof. For the first implication, let us show that $D(I)$ is stable under ascent. To this end, assume that I is pure and let P, Q be prime ideals of $\mathcal{L}$ such that $P \subseteq Q$ and $P \in D(I)$. Then, $I \nsubseteq P$, which implies that there is $x \in I \setminus P$. From the fact that $x \in I = \sigma(I)$, there are $a \in I$ and $b \in x^{\perp}$ such that $a \oplus b = 1$. But $b \in x^{\perp}$ implies that $b'' \wedge x'' = 0 \in P$. Since P is prime and $x \notin P$, we deduce that $b \in P \subseteq Q$. It yields that $Q \in D(I)$, otherwise we will have $I \subseteq Q$, implying that $a \in Q$ and $1 = a \oplus b \in Q$, contradicting the assumption that Q is proper.

On the other hand, assume that $D(I)$ is stable in $Spec_{Id}(\mathcal{L})$ and suppose by contrary that I is not a pure ideal of $\mathcal{L}$, that is, $\sigma(I) \subsetneq I$. Then, there exists $x \in I \setminus \sigma(I)$. Applying Proposition 2.10, there exists a minimal prime ideal P of $\mathcal{L}$ such that $\sigma(I) \subseteq P$ and $x \notin P$. This implies that $I \nsubseteq P$, that is, $P \in D(I)$.

Applying Proposition 3.8, I is not pure iff $x^{\perp} \vee I \neq L$, which implies that $x^{\perp} \vee I$ is proper. Thus, from the prime ideal theorem (see Theorem 2.9), there exists a prime ideal Q such that $x^{\perp} \vee I \subseteq Q$. This implies that $I \subseteq Q$, and therefore $Q \notin D(I)$. But $\sigma(I) \subseteq I \subseteq Q$, and by the minimality of P, we have $P \subseteq Q$. Since $D(I)$ is stable and $P \in D(I)$, it follows that $Q \in D(I)$, which is a contradiction. Hence, I is a pure ideal of $\mathcal{L}$.
□

Corollary 4.3. *For a residuated lattice $\mathcal{L}$, the assignment $I \rightsquigarrow D(I)$ is a bijection between the set of pure ideals of $\mathcal{L}$ and the set of open stable subsets of $Spec_{Id}(\mathcal{L})$.*

Theorem 4.2 yields the following separation property.

Theorem 4.4. *Let I be a pure ideal of $\mathcal{L}$, let P_1, P_2 be minimal ideals and P a prime ideal of $\mathcal{L}$ such that $P_1, P_2 \subseteq P$. Then, $I \subseteq P_1$ iff $I \subseteq P_2$.*

Proof. Suppose by contrary that $I \subseteq P_1$ and $I \nsubseteq P_2$. Then, $P_2 \in D(I)$. Since I is pure, we deduce from Theorem 4.2 that $D(I)$ is stable. Thus, from $P_2 \subseteq P$ and $P_2 \in D(I)$ it follows that $P \in D(I)$, which means that $I \nsubseteq P$. But, $I \subseteq P_1$ and $P_1 \nsubseteq P$ imply that $I \subseteq P$, which is a contradiction. □

For any maximal ideal M of $\mathcal{L}$, we set $\widehat{M} := \{P \in Spec_{Id}(\mathcal{L}) : P \subseteq M\}$.

Corollary 4.5. *For any pure ideal I and any maximal ideal M of $\mathcal{L}$, either $I \subseteq P$ for every $P \in \widehat{M}$, or $I \nsubseteq P$ for every $P \in \widehat{M}$.*

Proof. Assuming by contrary that there are $P_1, P_2 \in \widehat{M}$ such that $I \subseteq P_1$ and $I \nsubseteq P_2$, is in contradiction with Theorem 4.4.
□

To investigate the stable topology on i-local residuated lattices, we need the following results.

Proposition 4.6. *Let I be an ideal of $\mathcal{L}$. If $\sigma(I) \neq \{0\}$, then there is an element $a \in I$ such that $ord(a'') = \infty$.*

Proof. If $x \in \sigma(I)$ with $x \neq 0$, then there are $a \in I$ and $b \in x^{\perp}$ such that $a \oplus b = 1$. It follows that $a' \rightarrow b'' = 1$, which means that $a' \leq b''$. This implies that $(b')^n \leq (a'')^n$, for every $n \geq 1$. It is sufficient to show that $ord(b') = \infty$. We have $x'' \wedge b'' = 0$ (since $b \in x^{\perp}$) , which implies that $n.n(x'' \wedge b'') = 0$, for every $n \geq 2$. From (P13), $(nx) \wedge (nb) \leq n.n(x'' \wedge b'') = 0$, i.e., $(nx) \wedge (nb) = 0$, which is equivalent to $[(x')^n]' \wedge [(b')^n]' = 0$, for every $n \geq 2$. If by contrary $[(b')^n]' = 1$ for some $n \geq 2$, then $[(x')^n]' = 0$, i.e., $(nx) = 0$, implying that $x = 0$ (as $x \leq nx = 0$), contradicting the hypothesis. Thus, $[(b')^n]' \neq 1$. We deduce that $(b')^n \neq 0$, for every $n \geq 2$.
It is worth noticing that if $n = 1$, then $b' \neq 0$. Otherwise, we will have $b'' = 1$, implying from $x'' \wedge b'' = 0$ that $x'' = 0$, which is equivalent to $x = 0$, a contradiction to the hypothesis. Therefore, $ord(b') = \infty$ and hence $ord(a'') = \infty$.
□

Corollary 4.7. *Let I be a proper ideal of $\mathcal{L}$. If $\mathcal{L}$ is i-local, then $\sigma(I) = \{0\}$, that is, the unique pure ideals of $\mathcal{L}$ are $\{0\}$ and L.*

Proof. Let I be a proper ideal of the i-local residuated lattice $\mathcal{L}$. Assume by contrary that $\sigma(I) \neq 0$. Then, there exists $a \in I$ such that $ord(a'') = \infty$, from Proposition 4.6. Since $\mathcal{L}$ is i-local, we deduce from Corollary 3.20 that $ord(a') < \infty$, i.e., $(a')^n = 0$ for some $n \geq 1$. Therefore, $1 = [(a')^n]' = na \in I$, i.e., $I = L$, contradicting the hypothesis that I is proper. Hence the only pure ideals of $\mathcal{L}$ are 0 and L.
□

The theorem below states that the stable topology for an i-local residuated lattice is trivial.

Theorem 4.8. *If $\mathcal{L}$ is i-local, then the stable topology S_L on $\mathcal{L}$ is trivial.*

Proof. From Theorem 4.2, $D(I)$ is stable in $Spec_{Id}(\mathcal{L})$ iff I is a pure ideal of $\mathcal{L}$. But if $\mathcal{L}$ is i-local, then it follows from Corollary 4.7 that the only pure ideals of $\mathcal{L}$ are $\{0\}$ and L . Thus, either $I = \{0\}$ or $I = L$, and therefore $D(I) = \emptyset$, or $D(I) = Spec_{Id}(\mathcal{L})$. Hence, $S_L = \{\emptyset, Spec_{Id}(\mathcal{L})\}$. □

5 Conclusion

This work aimed to equip the set of prime ideals of a residuated lattice with the stable topology, a topology coarser than Zariski topology. To achieve, based on the notion of annihilator, we have introduced the concept of pure ideal in residuated lattices, along with its properties. After establishing a relation between pure ideals and pure filters of a residuated lattice, we have characterized open stable sets relative to the stable topology on prime ideals of a residuated lattice.

In our forthcoming research, following the approach in [27], we will explore some sheaf representations of i-normal residuated lattices described in [13]. Also, we plan to construct the Belluce lattice using the prime ideals of a residuated lattice to offer some additional characterizations of pure ideals and a better understanding of the prime ideals space of a residuated lattice. Recognizing the limitations of existing models such as those used for De Morgan residuated lattices [14], or MV algebras [16], we acknowledge the need for a novel approach.

Conflict of Interest: The authors declare no conflict of interest.

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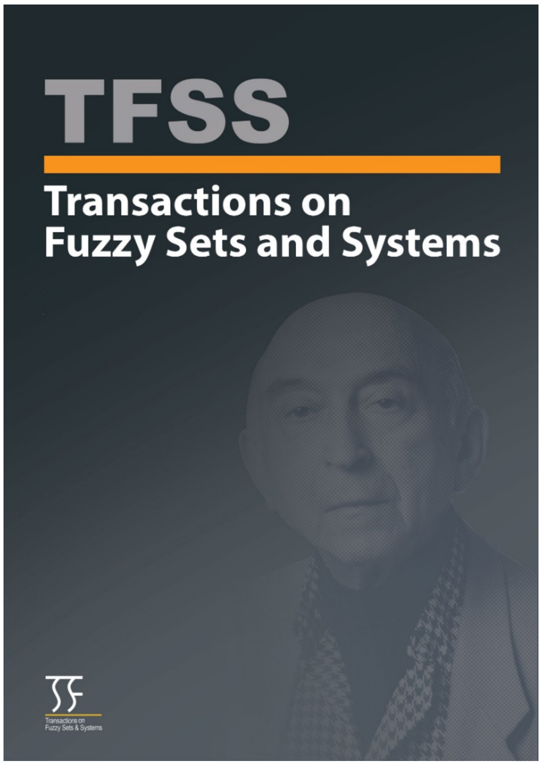

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A TOPSIS-Based Improved Weighting Approach with Evolutionary Computation

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A TOPSIS-Based Improved Weighting Approach with Evolutionary Computation

Mithat Zeydan*, **Murat Güngör**, **Burak Urazel**

Abstract. Although optimization of weighted objectives is ubiquitous in production scheduling, the literature concerning the determination of weights used in these objectives is scarce. Authors usually suppose that weights are given in advance, and focus on the solution methods for the specific problem at hand. However, weights directly settle the class of optimal solutions, and are of utmost importance in any practical scheduling problem. In this study, we propose a new weighting approach for single machine scheduling problems. First, factor weights to be used in customer evaluation are found by solving a nonlinear optimization problem using the covariance matrix adaptation evolutionary strategy (CMAES) under fuzzy environment that takes a pairwise comparison matrix as input. Next, customers are sorted using the technique for order of preference by similarity to ideal solution (TOPSIS) by means of which job weights are obtained. Finally, taking these weights as an input, a total weighted tardiness minimization problem is solved by using mixed-integer linear programming to find the best job sequence. This combined methodology may help companies make robust schedules not based purely on subjective judgment, find the best compromise between customer satisfaction and business needs, and thereby ensure profitability in the long run.

AMS Subject Classification 2020: 90B50; 90B35
Keywords and Phrases: Covariance matrix adaptation evolutionary strategy, Technique for order of preference by similarity to ideal solution, Weighted single machine scheduling, Mixed-integer linear programming.

1 Introduction

Companies should develop customer-focused strategies for being one step ahead in today's competitive market. The primary rule, which is an overwhelming and daunting task, is getting to know and identifying customers better. This not only helps companies fulfill their expectations, but also facilitates prioritization. Actually, some customers are more valuable than others. In production scheduling, this is reflected in the practice of assigning weights to orders or jobs. Each jobs contribution to the objective function thereby depends on its weight.

Although optimization of weighted objectives is ubiquitous in production scheduling, the literature concerning the determination of weights used in these objectives is scarce. Authors usually suppose that weights are given in advance, and focus on the solution methods for the specific problem at hand. However, weights directly settle the class of optimal solutions, and are of utmost importance in any practical scheduling problem.

Lin et al. [1] consider a hybrid flow shop scheduling problem with dynamic reentrant characteristics substantiated by the complexities in a repairing company. A genetic algorithm is applied to obtain near-optimal

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schedules, while the analytic hierarchy process (AHP) is used both to fulfill multiple criteria concerning the problem and to speed up the genetic algorithm's convergence. Deliktas et al. [2] propose an integrated approach for single machine scheduling with sequence-dependent setup times. In the first stage, job weights are determined by using AHP. In the second stage, a mixed-integer nonlinear programming model is built by considering three objective functions, namely the weighted number of tardy jobs, total weighted completion time, and makespan with sequence-dependent setup times. nemli [3] aims to create an algorithm to support the decision maker in the scheduling of customer orders for a box packaging production company in a make-to-order environment. In the first stage, the weighted tardiness of the orders is minimized, where the weights are determined by AHP, based on the knowledge and experience of experts. Ortiz-Barrios et al. [4] propose an integrated and enhanced method of a dispatching algorithm for scheduling flexible job shops based on fuzzy AHP and the technique for order of preference by similarity to ideal solution (TOPSIS). Fuzzy AHP is used to calculate the criteria weights under uncertainty, and TOPSIS is later applied to rank the eligible operations. Utku et al. [5] develop a mixed-integer programming model to minimize total lateness and total completion time of jobs in an automotive company. AHP is used to determine the weights of the two objectives.

Ignorance of weight determination in scheduling literature might be partly attributed to the gap between the theory and practice of scheduling. Stoop and Wiers [6] give an overview of the problems related to the complexity of scheduling in practice. Alternative suggestions to improve scheduling are proposed. First a description of scheduling and how it relates to planning and sequencing is presented. Then a description of problems that cause the scheduling function in practice to be very complex, and also an overview of shop floor models and scheduling techniques are given. Next, the problem of measuring schedule performance is discussed. Then possible solutions to the problems discussed are provided. Wiers [7] gives an overview of the applicability of techniques and the role of humans in production scheduling. He indicates that most of the literature reports give little indication of whether the system has been implemented in manufacturing practice, and for those systems that have been implemented, what types of implementation problems were encountered. The success of scheduling techniques in practice can only improve when researchers are aware of the implementation pitfalls through learning from each other's experiences. McKay and Wiers [8] argue that the gap between theory and practice in production scheduling has been confounded by the traditional view of scheduling as sequencing. This definition has focused researchers on the sequencing issue at the expense of the larger scheduling problem faced by practitioners dealing with the problems of partiality, temporality, and predictiveness. Namely, a scheduling process generates partial solutions for partial problems; anticipates, reacts to, and adjusts for disturbances in the process and environment; and is sensitive to and adjusts to the meaning of time in the production situation. The authors present an extended view of scheduling that unifies the traditional definition used in operations research and a number of key aspects of real-world scheduling. Dudek et al. [9] claim in the context of flow shop scheduling that scores of person-years of research time have been wasted on an intractable problem of little practical consequence. Although Gupta and Stafford [10] disagree with the viewpoint expressed by Dudek et al. [9], they admit that the mathematical theory of flow shop scheduling suffers from too much abstraction and too little application.

The purpose of this paper is to propose a new weighting approach with evolutionary computation for single machine scheduling problems. First, a pairwise comparison matrix that shows the relative importance of the criteria to be used in assessing customers is formed by having recourse to expert opinion, and criteria weights are determined by solving a nonlinear optimization problem via covariance matrix adaptation evolutionary strategy (CMAES) under fuzzy environment. Second, customer orders are ranked according to these criteria with TOPSIS. Finally, orders are sequenced so as to minimize total weighted tardiness by mixed-integer linear programming, where TOPSIS performance scores are taken as input.

The remainder of the paper is organized as follows. Section 2 presents the workflow regarding the application of the new approach. Section 3 defines the total weighted tardiness minimization problem, presents its mixed-integer linear program and suggests one possible way of handling the problem of quoting due dates for new orders in this context. Section 4 gives the numerical solution for the case study in a Turkish textile firm. Finally, we summarize our findings in Section 5.

2 TOPSIS-Based Improved Weighting Approach with Evolutionary Algorithm

Workflow of the proposed approach consists in (1) forming a pairwise comparison matrix for the criteria to be used in assessing customers, (2) determination of the weights of these criteria by solving an optimization problem via CMAES taking the pairwise comparison matrix as input, and (3) finding customers' scores with respect to these weights by using TOPSIS. The steps will be explained in detail below.

2.1 Forming Pairwise Comparison Matrix

We assume that preference of criterion i over j is given by a triple $(x^l_{ij}, x^m_{ij}, x^u_{ij})$. We call this a "fuzzy triangular number." Here the superscripts l, m, u stand for lower, middle, and upper, respectively. The middle coordinate x^m_{ij} may take an integer value in between 1 and 9. The equality $x^m_{ij} = 1$ implies that the criteria in question are equally important, whereas the equality $x^m_{ij} = 9$ implies that criterion i is extremely important compared to j. Unless $x^m_{ij} = 1$ or $x^m_{ij} = 9$, the first coordinate x^l_{ij} is 1 less than x^m_{ij}, and the third coordinate x^u_{ij} is 1 more than x^m_{ij}. If $x^m_{ij} = 1$, then all three coordinates are 1; if $x^m_{ij} = 9$, then all three coordinates are 9. Formally, pairwise comparison matrix formation using triangular numbers is composed of the following steps, where n denotes the number of criteria:

1. Form the tentative pairwise comparison matrix for the criteria:

$$\begin{pmatrix} (1,1,1) & (x^l_{12}, x^m_{12}, x^u_{12}) & \cdots & (x^l_{1n}, x^m_{1n}, x^u_{1n}) \\ (x^l_{21}, x^m_{21}, x^u_{21}) & (1,1,1) & \cdots & (x^l_{2n}, x^m_{2n}, x^u_{2n}) \\ \vdots & \vdots & \ddots & \vdots \\ (x^l_{n1}, x^m_{n1}, x^u_{n1}) & (x^l_{n2}, x^m_{n2}, x^u_{n2}) & \cdots & (1,1,1) \end{pmatrix}.$$

2. Perform defuzzification according to the formula

$$x_{ij} := \frac{x^l_{ij} + 4x^m_{ij} + x^u_{ij}}{6}.$$

3. Calculate the consistency index (CI) of the matrix (x_{ij}):

$$\text{CI} := \frac{\lambda_{\max} - n}{n - 1}.$$

   Here $\lambda_{\max}$ denotes the principal eigenvalue.

4. Calculate the consistency ratio (CR) of the matrix (x_{ij}):

$$\text{CR} := \frac{\text{CI}}{\text{RI}}.$$

   Here RI is the random index associated with dimension n.

5. If CR is less than 0.1, then proceed to obtain criteria weights; otherwise, revise the pairwise comparison matrix.

2.2 Determination of Criteria Weights

Many methods exist for deriving preference values from judgment matrices [11]. Basically, the idea is to obtain weights w_i such that

$$\frac{w_i}{w_j} \approx x_{ij}$$

for all i, j where (x_{ij}) denotes the pairwise comparison matrix [12]. Let μ_{ij} be a piecewise linear function of weights defined as

$$\mu_{ij}(w_1, \ldots, w_n) = \begin{cases} \frac{(w_i/w_j) - x_{ij}^l}{x_{ij}^m - x_{ij}^l}, & w_i/w_j \le x_{ij}^m; \\ \frac{x_{ij}^u - (w_i/w_j)}{x_{ij}^u - x_{ij}^m}, & w_i/w_j > x_{ij}^m. \end{cases}$$

Note that μ_{ij} is an indicator of how well the weights w_j, w_j comply with the pairwise comparison value x_{ij}. We have

- $\mu_{ij} = 1$ if and only if $w_i/w_j = x_{ij}^m$,
- $\mu_{ij} \in (0, 1)$ for $w_i/w_j \in (x_{ij}^l, x_{ij}^u) \setminus \{x_{ij}^m\}$, and
- $\mu_{ij} \le 0$ whenever $w_i/w_j \le x_{ij}^l$ or $w_i/w_j \ge x_{ij}^u$.

Therefore, all μ_{ij} shall be as large as possible. One possible way towards this end is to maximize the minimum of the μ_{ij}. So we define

$$G(w_1, \ldots, w_n) := \min_{i<j} \mu_{ij}(w_1, \ldots, w_n).$$

Hence, weights can be determined by solving the following nonlinear optimization problem:

$$\begin{aligned} \text{maximize} \quad & G(w_1, \ldots, w_n) \\ \text{such that} \quad & w_1 + \cdots + w_n = 1. \end{aligned}$$

Note that, rewriting w_n in terms of $w_1, \ldots, w_{n-1}$, the problem can be converted into an unconstrained maximization problem. As in Zeydan et al. [13], we solve this by CMAES under fuzzy environment, which is a derivative-free stochastic global search algorithm developed recently [14]. It works iteratively by adapting the resulting search distribution to the contours of the objective function by updating the covariance matrix deterministically using information from evaluated points [14]. We refer the reader to Hansen [15] for details of the CMAES algorithm.

2.3 Ranking Customers with TOPSIS

Let there be m alternatives and n criteria indexed respectively by i and j. Criteria weights w_j are assumed to be given. Steps for ranking customers with the TOPSIS method can be stated as follows [16]:

1. Form the decision matrix (x_{ij}). (This is not to be confused with the matrix obtained in §2.1 after defuzzification.)
2. Construct the normalized decision matrix (r_{ij}):

$$r_{ij} := \frac{x_{ij}}{\sqrt{\sum_{k=1}^{m} x_{kj}^2}}.$$

3. Construct the weighted normalized decision matrix (v_{ij}):

$$v_{ij} := w_j \times r_{ij}.$$

4. Determine the positive and negative ideal rows $(v_1^+, \ldots, v_n^+)$ and $(v_1^-, \ldots, v_n^-)$.

5. Measure the distance of each alternative from the ideal rows:

$$d_i^+ := \sqrt{\sum_{j=1}^{n} (v_{ij} - v_j^+)^2}, \qquad d_i^- := \sqrt{\sum_{j=1}^{n} (v_{ij} - v_j^-)^2}.$$

6. Calculate the closeness of the alternatives to the ideal solution, namely the TOPSIS scores:

$$\text{Score}_i := \frac{d_i^-}{d_i^+ + d_i^-}.$$

3 Total Weighted Tardiness Minimization Problem on a Single Machine

Let there be n jobs to be processed on a single machine. We index jobs by j. Each job has a processing time p_j, due date d_j, and weight w_j. Preemptions are not allowed; in other words, processing of a job cannot be interrupted until it is completed. Let C_j denote the completion time of job j. Tardiness is defined as

$$T_j := \max\{C_j - d_j, 0\}.$$

Thus, tardiness equals lateness if the job is late, and it is zero otherwise. The question is to find a schedule that minimizes total weighted tardiness. In the common three-field notation, the problem is $1 \mid\mid \sum w_j T_j$ [17, 18]. The objective function is nondecreasing in completion times; i.e., it is regular. So there exists an optimal schedule in which the machine is never kept idle. Therefore, the problem amounts to finding the best job sequence with respect to total weighted tardiness. Table 1 shows the indices, parameters, and decision variables for $1 \mid\mid \sum w_j T_j$.

Table 1: Indices, parameters, and decision variables for $1 \mid\mid \sum w_j T_j$.

Symbol	Explanation
j	job index
n	number of jobs
p_j	processing time of job j
d_j	due date of job j
w_j	weight of job j
C_j	completion time of job j
T_j	tardiness of job j

3.1 Mixed-Integer Linear Programming Formulation

Minimization of total weighted tardiness on a single machine, which is strongly NP-hard in terms of computational complexity [19], has received much attention in the literature [20, 21, 22]. Branch-and-bound and dynamic programming approaches have been proposed to obtain optimal solutions. The problem can also be modeled as a mixed-integer linear program (MILP). One can build a model based on precedence or time-indexed decisions [23]. We shall focus on the former in this section. Let x_{jk} be a binary variable defined as follows:

$$x_{jk} = \begin{cases} 1, & \text{if job } j \text{ precedes job } k; \\ 0, & \text{otherwise.} \end{cases}$$

If $x_{jk} = 1$, then $C_j \leq C_k - p_k$; if $x_{jk} = 0$, then $C_k \leq C_j - p_j$. These conditional statements can be expressed respectively as

$$C_j \leq C_k - p_k + M(1 - x_{jk}),$$
$$C_k \leq C_j - p_j + Mx_{jk},$$

where M is a sufficiently large number. Note that the first inequality becomes redundant when $x_{jk} = 0$, and the second becomes redundant when $x_{jk} = 1$. It is enough to define x_{jk} for each pair of jobs, so there are $\binom{n}{2} = n(n-1)/2$ such variables. Below is the MILP formulation of $1 \mid\mid \sum w_j T_j$ using precedence constraints:

$$\begin{aligned}
\text{minimize} \quad & \sum_j w_j T_j && && \text{(1a)}\\
\text{subject to} \quad & C_j \leq C_k - p_k + M(1 - x_{jk}) && \text{for all } j < k && \text{(1b)}\\
& C_k \leq C_j - p_j + Mx_{jk} && \text{for all } j < k && \text{(1c)}\\
& T_j \geq C_j - d_j && \text{for all } j && \text{(1d)}\\
& C_j \geq p_j && \text{for all } j && \text{(1e)}\\
& C_j, T_j \geq 0 && \text{for all } j && \text{(1f)}\\
& x_{jk} \in \{0, 1\} && \text{for all } j < k. && \text{(1g)}
\end{aligned}$$

The objective (1a) is to minimize the sum of weighted tardiness. Constraints (1b) and (1c) relate the precedence decisions to jobs' completion times as explained above. Inequality (1d) must hold as an equality in view of the objective whenever job j is late. The next constraint (1e) guarantees that the completion time of the first job in the sequence is nonzero. Note that in the formulation there are $2n$ continuous variables, namely C_j and T_j, in addition to the $n(n-1)/2$ binary variables x_{jk}. There are $n^2 + n$ constraints in total (except binary and nonnegativity restrictions). The big M can be taken as the sum of all processing times.

3.2 Due Date Quotation Problem

Although the number of tardy jobs is a common performance criterion in practice, its minimization may be an unrealistic objective as it may lead to schedules with unacceptably late jobs. The same is true of weighted tardiness minimization: if a job has a small weight relative to others, it may be overly late in an optimal schedule. Therefore, it makes sense to assume deadlines $\overline{d_j}$ for real-life applications. Deadlines represent hard constraints: in any feasible schedule, all deadlines must be met. Mathematically, the inequality $C_j \leq \overline{d_j}$ must be satisfied for all j. We assume that deadlines are defined as translations of due dates by a specified constant δ:

$$\overline{d_j} = d_j + \delta.$$

Then inequalities $C_j \leq \overline{d_j}$ can be stated equivalently as

$$T_j \leq \delta.$$

Thus, a job is ideally to be completed by its due date, but if it somehow happens to be late, the lateness cannot exceed δ. In the three-field notation, we express this problem by $1 \mid \overline{d_j} = d_j + \delta \mid \sum w_j T_j$.

In scheduling literature it is often assumed that due dates are given beforehand. However, in many circumstances, determination of due dates itself is a problem: what due date should be assigned to a new customer order? This is called the due date quotation or management problem. A comprehensive literature survey thereon is given by Kaminsky and Hochbaum [24].

We consider due date quotation in the setting $1 \mid \overline{d_j} = d_j + \delta \mid \sum w_j T_j$. More precisely, let there be n jobs, already sequenced to minimize $\sum w_j T_j$ respecting the deadlines. Suppose a new customer order, namely the $(n+1)$st job, has arrived. Its processing time p_{n+1} and weight w_{n+1} are known. What due date d_{n+1} should be assigned to this job? There is an intrinsic trade-off to be faced here: if d_{n+1} is large, the existing schedule will not be affected much and the objective value $\sum_{j=1}^{n+1} w_j T_j$ will be small, but the new customer's satisfaction will be less; if d_{n+1} is small, the situation is the other way around.

Let $d_{\min}$ and $d_{\max}$ be the minimum and maximum possible due dates for the $(n+1)$st job. If it is sequenced first, it will be completed by p_{n+1}; if it is sequenced last, it will be completed by $\sum_{j=1}^{n+1} p_j$. So we naturally have $d_{\min} := p_{n+1}$ and $d_{\max} := \sum_{j=1}^{n+1} p_j$. For each possible due date $d \in [d_{\min}, d_{\max}]$ for the $(n+1)$st job, let $z^*(d)$ be the objective value $\sum_{j=1}^{n+1} w_j T_j^*(d)$ associated with the optimal sequencing of all $n+1$ jobs (we assume δ is large enough to guarantee that there always exists a feasible solution). As mentioned above, z^* is a nonincreasing function of d; that is, for all d, d',

$$d \leq d' \;\; \Rightarrow \;\; z^*(d) \geq z^*(d').$$

It follows that in this context the due date quotation, in essence, is a multi-objective optimization problem. Namely, we are to find the best compromise between the due date d_{n+1} to be assigned and the objective value $z^*(d_{n+1})$ associated with it.

Now we discuss one way to do this. Let $\sum_{j=1}^{n} w_j T_j^*$ be the optimal total weighted tardiness for the existing n jobs, let $\sum_{j=1}^{n} w_j T_j^*(d)$ be the updated value of this sum after the arrival of the $(n+1)$st job given that its quoted due date is d, and let

$$\Delta z(d) := \sum_{j=1}^{n} w_j T_j^*(d) - \sum_{j=1}^{n} w_j T_j^*$$

denote the difference of these two sums. Clearly, as d gets smaller, $\Delta z(d)$ gets larger. Suppose, without loss of generality, that $\sum_{j=1}^{n} w_j = 1$. Then an increase of $\Delta z(d)$ by 1 means that the tardiness of each one of the existing n jobs has increased on average by 1 time unit. What is the utility of this in terms of assigning a better due date to the $(n+1)$st job? In other words, if the pairs of solutions $(d, \Delta z(d))$ and $(d', \Delta z(d)+1)$ are equivalent, what is $d - d'$? Ultimately, this depends on the decision-maker, but a possible answer would be w_{n+1}. Then the due date quotation problem can be written concisely as

$$\min_{d} \;\; w_{n+1} d + \Delta z(d).$$

4 Application in a Textile Company

In this section, we present a numerical demonstration of the weighting approach introduced above with data obtained from a textile firm in Turkey. The problem is to find an optimal sequence of 11 customer orders that minimizes total weighted tardiness. Customers are to be assessed with respect to five criteria: profitability (%), average order quantity (meters), unit selling price (dollars per meter), payment performance (lateness per order), risk limit (dollars). First of all, managers from three distinct departments—production, marketing, and finance—are consulted in order to construct pairwise comparison matrices. Then an aggregate matrix has been built as shown in Table 2.

Consistency ratio for the matrix in Table 2 turns out to be 0.089, so it is convenient to use this matrix as an input to CMAES to find criteria weights. We coded the 29 April 2014 version of CMAES algorithm using

Table 2: Aggregate pairwise comparison matrix for the five criteria.

	C1	C2	C3	C4	C5
C1	1.000	2.000	3.000	3.000	6.000
C2	0.556	1.000	2.000	2.000	7.000
C3	0.347	0.556	1.000	1.000	8.000
C4	0.347	0.556	1.000	1.000	8.000
C5	0.168	0.144	0.126	0.126	1.000

MATLAB 2018. Convergence of the method with respect to the objective function value and the weights are given in Figure 1. The resulting weights are

$$w_1 = 0.284, \quad w_2 = 0.335, \quad w_3 = 0.171 \quad w_4 = 0.164 \quad w_5 = 0.040.$$

Table 3 shows the decision matrix for the first step of the TOPSIS algorithm, namely numerical values associated with the 11 customers for the five aforementioned criteria. Tables 4 and 5 show the normalized and the weighted normalized versions thereof, respectively. Table 6 shows positive and negative ideal rows obtained from the weighted normalized decision matrix. Finally, Table 7 shows the distance of the alternatives (customers) to the ideal rows, their TOPSIS scores, and the relevant ranking.

Table 8 shows the data and the optimal solution of the single machine weighted tardiness minimization problem. The processing times and due dates are randomly generated following the weighted tardiness instance generation routine in the OR-Library maintained by John Beasley. We took the range of due dates (RDD) and the average tardiness factor (TF) parameters in this routine as 0.6. Weights are assumed to be the TOPSIS scores computed in Table 7. Solving the mixed-integer linear program (1), the optimal job sequence turns out to be (11, 8, 7, 2, 10, 5, 4, 1, 3, 6, 9) with an objective value of 204.19.

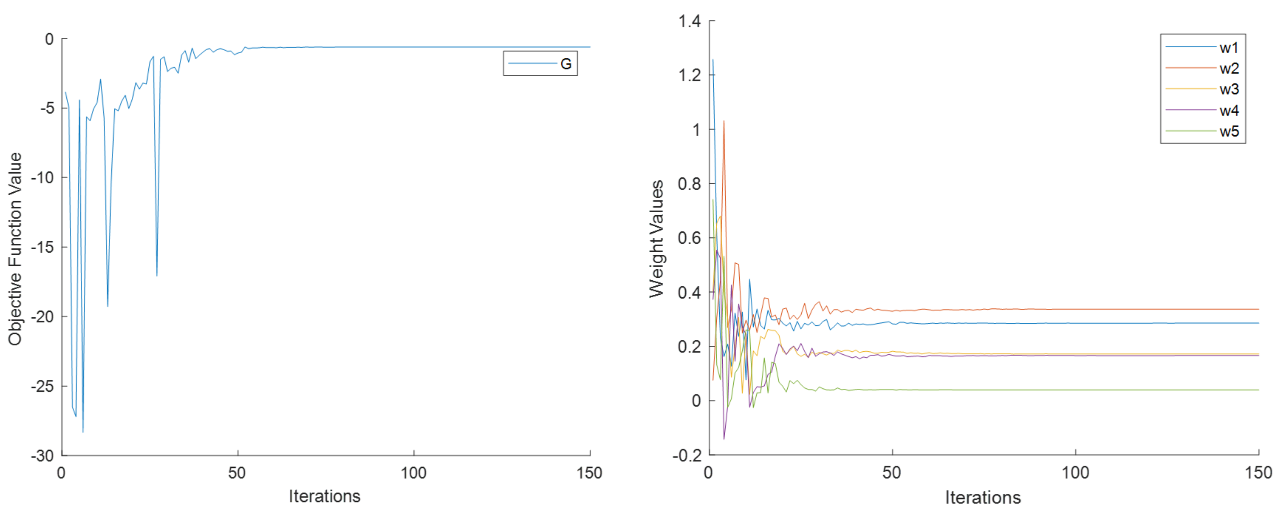

Figure 1: Convergence of CMAES with respect to the objective function value (on the left) and the weights (on the right).

Table 3: Decision matrix for TOPSIS.

Customer	Profitability	Average order quantity	Unit selling price	Payment performance	Risk limit
1	1	17976	2.67	2	200000
2	1	35766	2.29	29	100000
3	7	1966	1.95	10	50000
4	4	9306	1.63	9	15000
5	1	43721	1.55	18	50000
6	1	25030	2.20	20	200000
7	5	72609	2.19	2	50000
8	3	19444	2.52	34	30000
9	1	9515	1.81	7	200000
10	1	12961	5.81	7	150000
11	5	13921	3.76	2	200000

Table 4: Normalized decision matrix.

Customer	Profitability	Average order quantity	Unit selling price	Payment performance	Risk limit
1	0.0877	0.1768	0.2839	0.0364	0.4459
2	0.0877	0.3518	0.2435	0.5284	0.2229
3	0.6134	0.0193	0.2074	0.1822	0.1114
4	0.3508	0.0915	0.1733	0.1639	0.0334
5	0.0877	0.4301	0.1648	0.3279	0.1114
6	0.0877	0.2462	0.2340	0.3644	0.4459
7	0.4385	0.7142	0.2329	0.0364	0.1114
8	0.2631	0.1912	0.2680	0.6195	0.0668
9	0.0877	0.0936	0.1925	0.1275	0.4459
10	0.0877	0.1275	0.6179	0.1275	0.3344
11	0.4385	0.1369	0.3999	0.0364	0.4459

Table 5: Weighted normalized decision matrix.

Customer	Profitability	Average order quantity	Unit selling price	Payment performance	Risk limit
1	0.0250	0.0595	0.0488	0.0060	0.0175
2	0.0250	0.1185	0.0418	0.0878	0.0087
3	0.1752	0.0065	0.0356	0.0302	0.0043
4	0.1001	0.0308	0.0298	0.0272	0.0013
5	0.0250	0.1449	0.0283	0.0545	0.0043
6	0.0250	0.0829	0.0402	0.0605	0.0175
7	0.1251	0.2406	0.0400	0.0060	0.0043
8	0.0750	0.0644	0.0460	0.1029	0.0026
9	0.0250	0.0315	0.0330	0.0211	0.0175
10	0.0250	0.0429	0.1062	0.0211	0.0131
11	0.1251	0.0461	0.0687	0.0060	0.0175

Table 6: Positive and negative ideal rows.

	Profitability	Average order quantity	Unit selling priceprice	Payment performance	Risk limit
v^+	0.1752	0.2406	0.1062	0.0060	0.0175
v^-	0.0250	0.0065	0.0283	0.1029	0.0013

Table 7: Distance to ideal rows and composite indices of the customers.

Customer	d_i^+	d_i^-	Score	Ranking
1	0.2421	0.1135	0.3192	7
2	0.2199	0.1140	0.3414	5
3	0.2460	0.1670	0.4044	4
4	0.2370	0.1093	0.3156	8
5	0.2007	0.1466	0.4221	3
6	0.2339	0.0896	0.2770	9
7	0.0840	0.2727	0.7645	1
8	0.2330	0.0785	0.2520	10
9	0.2680	0.0871	0.2453	11
10	0.2487	0.1192	0.3240	6
11	0.2043	0.1512	0.4253	2

Table 8: Data and optimal solution of the single machine weighted tardiness minimization problem.

Customer	p_j	d_j	w_j	C_j^*	T_j^*
1	50	393	0.3192	378	0
2	90	215	0.3414	224	9
3	58	416	0.4044	436	20
4	50	332	0.3156	328	0
5	41	330	0.4221	278	0
6	79	214	0.2770	515	301
7	74	151	0.7645	134	0
8	44	179	0.2520	60	0
9	82	150	0.2453	597	447
10	13	386	0.3240	237	0
11	16	68	0.4253	16	0

5 Conclusion

In this paper, we proposed a novel bottom-up approach for solving weighted single machine scheduling problems. First, a pairwise comparison matrix that shows the relative importance of the criteria to be used in evaluating customers is formed through expert opinion, and criteria weights are calculated by optimizing a nonlinear function via the covariance matrix adaptation evolutionary strategy (CMAES) under fuzzy environment. Second, customer orders are sorted with respect to these criteria with the technique for order of preference by similarity to ideal solution (TOPSIS). Finally, orders are sequenced by mixed-integer linear programming with the objective of minimizing total weighted tardiness, where TOPSIS scores are taken as weights. This combined methodology may help companies make robust schedules not based purely on subjective judgment, find the best compromise between customer satisfaction and business needs, and thereby ensure profitability in the long run. As a topic of future study, it is worthwhile to investigate how the proposed methodology works in practice for due date quotation as discussed in Section 3.2.

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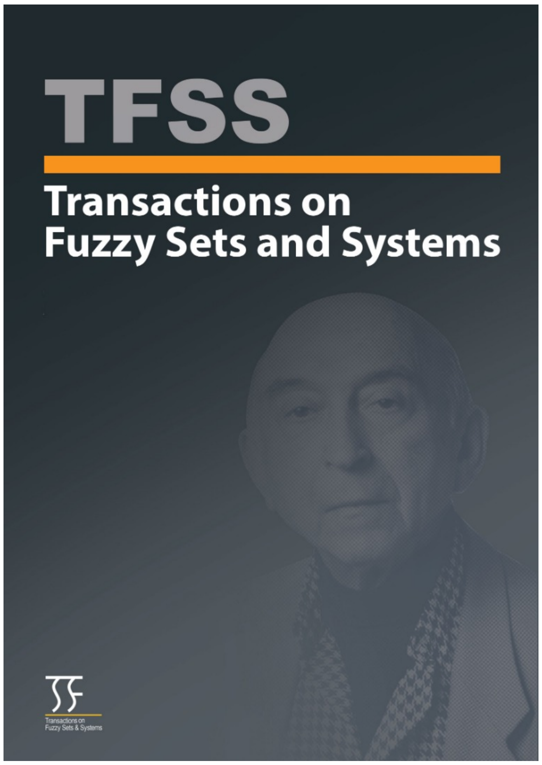

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Families of Fuzzy Sets and Lattice Isomorphisms Preparation

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Families of Fuzzy Sets and Lattice Isomorphisms Preparation

John N Mordeson, **Sunil Mathew***

Abstract. In this paper, we discuss how theoretical results from one family of fuzzy sets can be carried over immediately to another family of fuzzy sets by the use of lattice isomorphisms. We also show that these families can occur naturally and that applications may not necessarily be carried over using these isomorphisms. We illustrate this using techniques from the study of human trafficking and its analysis using mathematics of uncertainty. We also consider the new definition of fuzzy set provided by Trillas and de Soto.

AMS Subject Classification 2020: 03B52; 03E72
Keywords and Phrases: Modern slavery, Government response, Vulnerability, Lattice isomorphisms, Fuzzy sets.

1 Introduction

In this paper, we discuss how theoretical results from one family of fuzzy sets can be carried over immediately to another family of fuzzy sets by the use of lattice isomorphisms. We also show that these families can occur naturally and that applications may not necessarily be carried over using these isomorphisms. We illustrate this using techniques from the study of human trafficking and its analysis using mathematics of uncertainty. Mathematics of uncertainty is a very appropriate tool to use in the study of trafficking. This is because accurate data concerning trafficking in persons is impossible to obtain. The goal of the trafficker is to be undetected. The size of the problem also makes it very difficult to obtain accurate data. Victims are reluctant to report crimes or testify for fear of reprisals, disincentives, both structural and legal, for law enforcement to act against traffickers, a lack of harmony among existing data sources, and an unwillingness of some countries and agencies to share data. We also generalize some results concerning families of fuzzy sets involved with these lattice isomorphisms.

2 Fuzzy Sets and Lattice Isomorphisms

One of the most important papers concerning fuzzy set theory in recent years is one by Klement and Mesiar, [1]. In this paper, it is shown that differently defined families of fuzzy sets have lattice structures that are actually isomorphic and so theoretical results for one family can be carried over to another family.

We show by using a real world problem with real world data that even though theoretical results can be obtained for one family from another, the two families may arise naturally in an application.

We use the concepts of vulnerability and government response to modern slavery to illustrate our findings. In [2], it is stated that the departing point is the fact that not only fuzzy sets originate in Language, but

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that they are just 'linguistic entities' genetically different from the concept of 'crisp sets' whose origin is either in a physical collection of objects, or in a list of them. A new definition of a fuzzy set is presented by means of two magnitudes: A qualitative one, called a graph, the basic magnitude, and a quantitative one, a scalar magnitude. If the first reflects the language's relational ground of the fuzzy set, the second reflects the (numerical) extensional state in which it currently appears.

We next illustrate these ideas using the concepts of vulnerability and government response with respect to modern slavery, [3].

Vulnerability Measures

(1) Government issues

(2) Nourishment and access

(3) Inequality

(4) Disenfranchised groups

(5) Effect of conflicts

Countries are scored with respect to these five measures. Then a weighted average of these scores is taken to provide a single score for each number. For example, the final score for Brazil is 36.4. The countries are placed into regions. Brazil is in the Americas. For this region, the highest score was 69.6 and the smallest was 10.2. The country scores were normalized using the formula (number - minimum)/(maximum - minimum) to obtain $(36.4 - 10.2)/(69.6 - 10.2) = 0.441$.

Government Response

(1) Support for survivors

(2) Criminal justice

(3) Coordination

(4) Response

(5) Supply chains

Similarly, as for the vulnerability measures, a final score is determined for each country with respect to government response. For example the final score for Brazil is 55.6. For the Americas, the maximum score was 71.7 and the minimum was 20.8. Hence the normalized value for Brazil was $(55.6-20.8)/(71.7-20.8) = 0.684$.

In [2], it is stated that shortening the statement x is less P, where P is a predicate, by $x \prec_P y$ facilitates the basic magnitude. That is, $x \prec_P y \subseteq X \times X$.

For our illustration, we let P denote the predicate vulnerable and X denote the set of countries under consideration. Now the final vulnerable score for Mexico was 57.3. Brazil's was 36.4. Hence Brazil $\prec_P$Mexico. The final value for government response for Mexico was 52.4 and for Brazil 55.6. In the case, we have Mexico $\prec_P$ Brazil if P denotes government response and $\prec_P$is the linguistic relation x has less government response than y.

In [2], a membership function $m_P : X \times X \to [0, 1]$ was introduced. It provides a numerical value for measuring the degree to which x is P. The membership function is required to satisfy the following three properties:

(i) $x \prec_P y$ implies $m_P(x) \leq m_P(y)$.

(ii) If z is minimal, then $m_P(z) = 0$.

(iii) If w is maximal, then $m_P(w) = 1$.

We see that our membership function $m_P(x) = \frac{\#(x)-\min}{\max-\min}$, where $\#(x)$ denotes the final score of x, satisfies these three properties. Thus $m_V(Brazil) = 0.441$, where V denotes vulnerable and $m_G(Brazil) = 0.684$, where G denotes government response. For Mexico, we have $m_V(Mexico) = 0.793$ and $m_G(Mexico) = 0.621$.

We present some isomorphisms and other methods in fuzzy set theory to obtain results from one family for another.

The following table is from [3]. See also [[4], p. 104].

Table 1: Global slavery index Americas

Country	Government Response	Vulnerability	Prevalence
Argentina	0.821	0.297	0.156
Barbados	0.365	0.533	0.431
Bolivia	0.402	0.570	0.313
Brazil	0.684	0.441	0.294
Canada	0.742	0.000	0.000
Chile	0.815	0.259	0.058
Columbia	0.398	0.696	0.431
Costa Rica	0.573	0.306	0.156
Cuba	0.000	0.710	0.647
Dominican Rep.	0.730	0.553	0.686
El Salvador	0.326	0.681	0.392
Ecuador	0.502	0.523	0.372
Guatemala	0.479	0.705	0.470
Guyana	0.210	0.592	0.509
Haiti	0.371	1.000	1.000
Honduras	0.318	0.762	0.568
Jamaica	0.742	0.572	0.411
Mexico	0.621	0.793	0.431
Nicaragua	0.500	0.567	0.470
Paraguay	0.394	0.516	0.215
Panama	0.453	0.441	0.313
Peru	0.622	0.574	0.411
Suriname	0.123	0.537	0.352
Trinidad and Tobago	0.571	0.486	0.490
United States	1.000	0.095	0.156
Uruguay	0.581	0.159	0.098
Venezuela	0.145	0.803	1.000

Neutrosophic fuzzy sets and Pythagorean fuzzy sets: Recall that a neutrosophic fuzzy set is a triple (σ, τ, μ) of fuzzy subsets of a set. It is based on the lattice of elements $(x_1, x_2, x_3) \in [0,1]^3$,where $(x_1, x_2, x_3) \leq (y_1, y_2, y_3)$ if and only if $x_1 \leq y_1, x_2 \leq y_2$, and $x_3 \geq y_3$. Also, a Pythagorean fuzzy set is a pair of fuzzy subsets (σ, τ) of a set X such that for all $x \in X, \sigma(x)^2 + \tau(x)^2 \leq 1$, [5].We can see that vulnerability and government response corresponding to modern slavery are opposites, [5]. That is, an increase in government response by a country would lower the country's vulnerability. However, $m_V(Brazil) + m_G(Brazil) = 0.441 + 0.684 > 1$. This gives meaning to neutrosophic fuzzy sets, [6], even though certain theoretical results can following immediately from other types of fuzzy sets. Also, $(0.441)^2 + (0.684)^2 = 0.194 + 0.468 < 1$. Consequently, similar comments might be able to be made here even though Pythagorean fuzzy sets and intuitionistic fuzzy sets, [7], have corresponding isomorphic lattices. However, this isomorphism may make the situation different to the neutrosophic case since it is so straight forward. The lattice isomorphism f involved here is $f : P^* \to L^*$ defined by $f((x_1, x_2)) = (x_1^2, x_2^2)$, where $L^* = \{(x_1, x_2)|x_1, x_2, \epsilon[0, 1], |x_1 + x_2 \leq 1\}$ and $P^* = \{(x_1, x_2)|x_1, x_2\epsilon[0, 1], x_1^2 + x_2^2 \leq 1\}$. The paper by Klement and Mesiar contains many other cases, where various families of fuzzy sets have isomorphic lattices.

Let X be a set with n elements, say $X = \{x_1, ..., x_n\}$. Let μ, ν be fuzzy subsets of X. Consider the fuzzy similarity measures, $M(\mu, \nu) = \frac{\sum_{x\in X} \mu(x)\wedge\nu(x)}{\sum_{x\in X} \mu(x)\vee\nu(x)}$ and $S(\mu, .\nu) = 1 - \frac{\sum_{x\in X} |\mu(x)-\nu(x)|}{\sum_{x\in X}(\mu(x)+\nu(x))}$. Let m be a positive real number. Then $\frac{\sum_{x\in X} \mu(x)\wedge\nu(x)}{\sum_{x\in X} \mu(x)\vee\nu(x)} = \frac{\sum_{x\in X} \mu(x)/m\wedge\nu(x)/m}{\sum_{x\in X} \mu(x)/m\vee\nu(x)/m}$ and $\frac{\sum_{x\in X} |\mu(x)-\nu(x)|}{\sum_{x\in X}(\mu(x)+\nu(x))} = \frac{\sum_{x\in X} |\mu(x)/m-\nu(x)/m|}{\sum_{x\in X}(\mu(x)/m+\nu(x)/m)}$. Suppose there exists $x \in X$ such that $\mu(x) + \nu(x) > 1$. Let m denote the maximal such $\mu(x) + \nu(x)$. Then we see that we get the same M and S values if we divide all the $\mu(x)$ and $\nu(x)$ by m.

Example 2.1. Let $X = \{x_1, x_2\}$. Define the fuzzy subsets μ, ν of X as follows:

	μ	ν
x_1	0.1	0.1
x_2	0.2	0.91

Then $\mu(x_2) + \nu(x_2) = 0.2 + 0.91 = 1.11 > 1$. Now $M(\mu, \nu) = \frac{0.1\wedge0.1+0.2\wedge0.91}{0.1\vee0.1+0.2\vee0.91} = \frac{0.3}{1.01}$.

Define the fuzzy subsets μ', ν' of X as follows:

	μ'	ν'
x_1	0.1	0.2
x_2	0.2	0.82

Then $\mu'(x_2) + \nu'(x_2) = 0.2 + 0.82 = 1.02 > 1$. Now $M(\mu', \nu') = \frac{0.1\wedge0.2+0.2\wedge0.82}{0.1\vee0.2+0.2\vee0.82} = \frac{0.3}{1.02}$.

Thus $M(\mu, \nu) > M(\mu', \nu')$. We have a Pythagorean situation since $(0.2)^2 + (0.91)^2 < 1$ and $(0.2)^2 + (0.82)^2 < 1$.

Squaring the values of μ and ν, we obtain $\mu_2(x_1) = 0.01, \mu_2(x_2) = 0.04$ and $\nu_2(x_1) = 0.01, \nu_2(x_2) = 0.8281$. Hence

$$M(\mu_2, \nu_2) = \frac{0.01 \wedge 0.01 + 0.4 \wedge 0.8281}{0.01 \vee 0.01 + 0.4 \vee 0.8281} = \frac{0.01 + 0.04}{0.01 + 0.8281}.$$

Also, $\mu_2'(x_1) = 0.01, \mu_2'(x_2) = 0.04, \nu_2'(x_1) = 0.04, \nu_2'(x_2) = 0.6724$. Thus

$$M(\mu_2', \nu_2') = \frac{0.01 \wedge 0.04 + 0.04 \wedge 0.6724}{0.01 \vee 0.04 + 0.04 \vee 0.6724} = \frac{0.01 + 0.04}{0.04 + 0.6724}.$$

Hence $M(\mu_2, \nu_2) < M\mu_2', \nu_2')$. That is, the inequalities have switched. They were not preserved.

We next consider $S(\mu, \nu) = 1 - \frac{\sum_{x\in X} |\mu(x)-\nu(x)|}{\sum_{x\in X}(\mu(x)+\nu(x))}$. For the previous situation, we have $S(\mu, \nu) = 1 - \frac{0+0.71}{0.2+1.11} = \frac{0.71}{1.31} = 1 - 0.542$ and $S(\mu', \nu') = 1 - \frac{0.1+0.62}{0.3+1.02} = 1 - \frac{0.72}{1.32} = 1 - 0.545$. Thus $S(\mu, \nu) > S(\mu', \nu')$.

We also have $S(\mu_2, \nu_2) = 1 - \frac{0+0.7881}{0.04+0.8281} = 1 - \frac{0.7881}{0.8681} = 1 - 0.9078$ and $S(\mu_2', \nu_2') = 1 - \frac{0.03+.6324}{0.05+0.7164} = 1 - \frac{0.6624}{0.7664} = 1 - 0.8643$. Hence $S(\mu_2, \nu_2) < S(\mu_2', \nu_2')$. Once again the inequalities were not preserved.

We have shown with this example that the isomorphism $f : P^* \to L^*$ defined by $f((x_1, x_2)) = (x_1^2, x_2^2)$, where $L^* = \{(x_1, x_2)|x_1, x_2\epsilon[0, 1], x_1 + x_2 \leq 1\}$ and $P^* = \{(x_1, x_2)|x_1, x_2 \in [0, 1]x_1^2 + x_2^2 \leq 1\}$, [1], shows that although theoretical results can determined between Pythagorean fuzzy sets and intuitionistic fuzzy sets, the isomorphism may not be suitable in changing a data set from one to another in applications. Also, isomorphisms in general preserve certain structural properties, but not all outside functions defined on the sets.

3 Lack of Accurate Data

Linguistic variables: The size of flow of trafficked people from country to country is given in [8]. It is reported in linguistic terms since accurate data concerning the size of the flow is impossible to obtain. Information is provided with respect to the reported human trafficking in terms of origin, transit, and/or destination according to the citation index. The data is provided in two columns. Information in the left column as to whether a country ranks (very) low, medium (very) high depends upon the total number of sources which made reference to this country as one of origin, transit, or destination. Information provided in then the right column provides further detail to the information provided in the left column. If a country in the right column was mentioned by one or two sources, the related country was ranked low. If linkage between the countries in the two columns was reported by 3-5 sources, the related country was ranked medium. If 5 or more sources linked the two countries, the country in the right was ranked high. This method of combining linguistic data provides an ideal reason for the use of mathematics of uncertainty to study the problem of trafficking by persons. For example, by assigning numbers in the interval $[0, 1]$ to the linguistic data, the data can be combined in a mathematical way. In [9], the notions of t-norms and t-conorms were used. The number 0.1 can be assigned very low, 0.3 to low. 0.5 to medium, 0.7 to high and 0.9 to very high. Using the notation and ideas from [10], we have $x \prec_P y$ if and only if country x's linguistic rank is less than country y's linguistic rank. We have $m_P(x) = 0.1, 0.3, 0.5, 0.7$, or 0.9 if x is assigned very low, low, medium, high, or very high, respectively. We note that here m_P does not satisfy (ii) and (iii).

Colors: In [10], colors are used to determine how well a country is achieving the Sustainable Development Goals (SDGs). A green rating on the SDG dashboard is assigned to a country if all the indicators under that goal are labeled green. Yellow, orange and red indicate increasing distance from the SDG achievement. The worst two colors of a target were averaged to determine the color for its SDG. In [11], the numbers $0.2, 0.4, 0.6.0.8$ are assigned to the colors red, orange, yellow, and green, respectively. Consequently, the results in [10] are placed into the context of mathematics of uncertainty.

4 Theoretical Results

Let m and n be positive real numbers. Let $P_{m,n} = \{(x, y)|x, y \in [0, 1]$ and $x^m + y^n \leq 1\}$. Let $L^* = \{(x, y)|x, y \in [0, 1]$ and $x + y \leq 1\}$. Define $\leq_{P\,m,n}$ on $P_{m,n}$ by for all $(x, y), (u, v) \epsilon P_{m,n}(u, v)$ if and only if $x \leq u$ and $y \geq v$. This includes L^* since $L^* = P_{1,1}$. The following result extends the theory for Pythagorean fuzzy sets to m, n-rung fuzzy sets, [12], since $P_{2,2}$ is a Pythagorean fuzzy set.

The following result follows from Theorem 4.2, but we place it here since it motivates Theorem 4.2.

Theorem 4.1. *Define $f : P_{m,n} \to L^*$ by for all $(x, y) \in P_{m,n}, f((x, y)) = (x^m, y^n)$. Then f is a lattice isomorphism of $P_{m,n}$ onto L^*.*

In the above table, we see that for Mexico, $(0.621, 0.793) \notin P_{2,2} \cup P_{3,1} \cup P_{1,3}$. We have $(0.62, 1.793) \in P_{2,3} \cap P_{3,2}$.

For the United States, $\nexists m, n$ such that $(1^m, 0.095^n) \leq 1$.

Consider Mexico again. Now $P_{2,2}$ is the set of all Pythagorean fuzzy sets. Define $g : P_{2,3} \to P_{2,2}$ by for all $(x, y) \in P_{2,3}, g((x, y)) = (x^{\frac{3}{2}}, y)$. Note $(x^{\frac{3}{2}}, y) \in P_{2,2}$ since $(x^{\frac{3}{2}})^2 + y^2 = x^3 + y^2 \leq 1$.

Let $f_1, f_2 : [0, 1] \to [0, 1]$ be one-to-one functions of $[0, 1]$ onto $[0, 1]$ such that for all $x, y \epsilon [0, 1], x \leq y$ implies $f_i(x) \leq f_i(y), i = 1, 2.$Then $f_i(0) = 0$ and $f_i(1) = 1, i = 1, 2$. Assume $f_i(x \wedge y) = f_i(x_i) \wedge f_i(y)$ and $f_i(x \vee y) = f_i(x) \vee f_i(y), i = 1, 2$.

Let $\widehat{L}(f_1, f_2) = \{(x, y)|x, y\epsilon[0, 1], f_1(x) + f_2(y) \leq 1\}$. For ease of notation, let $\widehat{L} = \widehat{L}(f_1, f_2)$. Define $\leq_{\widehat{L}}$ on $\widehat{L}$ by for all $(x, y), (w, z)\epsilon\widehat{L}, (x, y) \leq_{\widehat{L}} (w, z)$ if and only if $x \leq w$ and $y \geq z$. Then $\leq_{\widehat{L}}$ is a partial order on $\leq_{\widehat{L}}$ such that any two elements $\leq_{\widehat{L}}$ have a greatest lower bound and a least upper bound. Thus $\leq_{\widehat{L}}$ is a lattice. The greatest lower bound of $(x, y), (w, z)\epsilon \leq_{\widehat{L}}$ is $(x \wedge w, y \vee z)$ and the least upper bound is $(x \vee w, y \wedge z)$.

Theorem 4.2. *Define* $f : \widehat{L} \to L^*$ *by for all* $(x_1, x_2)\epsilon\widehat{L}, f((x_1, x_2)) = (f(x_1), f_2(x_2))$. *Then* f *is a lattice isomorphism of* $\widehat{L}$ *onto* L^*.

Proof. Clearly, f maps $\widehat{L}$ into L^*. Now $(x_1, x_2) = (y_1, y_2) \Leftrightarrow x_1 = y_1$ and $x_2 = y_2 \Leftrightarrow f_1x_1) = f_1(y_1)$ and $f_2(x_2) = f_2(y_2) \Leftrightarrow f((x_1, x_2)) = (f_1(x_1), f_2(x_2)) = (f_1(y_1), f_2(y_2)) = f((y_1, y_2))$. Hence f is single-valued and one-to-one. Let $(x_1, x_2)\epsilon L^*$. Then $(f_1^{-1}(x_1), f_2^{-1}(x_2))\epsilon\widehat{L}$. Thus f maps $\widehat{L}$ onto L^*. Let $(x_1, x_2), (y_1, y_2)\epsilon\widehat{L}$. Then

$$\begin{aligned} f((x_1, x_2) \wedge_{\widehat{L}} (y_1, y_2)) &= f((x_1 \wedge y_1, x_2 \vee y_2)) \\ &= (f_1(x_1 \wedge y_1), f_2(x_2 \vee y_2)) \\ &= (f_1(x_1) \wedge f_1(y_1)), (f_2(x_2) \vee f_2(y_2)) \\ &= (f_1(x_1), f_2(x_2)) \wedge_{L^*} (f_1(y_1), x_{1x} f_2(y_2)) \\ &= f((x_1, x_2) \wedge_{L^*} f(y_1, y_2)). \end{aligned}$$

Similarly, $f((x_1, x_2) \vee_{\widehat{L}} (y_1, y_2)) = f((x_1, x_2) \vee_{L^*} f(y_1, y_2))$.

Suppose $(x_1, x_2) \leq_{\widehat{L}} (y_1, y_2)$. Then $x_1 \leq y_1$ and $x_2 \geq y_2$ and so $f_1(x_1) \leq f_1(y_1)$ and $f_2(x_2) \geq f_2(y_2)$. Thus $(f_1(x_1), f_2(x_2)) \leq_{\widehat{L}} (f_1(y_1), f_2(y_2))$. That is, $f((x_1, x_2) \leq_{\widehat{L}} f((y_1, y_2))$. □

If we let m, n be positive real numbers. Define $f_1, f_2 : [0, 1] \to [0, 1]$ by for all $x\epsilon[0, 1], f_1(x) = x^m$ and $f_2(x) = x^n$, then Theorem 4.1 follows from Theorem 4.2.

Example 4.3. Let i, j be positive real numbers. Define $f_1, f_2 : [0, 1] \to [0, 1]$ by for all $x\epsilon[0, 1], f_1(x) = x^i$ and $f_2(x) = x^j$. Then f_1 and f_2 satisfy the above properties. Thus the above Theorem holds for (m, n)-rung fuzzy sets, where m, n are positive integers.

Let $f_1, f_2 : [0, 1] \to [0, 1]$ be defined by for all $x\epsilon[0, 1], f_1(x) = x$ and $f_2(x) = 1 - x$. Let $\mathcal{I} = \{[x, y]|0 \leq x \leq y \leq 1\}$. Define $f : L^* \to \mathcal{I}$ by for all $(x, y)\epsilon L^*, f((x, y)) = [f_1(x), f_2(y)]$. Then $f((x, y)) = [x, 1 - y]$. Clearly f is single-valued. Note that since $x + y \leq 1, x \leq 1 - y$. Let $[x, y]\epsilon\mathcal{I}$. Then $f((x, 1 - y)) = [x, y]$ and $(x, 1 - y)\epsilon L^*$ since $x + 1 - y \leq 1$, i.e., $(x, 1 - y)\epsilon L^*$. Thus f maps L^* onto $\mathcal{T}$. (Note $x \leq y$ so $x + 1 - y \leq 1$.). Now f is one-to-one since f_1 and f_2 are.

Define $\wedge_{\mathcal{I}}, \vee_{\mathcal{I}}$ on $\mathcal{I}$by for all $[x, 1 - y], [w, 1 - z]\epsilon\mathcal{I}$, $[x, 1 - y] \wedge_{\mathcal{I}} [w, 1 - z] = [x \wedge w, (1 - y) \wedge (1 - z)]$, and $[x, 1 - y] \vee_{\mathcal{I}} [w, 1 - z] = [\, x \vee w, (1 - y) \vee (1 - z)]$. Define $\leq_{\mathcal{I}}$by on $\mathcal{I}$ by for all $[x, 1 - y], [w, 1 - z]\epsilon I$, $[x, 1 - y] \leq_{\mathcal{I}} [w, 1 - z]$ if and only if $x \leq w$ and $y \geq z$.

Theorem 4.4. f *is a lattice isomorphism of* L^* *onto* $\mathcal{I}$.

Proof. By the discussion above f is a one-to-one function of L^* onto $\mathcal{I}$. Let $(x_1, x_2), (y_1, y_2)\epsilon L^*$. Then

$$\begin{aligned} f((x_1, x_2) \wedge_{L^*} (y_1, y_2)) &= f((x_1 \wedge y_1, x_2 \vee y_2)) = (f_1(x_1 \wedge y_1), , f_2(x_2 \vee y_2)) \\ &= (x_1 \wedge y_1, 1 - (x_2 \vee y_2)) = (x_1 \wedge y_1, (1 - x_2) \wedge (1 - y_2)) \\ &= (x_1, (1 - x_2)) \wedge_{\mathcal{I}} (y_1, (1 - y_2)) \\ &= (f_1(x_1), f_2(x_2)) \wedge_{\mathcal{I}} (f_1(y_1), f_2(y_2) \\ &= f((x_1, x_2) \wedge_{\mathcal{I}} f((y_1, y_2)) \end{aligned}$$

Similarly, $f((x_1, x_2) \vee_{L^*} (y_1, y_2)) = f((x_1, x_2) \vee_{\mathcal{I}} f((y_1, y_2))$.

Now $(x_1, x_2) \leq_{L^*} (y_1, y_2) \Leftrightarrow x_1 \leq y_1$ and $x_2 \geq y_2 \Leftrightarrow x_1 \leq y_1$ and $1 - x_2 \leq 1 - y_2 \Leftrightarrow [x_1, 1 - x_2] \leq_{\mathcal{I}^*} [y_1, 1 - y_2] \Leftrightarrow f((x_1, x_2)) \leq_{\mathcal{I}^*} f((y_1, y_2))$. □

5 Conclusion

In this paper, we discussed the important paper by Klement and Mesiar that shows, using lattice isomorphisms, how theoretical results can be carried over immediately from one family of fuzzy sets to another. We show that these families of fuzzy sets can arise naturally in applications. We also show that newly developed families of fuzzy sets may also have these isomorphic lattices.

Conflict of Interest: The authors declare no conflict of interest.

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