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A Primal Simplex Algorithm for Solving Linear Programming Problem with Grey Cost Coefficients

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Abstract
In this paper, a linear programming problem is considered involving interval grey numbers as an extension of the classical linear programming problem to an inexact environment as well as fuzzy and stochastic environment. Hence, here a new approach for solving interval grey number linear programming problems is introduced without converting them to classical linear programming problems. The proposed method is established based on the primal simplex algorithm where the cost coefficient row includes grey numbers. As an essential tool in the solving process, a theoretical discussion on grey arithmetic and in particular an ordering role for grey numbers is necessary to evaluate the optimality conditions of a candidate feasible solution. It is also emphasized that the discussed model and the solution process is useful for real situations and practical cases, when a kind of the grey number linear programming is appeared, such as Water Resource Management and Planning, Economics, etc. Finally, the proposed approach is illustrated by a numerical example.

Keywords: Grey linear programming, Grey system theory, Interval Grey number, Primal simplex algorithm, Uncertainty.

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1. Introduction

Linear Programming (LP) is a most widely and successfully used decision tool in the quantitative analysis of practical problems where rational decisions have to be made. In order to solve a LP Problem, the decision parameters of the model must be fixed at as crisp values. However, to model real-life problems and perform computations, we must deal with uncertainty and inexactness. These uncertainties and inexactness are due to measurement inaccuracy, simplification of physical models, variations of the parameters of the system, computational errors, etc. [35]. Over the last three decades, varieties of techniques have surfaced to optimize problems in the face of uncertainty. Techniques such as chance constraints, grey numbers, fuzzy numbers, probabilistic, possibility, flexible and stochastic programs with recourse have been presented to systematically and proactively incorporate numerical uncertainties in optimization models [41]. Traditional methods. based on exact values of parameters cannot satisfy the planners need for flexibility. Furthermore, stochastic and fuzzy methods, which are often used to handle uncertainty and flexibility, must specify probability and possibility distributions, which may be difficult to obtain in practice [27].

As a different model for uncertainty representation, Professor Julong Deng proposed Grey Systems Theory (GST) in 1982 [8]. GST focuses on the study of such uncertain systems with partially known and partially unknown information that matters of the characteristics of poor information are seen as its research subjects [22]. The scientific fields covered by GST include systems analysis, data processing, modeling, prediction, decision-making and control; these fields are neither deterministic nor totally unknown, but rather they are partially known [47]. Grey mathematics is the mathematical foundation of GST and its unit is a grey number. Grey numbers have been applied to a variety of programs with applications including hypothetical numerical examples of solid waste management [14, 17, 18], water resources allocation [29, 35], optimized urban water supply scheme [1], chemical integrated system [38], buildings deformation monitoring [53], web service selection [48]. By combining the grey theory with the principle and method of linear programming problem, the linear programming model is established based on the grey theory. Grey Linear Programming (GLP) is a model of grey systems analysis for decision making under uncertainty. It is a development of the traditional LP method [20]. A grey linear programming model has many advantages for deterministic programming. Grey programming has the ability to deal with poor, incomplete, or uncertainty problems with systems, and it has been widely used in many aspects such as economics, agriculture, medicine, geography, industry, etc. [25]. Huang et al. [18] and Huang and Moore [20] designed GLP, in which parameters and variables are both grey numbers. This was followed by grey fuzzy programming [4], grey integer programming [19], grey nonlinear programming [3] and grey multi-objective programming [16, 36], grey quadratic programming [17] and stochastic [15]. Early applications of GLP incorporated grey numbers in the objective function [19], constraint matrix [20], right-hand sides of constraints, and all of the above [18, 19]. Deng studied the grey drafting LP [9]. The GLP of prediction type is studied by Chen and Wu [6]. Chen et al. used mathematical programming problems, with grey interval and developed grey forecasting [5]. The confidence degree solution of GLP problems has been studied by Xiao [51] and Wang [43]. Li provided a covered solution for GLP [21]. However, the inverse of a grey matrix can be obtained only when the column or row-sum norm of a selected matrix satisfies given conditions. Li et al. [24] proposed a covered solution for a GLP that is suitable for any norm. Liu and Forrest [28] put forward a series of new conceptions such as the positioned linear programming, the ideal model, the critical model, and some other new ideas for solving the LP with grey parameter problems. Many researchers have applied GLP [12] and utilized a whitening technique to solve it [20, 44, 45, 51]. Because the whitening technique only provides a crisp model and does not reflect the evolutionary characteristics of a grey set, it does not yield the ranges of the optimal values and optimal solutions. Razavi et al. [36] proposed a multi objective model to solve GLP problems in general form. The suggested approach is mainly based on the concept of order relations between grey numbers. Based on this notion, a modeling process is proposed to convert the original GLP problem to a bi-objective problem. Xia et al. [50] proposed a grey general data envelopment analysis model, which is based on the GLP, and they got the optimal solution of the grey general data envelopment analysis by using a method of θ-positioned programming solution.

In the above discussed works, the most of the researchers have transformed the interval GLP problems into one or a series of classical LP problems and then obtained an optimal solution. In this paper, a method based on comparison of interval grey numbers and arithmetic on these numbers is proposed. An algorithm like well-known primal simplex algorithm for solving interval GLP problems is proposed where it can solve the problem without converting it to the classical LP problems. The rest of the paper is organized as follows. Some necessary backgrounds and definitions of the grey system theory which are needed in next sections are presented in Section 2. A definition of grey number linear programming problem is given in Section 3, and then the proposed approach and its related issues are explained. In Section 4, a numerical example is solved by the proposed procedure. Finally, Section 5 consists of conclusions.

2. Preliminaries

In this section, some definitions, concepts, notions and results which are useful in our further consideration about grey system theory and in particular, for the interval grey numbers calculus as a tool for solving the GLP, is introduced [2, 10, 11, 23, 30-32, 34, 54].

2.1. Grey system and Grey numbers

Grey theory is one of the methods used to study uncertainty, being superior in the mathematical analysis of systems with uncertain information.

Definition 2.1. A grey system is defined as a system containing uncertain information presented by a grey number and grey variables.

Definition 2.2. Let X be the universal set $X = \mathbb{R}$, the set of all real numbers. Then a grey system G of X is defined by the two mappings $\overline{\mu}_G$ and $\underline{\mu}_G$ where $\overline{\mu}_G : X \to [0,1]$, $\underline{\mu}_G : X \to [0,1]$, $\overline{\mu}_G \geq \underline{\mu}_G$.

where $\overline{\mu}_G$ and $\underline{\mu}_G$ are the upper and lower membership functions in G respectively.

Note: When $\overline{\mu}_G = \underline{\mu}_G$, the grey set G becomes a fuzzy set. Thus, the grey theory considers the condition of the fuzziness and can deal with the fuzziness situation.

GST introduces the concept of interval grey numbers. Grey numbers are regarded as the basic unit of grey systems to participate in the construction of grey model. It plays an important role in grey theory [46]. Let x denote a closed and bounded set of real numbers.

Definition 2.3. A grey number is a number with clear upper and lower boundaries, but which has an unknown position within the boundaries. A grey number in the system is expressed mathematically as [7]

$$\otimes x \in [\underline{x}, \overline{x}] = \{t \in x \mid \underline{x} < t < \overline{x}\} \qquad (2.1)$$

where $\otimes x$ is a grey number, t is information, $\underline{x}$ and $\overline{x}$ are the upper and lower limits of the information.

There are several types of grey numbers [29] among them the interval grey numbers are considered here as the convenient kind in the literature.

A grey number is a number whose exact value is unknown, but a range within which the value lies in is known [26].

There are the several types of grey numbers.

Definition 2.4. Grey numbers with only lower limits: $\otimes x \in [\underline{x}, \infty)$ or $\otimes(\underline{x})$, where a fixed real value $\underline{x}$ represents the lower limit of the grey number $\otimes x$.

Definition 2.5. Grey numbers with only upper limits: $\otimes x \in (-\infty, \overline{x}]$ or $\otimes(\overline{x})$, where $\overline{x}$ is a fixed real value number or an upper limit of grey number $\otimes x$.

Definition 2.6. Interval grey number is the number with both lower limit $\underline{x}$ and upper limit $\overline{x}$: $\otimes x \in [\underline{x}, \overline{x}]$.

Definition 2.7. (Continuous grey numbers and discrete grey numbers): The grey numbers continuously taking values, which cover an interval, are continuous. The grey numbers taking a finite number of values or a countable number of values in an interval are called discrete.

Definition 2.8. (Black and white numbers): When $\otimes x \in (-\infty, +\infty)$, i.e., when $\otimes x$ has neither an upper limit nor lower limit or the upper and the lower limits are all grey numbers, $\otimes x$ is called a black number. When $\otimes x \in [\underline{x}, \overline{x}]$ and $\underline{x} = \overline{x}$, $\otimes x$ is called a white number.

In the rest of the paper, we consider interval grey numbers and for simplify we shortly called the just by grey number.

Remark 2.1. We denote the set of all grey numbers by $R(\otimes)$. We also show an element of $R(\otimes)$, that is $\otimes x \in [\underline{x}, \overline{x}]$ by $[\underline{x}, \overline{x}]_G$.

Definition 2.9. A length of grey number $\otimes x \in [\underline{x}, \overline{x}]$ is defined as $L(\otimes x) = |\overline{x} - \underline{x}|$. it is obvious that $L(\otimes x) : R(\otimes) \mapsto R^+$.

Definition 2.10. [36] For any grey number, the center, $\otimes \hat{x}$ and width, $\otimes x_W$ of a grey number $\otimes x$ is defined as follows:

$$\otimes \hat{x} = \frac{1}{2}(\bar{x} + \underline{x}),\ \otimes x_W = \frac{1}{2}(\bar{x} - \underline{x}).$$

It is easily verifiable that $\bar{x} = \otimes\hat{x} + \otimes x_W$ and $\underline{x} = \otimes\hat{x} - \otimes x_W$.

The main arithmetic operations can be defined on grey numbers. Let $\otimes x_1 \in [\underline{x}_1, \bar{x}_1]$ and $\otimes x_2 \in [\underline{x}_2, \bar{x}_2]$ be two grey numbers. The following operations can be defined [34]:

$$\otimes x_1 + \otimes x_2 = [\underline{x}_1 + \underline{x}_2, \bar{x}_1 + \bar{x}_2] \quad (2.2)$$

$$\otimes x_1 - \otimes x_2 = [\underline{x}_1 - \bar{x}_2, \bar{x}_1 - \underline{x}_2] \quad (2.3)$$

$$\otimes x_1 \times \otimes x_2 = [\min(\underline{x}_1\underline{x}_2, \underline{x}_1\bar{x}_2, \bar{x}_1\underline{x}_2, \bar{x}_1\bar{x}_2), \max(\underline{x}_1\underline{x}_2, \underline{x}_1\bar{x}_2, \bar{x}_1\underline{x}_2, \bar{x}_1\bar{x}_2)] \quad (2.4)$$

$$\otimes x_1 \div \otimes x_2 = [\underline{x}_1, \bar{x}_1] \times [\frac{1}{\bar{x}_2}, \frac{1}{\underline{x}_2}], \text{where } 0 \notin [\underline{x}_2, \bar{x}_2]. \quad (2.5)$$

For scalar multiplication, let $\otimes x \in [\underline{x}, \bar{x}]$, where $\underline{x} < \bar{x}$ and k is a positive real number, then we have $k.\otimes x \in [k\underline{x}, k\bar{x}]$. Note that when $k \le 0$ we have $k.\otimes x \in [k\bar{x}, k\underline{x}]$.

The transformation of an interval grey number to the appropriate crisp value can be made by using the whitening function, which can be shown as follows $\tilde{\otimes} x = \lambda \underline{x} + (1-\lambda)\bar{x}$ with λ as whitening coefficient and $\lambda \in [0,1]$ [29].

Theorem 2.1. [31] The set of all grey numbers forms a field.

Proof. It is straight forward as given in [31].

Definition 2.11. [31] Assume that $R(\otimes)$ is the set of all grey numbers. If for any $\otimes_i$, $\otimes_j$, $\otimes_k \in R(\otimes)$ the following hold true:

1) $\otimes_i + \otimes_j = \otimes_j + \otimes_i$

2) $(\otimes_i + \otimes_j) + \otimes_k = \otimes_i + (\otimes_j + \otimes_k)$

3) There is a zero element $\otimes 0 \in R(\otimes)$ such that $\otimes_i + \otimes 0 = \otimes_i$

4) For any $\otimes \in R(\otimes)$, there is a $(-\otimes) \in R(\otimes)$ such that $\otimes + (-\otimes) = \otimes 0$

5) $\otimes_i \times (\otimes_j \times \otimes_k) = (\otimes_i \times \otimes_j) \times \otimes_k$

6) There is unit element $\otimes 1 \in \mathrm{R}(\otimes)$ such that $\otimes 1 \times \otimes_i = \otimes_i \times \otimes 1 = \otimes_i$

7) $(\otimes_i + \otimes_j) \times \otimes_k = (\otimes_i \times \otimes_k) + (\otimes_j \times \otimes_k)$

8) $\otimes_i \times (\otimes_j + \otimes_k) = (\otimes_i \times \otimes_j) + (\otimes_i \times \otimes_k)$

Then $R(\otimes)$ is referred to as a grey linear space.

Theorem 2.2. [31] The totality of all grey numbers constitutes a grey linear space.

Theorem 2.3. [36] If $\otimes x \in [\underline{x}, \bar{x}]$ and $\otimes y \in [\underline{y}, \bar{y}]$ be two grey numbers, then if the conditions $\bar{x} \le \bar{y}$ and $\otimes\hat{x} \le \otimes\hat{y}$ are hold, it can be guaranteed that $\otimes x \leq_G \otimes y$.

Proof. It is straight forward as given in [36].

2.2. Possibility degree

Definition 2.12. [42] For two grey numbers $\otimes x \in [\underline{x}, \overline{x}]$ and $\otimes y \in [\underline{y}, \overline{y}]$, the possibility degree of that $\otimes x$ is less (greater) than or equal $\otimes y$ can be expressed as follows:

$$\rho = \frac{\max(0, L^* - \max(0, \overline{x} - \underline{y}))}{L^*} \qquad (2.6)$$

where $L^* = L(\otimes x) + L(\otimes y)$.

The relationship between $\otimes x$ and $\otimes y$ is determined as follows:

1) If $\underline{x} = \underline{y}$ and $\overline{x} = \overline{y}$, we say that $\otimes x$ is equal to $\otimes y$, denoted as $\otimes x = \otimes y$, when $\rho = 0.5$.

2) If there is the intersection, when $\rho > 0.5$, we say that $\otimes y$ is greater than $\otimes x$, denoted as $\otimes x <_G \otimes y$. when $\rho < 0.5$, we say that $\otimes y$ is less than $\otimes x$, denoted as $\otimes x >_G \otimes y$.

Grey number is the basic unit of grey system and the operations of grey numbers are different from regular interval numbers. Interval number refers to a special one in grey number conception terms [52]. It's easy to prove the following results.

Theorem 2.4. [49] Let $\otimes x \in [\underline{x}, \overline{x}]$, $\otimes y \in [\underline{y}, \overline{y}]$ and $\otimes z \in [\underline{z}, \overline{z}]$ be grey numbers. If $L(\otimes x)$, $L(\otimes y)$ is not 0 at the same time, then the possibility degree has the following properties:

(1) $0 \leq \rho(\otimes x \leq_G \otimes y) \leq 1$;

(2) $\rho(\otimes x \leq_G \otimes y) = 1$, if $\overline{x} \leq \underline{y}$;

(3) $\rho(\otimes x \leq_G \otimes y) = 0$, if $\underline{x} \geq \overline{y}$;

(4) (Complementation)

$\rho(\otimes x \leq_G \otimes y) + \rho(\otimes y \leq_G \otimes x) = 1$, especially

$\rho(\otimes x \leq_G \otimes y) = \frac{1}{2}$;

(5) (Transitive) If $\rho(\otimes x \leq_G \otimes y) \geq 0.5$ and $\rho(\otimes y \leq_G \otimes z) \geq 0.5$ then $\rho(\otimes x \leq_G \otimes z) \geq 0.5$.

Proof. It is straight forward as given in [49].

Theorem 2.5. [49] The set of grey numbers is a totally ordered set based on possibility degree. (A grey number set based on possibility degree is a totally ordered set.)

Proof. It is straight forward as given in

Lemma 2.1. For two grey numbers $\otimes x \in [\underline{x}, \overline{x}]$ and $\otimes y \in [\underline{y}, \overline{y}]$, $\otimes x \geq_G \otimes y$, if $-\otimes x \leq_G -\otimes y$.

Definition 2.13. A grey number $\otimes x \in [\underline{x}, \overline{x}]$ is said to be zero grey number, if $\otimes x \in [0,0]$.

Lemma 2.2. A grey number $\otimes x \in [\underline{x}, \overline{x}]$ is said to be nonnegative, if $\otimes \hat{x} \geq 0$.

Proof. Consider the definition of grey possibility degree as (2.6). Now let $\otimes x \in [\underline{x}, \overline{x}]$ and $\otimes 0 \in [0,0]$ if $\otimes x \geq_G \otimes 0$ then $\rho(0 \leq_G \otimes x) \geq 0.5$, conversely suppose that $\rho(0 \leq_G \otimes x) < 0.5$, this implies that: $\max(0, L^* - \max(0, 0 - \underline{x}) < 0.5 L^*$

1) if $\max(0, L^* - \max(0, 0 - \underline{x}) = 0$, it means that $L^* \leq \max(0, 0 - \underline{x})$. By substituting L^* definition, it require that $\overline{x} \leq 0$, which contradicts with positivity $\otimes x$.

2) if $\max(0, L^* - \max(0, \overline{x} - \underline{y}) \neq 0$, for $\rho(0 \leq_G \otimes x) < 0.5$ be true, it requires that $0.5 L^* < \max(0, -\underline{x})$, now

-if $\max(0, -\underline{x}) = 0$, it means that $L^* < 0$, which contradicts with its definition, which is given in (6)

-if $\max(0, -\underline{x}) = -\underline{x}$, it requires that $\frac{1}{2} L^* < -\underline{x}$. By the definition (2.6), we have

$$\frac{1}{2}\overline{x} - \frac{1}{2}\underline{x} < -\underline{x} \Rightarrow \frac{1}{2}\overline{x} < \frac{-1}{2}\underline{x} \Rightarrow \overline{x} < -\underline{x}$$

$$\Rightarrow \underline{x} + \overline{x} < 0.$$

This inequality holds only, we have $-\underline{x} > \overline{x}$ or $\underline{x} + \overline{x} < 0$, which contradicts with the proofivity $\otimes x$. Therefore, $\rho(0 \leq_G \otimes x) < 0.5$ cannot be true and proof is complete. ■

Remark 2.2. A grey number $\otimes x \in [\underline{x}, \overline{x}]$ is said to be positive, if $\otimes \hat{x} \geq 0$. it is clear that a grey number is said to be negative, if $-\otimes \hat{x} \geq 0$.

2.3. Grey degree and kernel

In this subsection, the concepts and definitions are taken from [13, 49, 51, 54].

Definition 2.14. Suppose grey number $\otimes x \in [\underline{x}, \overline{x}]$, $\underline{x} \leq \overline{x}$, in the case of lack of the distributing information of the values of grey number $\otimes x$, If a grey number $\otimes x$ is continuous, then $\otimes \hat{x} = \frac{1}{2}(\underline{x} + \overline{x})$ is called the Kernel of grey number $\otimes x$.

Definition 2.15. Suppose that the background which makes a grey number $\otimes x$ coming into being is Ω and $\mu(\otimes x)$ is the measure of Ω, then $g^{\circ}(\otimes x) = \mu(\otimes x) / \mu(\Omega)$ is called the degree of greyness of grey number $\otimes x$ (denoted as g° for short).

According to the properties of the measure and $\otimes x \subset \Omega$, the definition of the degree of greyness of grey number put forward in definition 2.15 satisfies normality, that is $0 \leq g^{\circ}(\otimes x) \leq 1$.

The degree of greyness of grey numbers reflects the uncertainty degree of the things described by these grey numbers. White numbers are completely assured numbers and their degrees of greyness are 0. Black numbers are completely unknown and their degrees of greyness are 1. Most of the degrees of greyness $g^{\circ}(\otimes x)$ are between 0 and 1. The nearer $g^{\circ}(\otimes x)$ close to 0, the smaller the uncertainty of the values of grey numbers is. And contrarily, the nearer $g^{\circ}(\otimes x)$ close to 1, the bigger the uncertainty of the values of grey numbers is.

Definition 2.16. Suppose Ω is the field of grey number $\otimes x$. When $\mu(\Omega)=1$, the corresponding grey number is called a standard grey number. The reduced form of the standard grey number is called standard form of grey number.

Proposition 2.1. Suppose that $\otimes x$ is a standard grey number, then $g^{\circ}(\otimes x)=\mu(\otimes x)$.

Definition 2.17. Let $\otimes \hat{x}$ and $g^{\circ}(\otimes x)$ be respectively the kernel and the degree of greyness of a grey number $\otimes x$. Then $\otimes x=\otimes \hat{x}_{(g^{\circ})}$ is seen as a simple form of the grey number $\otimes x$.

Example 2.1. Suppose that the interval grey numbers $\otimes_1 \in[-4,-2]$, $\otimes_2 \in[1,7]$, $\otimes_3 \in[-2,10]$, $\otimes_4 \in[1,5]$ are all in the field $\Omega \in[-5,20]$. If we take the length of the grey interval as the measure of the grey numbers, try to work out the reduced forms of these three grey numbers.

Solution: According to the known conditions, we can respectively work out the measures of field Ω, $\otimes_1$, $\otimes_2$, $\otimes_3$, $\otimes_4$:

$\mu(\Omega)=20-(-5)=25,\ \mu(\otimes_1)=2,$
$\mu(\otimes_2)=6,\ \mu(\otimes_3)=12,\ \mu(\otimes_4)=4.$

The kernel and the degree of greyness of these three grey numbers are:

$\otimes \hat{x}_1=-3,\ \otimes \hat{x}_2=4,\ \otimes \hat{x}_3=4,\ \otimes \hat{x}_4=3$;

$g^{\circ}(\otimes_1)=0.08,\ g^{\circ}(\otimes_2)=0.24,$
$g^{\circ}(\otimes_3)=0.48,\ g^{\circ}(\otimes_4)=0.16$ and their reduced forms are $\otimes_1=-3_{(0.08)}$, $\otimes_2=4_{(0.24)}$, $\otimes_3=4_{(0.48)}$, $\otimes_4=3_{(0.16)}$.

Proposition 2.2. For grey numbers, there is an one-to-one correspondence between the simplified forms $\otimes x=\otimes \hat{x}_{(g^{\circ})}$ and grey numbers $\otimes x\in[\underline{x},\overline{x}],\ \underline{x}<\overline{x}$.

Based on the simplified form $\otimes x=\otimes \hat{x}_{(g^{\circ})}$

of grey numbers, we can obtain the following properties of the arithmetic operations of grey numbers:

$$Ä_1\hat{x}_{(g^o)} = Ä_2\hat{x}_{(g^o)} \; Û$$

$$Ä_1\hat{x} = Ä_2\hat{x} \; and \; g^o(Ä_1 x) = g^o(Ä_2 x)$$

With the increasing development of grey system theory in various scientific fields and the need to compare grey numbers in different areas, ranking of grey numbers plays a very important role in decision-making and some other grey system application. Several strategies have been proposed for ranking of grey numbers. In this paper, we reviewed recent ranking method, which will be useful for the researchers who are interested in this area.

Definition 2.18. [13] Suppose the grey numbers $\otimes x$ and $\otimes y$, $\otimes_1 \hat{x}$ and $\otimes_2 \hat{x}$ are the kernel of $\otimes x$ and $\otimes y$, $g^\circ(\otimes_1 x)$ and $g^\circ(\otimes_2 x)$ are the degree of greyness of $\otimes x$ and $\otimes y$, so

If $\otimes \hat{x} < \otimes \hat{y}$, thus $\otimes x <_G \otimes y$;

If $\otimes \hat{x} = \otimes \hat{y}$, thus

(1) if $g^\circ(\otimes x) = g^\circ(\otimes y)$, thus $\otimes x =_G \otimes y$;

(2) if $g^\circ(\otimes x) < g^\circ(\otimes y)$, thus $\otimes x >_G \otimes y$;

(3) if $g^\circ(\otimes x) > g^\circ(\otimes y)$, thus $Äx <_G Äy$.

In this way, will be compared different interval grey numbers.

Example 2.2. Obviously, the ranking of grey numbers in Example 2.1 is $\otimes_1 <_G \otimes_4 <_G \otimes_3 <_G \otimes_2$.

3. Grey number linear programming

Original LP models are constructed based on four fundamental assumptions of proportionality, additivity, divisibility and deterministicity. The deterministicity assumption means that coefficients C, A and b are all known deterministically [39]. The LP method has two disadvantages. Firstly, many variables and coefficients are uncertain in the real world. They cannot be expressed by certain numbers. However, the LP model can only deal with certain messages. Secondly, solutions of LP models are often very sensitive to even very small changes of coefficients, which can affect the effectiveness of the programming [20].

3.1. The definition of Grey Number Linear Programming problem

LP problem with grey parameters is a one of convenient models of the well-known real problems as well as Water Resource Planning, Economics, Geometry and etc. In this section, we define LP problems involving grey numbers as follows:

$$\max \otimes Z =_G \sum_{j=1}^{n} (\otimes c_j) x_j$$
$$\textit{subject to} \quad \sum_{j=1}^{n} a_{ij} x_j \le b_i, \quad i = 1,2,3,\ldots,m, \qquad (3.1)$$
$$x_j \ge 0 \;, j = 1,2,3,\ldots,n,$$

Where

b_i, x_j, a_{ij} Î R, Äc_j Î R(Ä)$^{1´ n}$, $i = 1,2,3,\ldots,m$,
$j = 1,2,3,\ldots,n$.

We call the above problem as a Grey Number Linear Programming (GNLP) problem and it can be rewritten as $\max \otimes Z \approx \otimes c.x$ *subject to* $Ax \le b$, *and* $x \ge 0$, where $\otimes c$ is a $(1 \times n)$ matrix consisting of grey numbers and A, b, x are $(m \times n), (m \times 1)$ *and* $(n \times 1)$ real matrix.

Definition 3.1. Any x satisfying the set of constraints (3.1) of the GNLP problem is called a feasible solution. Let Q be the set of all feasible solutions of GNLP problem. Then, we say that $x_0 \in Q$ is an optimal feasible solution for the GNLP problem, if $\otimes c.x \le_G \otimes c.x_0$ all $x \in Q$.

Definition 3.2. If x be a the feasible solution of model (3.1) with maximization objective function, it is an optimal solution, if there are not any feasible x' that $\otimes Z(x) <_G \otimes Z(x')$.

Basic feasible solution

Consider the system $Ax = b$ and $x \ge 0$, where A is an $m \times n$ matrix and b is an m vector. Suppose that rank (A, b) = rank (A)= m. Partition A, after possibly rearranging the columns of A, as [B, N], where B, $m \times m$, is nonsingular. It is apparent that $x_B = (x_{B_1}, \ldots, x_{B_m})^T = B^{-1}b, x_N = 0$ is a solution of $Ax = b$. The vector $x = (x_B^T, x_N^T)^T$ where $x_N = 0$ is called a basic solution of the system. If $x_B \ge 0$, then x is called a Basic Feasible Solution (BFS) of the system and the corresponding grey objective value is $\otimes Z =_G \otimes c_B.x_B$, where $c_B = (c_{B_1}, \ldots, c_{B_m})$. Now, corresponding to every index j, $1 \le j \le n$, define: $y_j = B^{-1}a_j$ and $\otimes Z =_G \otimes c_B.y_j$. Observe that for any basic index $j = B_i, 1 \le j \le m$, we have:

Äz_j - Äc_j $=_G$ Ä$c_B B^{-1} a_j$ - Äc_j $=_G$ Ä$c_B e_i$ - Äc_j
$=_G$ Äc_j - Äc_j $=_G$ Ä0

where e_i is the i th unit vector. Note that B is called the basic matrix and N is called the non-basic matrix. The components of x_B are called basic variables and the components of x_N are called non-basic variables.

Theorem 3.1. Let the GNLP problem be non-degenerate. A basic feasible solution $x_B = B^{-1}b,\ x_N = 0$ is optimal to (3.1) if and only if $\otimes c_B.B^{-1}a_j \geq_G \otimes c_j$, for all j, $1 \leq j \leq n$.

Proof: Suppose that $x^* = (x_B^T, x_N^T)^T$ is a basic feasible solution to (3.1), where $x_B = B^{-1}b,\ x_N = 0,$ so that the optimal value of the objective function is $\otimes Z =_G \otimes c_B.x_B$. Then, $\otimes Z =_G \otimes c_B x_B =_G \otimes c_B B^{-1}b.$ on the other hand, for every feasible solution x, we have $b = Ax = Bx_B + Nx_N$. Hence, we obtain:

$$\otimes z =_G \otimes c.x =_G \otimes c_B x_B + \otimes c_N x_N =$$
$$_G \otimes c_B B^{-1} b - \sum_{j \neq B_i} (\otimes c_B B^{-1} a_j - \otimes c_j) x_j.$$

Then,

$$\otimes Z =_G \otimes Z^* - \sum_{j \neq B_i} (\otimes z_j - \otimes c_j) x_j. \qquad (3.2)$$

The proof can now be completed using (3.2) and Theorem 3.3 given in Subsection 3.2.■

In the next section, we devise a grey primal simplex algorithm for solving the GNLP problems.

3.2. Simplex Method for the GNLP Problems

A.GNLP simplex method in tableau format

Consider the GNLP problem as in (3.1). We rewrite the GNLP problem as:

$$Max \otimes Z =_G \otimes c_B x_B + \otimes c_N x_N$$
$$s.t. \quad Bx_B + Nx_N = b$$
$$x_B \geq 0,\ x_N \geq 0.$$

Hence, we have $x_B + B^{-1}Nx_N = B^{-1}b.$ Therefore

$$\otimes Z + (\otimes c_B B^{-1} N - \otimes c_N) x_N =_G \otimes c_B B^{-1} b.$$

Now with $x_N = 0$, we have $x_B = B^{-1}b = y_0$ and $\otimes Z =_G \otimes c_B B^{-1} b.$ Thus, we rewrite the above GNLP problem as in Table 1.

Remark 3.1. Table 1 gives all the information needed to proceed with the simplex method. The grey cost row in Table1 is

$\otimes z_N - \otimes c_N =_G \otimes c_B B^{-1} A - \otimes c,$ where

$$\otimes y_{0j} =_G \otimes c_B B^{-1} a_j - \otimes c_j =_G \otimes z_j - \otimes c_j,$$
$$1 \leq j \leq n,$$

with $\otimes y_{0j} =_G \otimes 0$ for $j = B_i,\ 1 \leq i \leq m.$ According to the optimality conditions (Theorem 3.1), we are at the optimal solution if $\otimes y_{0j} \geq_G \otimes 0$ for all $j \neq B_i,\ 1 \leq i \leq m.$ On the other hand, if $\otimes y_{0k} <_G \otimes 0,$ for some k ≠ B, $1 \leq i \leq m$, then the problem is either unbounded or an exchange of a basic variable x_{B_r} and the non-basic variable x_k can be made to increase the rank of the objective value (under nondegeneracy assumption).

Theorem 3.2. If in a GNLP simplex tableau, there is a column k (not in basis) so that $\otimes y_{0k} =_G \otimes z_k - \otimes c_k <_G \otimes 0$ and $y_{ik} \le 0,\ i = 1,..., m,$ then the GNLP problem is unbounded.

Proof: Suppose that x_B is a basic feasible solution to (3.1) i.e. $\sum_{i=1}^{m} x_{B_i} b_i = b.$ Also let $\otimes Z =_G \otimes c_B . x_B$, $\otimes z_k - \otimes c_k \le_G \otimes 0$ and $y_{ik} \le 0,\ i = 1,..., m.$ Then for any scalar $\theta > 0$ we have $\sum_{i=1}^{m} x_{B_i} b_i - \theta a_k + \theta a_k = b,$ i.e.

$$\sum_{i=1}^{m} (x_{B_i} - \theta y_{ik}) b_i + \theta a_k = b. \qquad (3.3)$$

It is clear that (3.3) is a feasible solution in which m+1 variables can be different from zero. Now, we compute the value of $\otimes \hat{z}$ for the feasible (but in general non-basic) solution (3.3):

$$\otimes \hat{Z} = \sum_{l} \otimes c_l x_l = \sum_{i=1}^{m} \otimes c_{B_i} (x_{B_i} - \theta y_{ik}) + \theta \otimes c_k \text{ or}$$

$$\otimes \hat{Z} = [\underline{z} - \theta \underline{z}'_k, \bar{z} - \theta \bar{z}'_k\ ] + \theta\ [\underline{c}_k, \bar{c}_k] \qquad (3.4)$$

Since $y_{ik} \le 0,\ i = 1,..., m,$ hence $\underline{z}'_k \ge 0,\ \bar{z}'_k\ \ge 0.$ we conclude that

$$\otimes \hat{Z} = [\underline{z}, \bar{z}] + \theta\ [-\underline{z}'_k, -\bar{z}'_k\ ] + \theta\ [\underline{c}_k, \bar{c}_k] \quad \text{Or}$$

$$\otimes \hat{Z} = \otimes Z + \theta\ [\underline{c}_k - \underline{z}'_k, \bar{c}_k - \bar{z}'_k\ ] \qquad (3.5)$$

Clearly if $\theta > 0$, $\otimes \hat{Z}$ can be made arbitrarily large. Hence, we have unbounded solutions.

Theorem 3.3. If in a GNLP simplex tableau, a non-basic index k exists such that $\otimes y_{0k} =_G \otimes z_k - \otimes c_k \le_G \otimes 0$ and there exists a basic index B_i such that $y_{ik} > 0$, then a pivoting row r can be found so that pivoting on y_{rk} yields a feasible tableau with a corresponding nondecreasing (increasing under nondegeneracy assumption) grey objective value.

■

Table 1. The GNLP simplex tableau

Basis	x_B	x_N	R.H.S.
$\otimes z$	0	$\otimes z_N - \otimes c_N =_G \otimes c_B B^{-1} N - \otimes c_N$	$\otimes y_{00} =_G \otimes c_B B^{-1} b$
x_B	I	$Y = B^{-1} N$	$y_0 = B^{-1} b$

Remark 3.2. If k exists such that $\otimes y_{0k} \leq_G \otimes 0$ and the problem is bounded, then r can be chosen so that

$$\frac{y_{k0}}{y_{rk}} = \min_{1\leq i\leq m}\left\{\frac{y_{i0}}{y_{ik}} | y_{ik} > 0\right\},$$

in order to replace x_{B_r} in the basis by x_k, resulting in a new basis $\hat{B}(a_{B_1},...,a_{B_{r-1}},a_k,a_{B_{k+1}},...,a_{B_m})$. The new basis is primal feasible and the corresponding grey objective value is nondecreasing (increasing under nondegeneracy assumption). It can be shown that the new simplex tableau is obtained by pivoting on y_{rk}, that is, doing Gaussian elimination on the kth column using the pivot row r, with the pivot y_{rk}, to transform the kth column to the unit vector e_r. It is easily seen that the new grey objective value is:

$$\otimes \hat{y}_{00} =_G \otimes y_{00} - \otimes y_{0k} \frac{y_{r0}}{y_{rk}} \geq_G \otimes y_{00},$$

Since $\otimes y_{0k} \leq_G \otimes 0$ and $\frac{y_{r0}}{y_{rk}} > 0$ (if the problem is non-degenerate, then $\frac{y_{r0}}{y_{rk}} > 0$ and hence $\otimes \hat{y}_{00} >_G \otimes y_{00}$).

We now describe the pivoting strategy.

B. Pivoting and change of basis

If x_k enters the basis and x_{B_r} leaves the basis, then pivoting on y_{rk} in the simplex tableau is carried out, as follows:

1) Divide row r by y_{rk}.

2) For $i = 0,1,...,m.$ and $i \neq r$, update the i th row by adding to it $-y_{ik}$ times the new r th row.

We now present the simplex algorithm for the GNLP problem.

C. The main steps of GNLP simplex algorithm

Algorithm 1: The Grey simplex method for the GNLP problem.

Assumption: A basic feasible solution with basis B and the corresponding simplex tableau is at hand.

1. The basic feasible solution is given by $x_B = y_0$ and $x_N = 0.$ the grey objective value is $\otimes z =_G \otimes y_{00}$.

2. Calculate

$$\otimes y_{0j} =_G \otimes z_j - \otimes c_j, j = 1,...,n, \ j \neq B_i, i = 1,...,m.$$

Let $\otimes y_{0k} =_G \min_{j=1,...,n} \{\otimes y_{oj}\}$. If $\otimes y_{0k} \geq_G \otimes 0$, then stop; the current solution is optimal.

3. If $y_k \leq 0$, then stop; the problem is unbounded. Otherwise, determine an index r corresponding to a variable x_{B_r} leaving the basis as follows:

$$\frac{y_{r0}}{y_{rk}} = \min_{1\leq i\leq m}\left\{\frac{y_{i0}}{y_{ik}} | y_{ik} > 0\right\}.$$

4. Pivot on y_{rk} and update the simplex tableau. Go to step 2.

We have seen that for a column vector a_j of A which is not in B, for which $(\otimes z_j - \otimes c_j) <_G \otimes 0$ and $y_{ij} > 0$, for some i, is alone considered for inserting into the basis. Let us now discuss the situation when there exists an a_j such that $(\otimes z_j - \otimes c_j) <_G \otimes 0$ and $y_{ij} \leq 0$ for all $i = 1,2,3,...,m$. If $x = [\underline{x}, \overline{x}] \geq_G \otimes 0$ and $\lambda > 0$, then $\lambda . \otimes x = [\lambda \underline{x}, \lambda \overline{x}] \geq_G \otimes 0$. Now λ can be made sufficiently large so that $\lambda . \otimes x \geq_G \otimes y$ for any interval grey number $\otimes y$. If $\lambda > 0$, $(\otimes z_j - \otimes c_j) <_G \otimes 0$ then $\lambda .(\otimes z_j - \otimes c_j) <_G \otimes 0$. Now the proof of the following theorem follows easily.

D. Conditions of Optimality

As in the classical LP problems, we can prove that the process of inserting and removing vectors from the basis matrix will lead to any one of the following situations:

(i) There exist j such that $(\otimes z_j - \otimes c_j) <_G \otimes 0$, $y_{ij} \leq 0$, i = 1, 2, 3,..., m. or

(ii) For all j, $(\otimes z_j - \otimes c_j) \geq_G \otimes 0$

In the first case, we get an unbounded case and also we have an optimal solution, if the second case occurs.

Theorem 3.4. If $x_B = B^{-1}b$ is a BFS to problem (3.1) and if $(\otimes z_j - \otimes c_j) \geq_G \otimes 0$ for every column a_j of A, then x_B is an optimal solution to (3.1).

Proof: Let $x_j \geq 0$, $j = 1,..., n$, be any feasible solution for (3.1), i.e.

$$x_1 a_1 + x_2 a_2 + ... + x_n a_n = b. \tag{3.6}$$

The corresponding grey value of the objective function, denoted by $\otimes z_j^*$, is

$$\otimes z^* = \otimes c_1 x_1 + \otimes c_2 x_2 + ... + \otimes c_n x_n. \tag{3.7}$$

On substituting $a_j = \sum_{j=1}^{n} y_{ij} b_i$ into (3.6), we obtain

$$(\sum_{j=1}^{n} x_j y_{1j}) b_1 + (\sum_{j=1}^{n} x_j y_{2j}) b_2 + ... + (\sum_{j=1}^{n} x_j y_{mj}) b_m = b.$$

Therefore,

$$x_{B_i} = \sum_{j=1}^{n} x_j y_{ij}. \tag{3.8}$$

Assume that a_j is in column i of B. Hence,

$\otimes z_j = c_B y_j = c_B e_i = c_j.$

Thus for every column of A we have $\otimes z_j - \otimes c_j \geq_G \otimes 0$. Consequently, using $\otimes z_j - \otimes c_j \geq_G \otimes 0$ in (3.7) we see that

$$\otimes z_1 a_1 + \otimes z_2 a_2 + ... + \otimes z_n a_n \geq_G \otimes z^*. \tag{3.9}$$

on substituting $\otimes z_j = \sum_{j=1}^{m} \otimes c_{B_i} y_{ij}$ into (3.9) we obtain

$$(\sum_{j=1}^{n} x_j y_{1j}) \otimes c_{B1} + (\sum_{j=1}^{n} x_j y_{2j}) \otimes c_{B2} + ... + (\sum_{j=1}^{n} x_j y_{mj}) \otimes c_{Bm} \geq_G \otimes z^*.$$

Now if $\otimes z_0 = \otimes c_B x_B$, we have

$$\otimes z_0 = x_{B_1} \otimes c_{B_1} + x_{B_2} \otimes c_{B_2} + \ldots + x_{B_m} \otimes c_{Bm} \geq_G \otimes z^*.$$

This proves that x_B is an optimal solution of (3.1).■

4. Application of grey linear programming

Ren et al. [38] used grey programming approach to planning model for the chemical integrated system under uncertainty and they have shown in the grey programming model, the decision makers can suggest the value of credibility, and the results of the production planning calculated by the model can help them to achieve their desired target. LP based on grey constrained interval was applied to solve the short-term hydro scheduling problem. To reach an optimal scheduling under the uncertain environment, a GLP model in which the hourly loads, the hourly natural inflows were all expressed in grey number notation, was developed. The developed GLP approach is applied to schedule the generation in the Taiwan power system [25]. The GLP is used as a decision tool for Time-cost-quality trade off problem with the grey information. A combination of fuzzy goal programming and GLP is also developed to solve the proposed model. The most important aspect of the proposed model is that it considers uncertainty of the project planning data in the form of grey numbers [37].

4.1. Numerical example

In this section, for an illustration of the above approach, a numerical example of the GNLP will be solved based on the proposed method.

Table 2. The first simplex tableau of the GNLP problems.

Basis	x_1	x_2	s_1	s_2	R.H.S.
$\otimes Z$	$[-3,-1]_G$	$[-5,-2]_G$	0_G	0_G	0_G
s_1	2	3	1	0	6
s_2	3	1	0	1	4

Table 3. The optimal simplex tableau of the GNLP problems.

Basis	x_1	x_2	s_1	s_2	R.H.S.
$\otimes Z$	$[-\frac{5}{3},\frac{7}{3}]_G$	0_G	$[\frac{2}{5},\frac{5}{3}]_G$	0_G	$[4,10]_G$
x_2	$\frac{2}{3}$	1	$\frac{1}{3}$	0	2
s_2	$\frac{7}{3}$	0	$-\frac{1}{3}$	1	2

Example 4.1. Consider the following GNLP problem:

$$Max \quad \otimes Z = \otimes c_1.x_1 + \otimes c_2.x_2$$
$$subject \quad to$$
$$2x_1 + 3x_2 \le 6$$
$$3x_1 + x_2 \le 4$$
$$x_1, x_2 \ge 0$$

where $\otimes c_1 \in [1,3], \otimes c_2 \in [2,5]$. The above problem calls as the primal problem. Solving the GNLP as follows:

The standard form of the given GNLP becomes

$$Max \quad \otimes Z - \otimes c_1.x_1 - \otimes c_2.x_2 = 0$$
$$subject \quad to$$
$$2x_1 + 3x_2 + s_1 = 6$$
$$3x_1 + x_2 + s_2 = 4$$
$$x_1, x_2, s_1, s_2 \ge 0$$

Since for all j, the current basic feasible solution is optimal. The optimal solution is $(x_1^*, x_2^*) = (0,2)$ and $\otimes Z^* \in [4,10]$.

5. Conclusion

The LP model is easy, but there are too many approximations and hypotheses in the model to acquire the accurate solution. The papers published before usually provide a model with many parameters, e.g. the price of the production, the price of the raw materials, the production capacity, the market demand etc. The assumption that these parameters are constants is not accurate because the price of the raw materials and the production capacity are mutative with time. Fuzzy programming can also deal with uncertainty problems, but membership functions are needed in fuzzy programming, meanwhile it cannot reflect the interval which the mutative parameter belongs to directly. The grey information (parameter belongs to an interval) is used instead of white information (parameter is a constant). Grey programming is different from interval programming. The main difference of grey programming and interval programming is the concept of grey parameter and interval parameter. Grey parameter is a parameter which belongs to an interval, but the interval parameter is a set in the form of an interval. Besides probability theory and fuzzy system theory, grey system theory is another important methodology that tackles uncertainty. These theories solve different problems of uncertainty. The field of GLP has recently attracted significant interest. GLP is a model of grey systems analysis for decision making under uncertainty.

In general, the GLP problems transformed into one or a series of the classical LP problems and then obtained an optimal solution. An algorithm like well-known primal simplex algorithm for solving GLP problems is proposed, where it can solve the problem without converting to the classical LP problems. The proposed approach will be useful for establishing the sensitivity analysis when a real problem is formulated as the GNLP problem.

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