Available online at http://jnrm.srbiau.ac.ir
Vol.1, No.2, Summer 2015

Journal of New Researches in Mathematics

Science and Research Branch (IAU)

Introducing a secondary goal for evaluating DMUs by cross efficiency in data envelopment analysis

R.Mehrjoo [a], N.Shoja [b*]

[(a)] *Department of Mathematics, Science and Research Branch, Islamic Azad University, Tehran, Iran*
[(b)] *Associate Professor, Islamic Azad University, Firoozkuh Branch, Firoozkuh, Iran*

Received Spring 2015, Accepted Summer 2015

Abstract

One way to rank DMUs in DEA is the cross efficiency method. In this method, the efficiency of each DMU is calculated by other DMUs optimum weights, which makes the ranking more acceptable for managers. Existing alternative optimum weights in cross efficiency method lead to several ranks for DMUs. Several secondary goals have introduced to avoid this problem, till now. In this paper, a new model is presented, that would be satisfying and acceptable for all DMUs. Therefore, by solving this model, the optimum weights are agreeable and fairy for DMUs.

Keywords: *Data Envelopment Analysis, Ranking, Cross Efficiency, Secondary Goal.*

---

*Corresponding author: Shoja@iaufb.ac.ir

1. Introduction

Ranking DMUs in data development analysis models was introduced by Sexton et al (1986), which is an effective way to rank DMUs. However, in this method, existence of alternative weights provides several ranks for the DMUs, makes the model unable to rank them correctly. In fact, existence of alternative weights caused to several rank. After that, many different methods have been presented to find the most appropriate weight among optimum weights, in each a secondary goal has been considered as an objective function. Regarding the secondary goal, a weight would be selected which achieve those goals. Doyle and Green (1994) introduced aggressive and benevolent models. Wang et al (2011) discussed the ways to define weights in cross efficiency. Furthermore, Wang et al (2011) applied the Ideal and Anti-Ideal DMUs for combination of DMUs' cross efficiency. Liang et al (2008) used Game Theory in cross efficiency and generated game efficiency. Rudder and Reucher (2011) represented a DEA model by using Peer-DMU which is optimized cross efficiency. More information would be found in Jahanshahloo et al (2008), and Alder et al (2002).

In this paper, a model is introduced to deal with such problem, in which the difference between cross efficiencies of DMUs obtained by DMUO would be minimized simultaneously. In this regard, by solving one model for DMUO, among all weights related to that, the one would be selected that minimizes the difference. Therefore, the obtained weight can be an equitable criterion for ranking the other DMUs, which makes the ranking process complete, satisfying, and acceptable. In fact, the weights extracted in this model, minimize the difference of upper-band and lower-band. The upper-band is the maximum efficiency score which could be given to other DMUs by the optimum weight of DMUO. On the other hand, the lower-band is the minimum efficiency score. Initially, the model obtaining in this way, is non-linear that would be illustrated that some constraints are redundant, so they can be omitted. Then the presented model would be solved for each DMU. Regarding the resulted weight for each DMU, the cross efficiency for all DMUs would be implied, and then ranking is calculated by combination of them.

2. CCR model

Suppose that we have N determining unit of DMU that use input M`s to produce output S`s . The vector of X_j (j = 1, … , n) is the corresponded vector of DMU_j`s inputs that $X_j=(X_{1j},\ldots,X_{mj})$ is an m vector and the vector of Y_j is a correspondent vector of DMU that $Y_j= (Y_{1j},\ldots,Y_{sj})$ is a kind of vector with S and vector relating to DMU_j is defined in the form of (X_j,Y_j) j=1,…,n . In self – assessment method with solving the following model, an efficiency amount for DMU_0 is obtained. The following model is a CCR multiple model.

$$Max \quad \theta_o = \frac{u^t y_o}{v^t x_o}$$

$$s.t. \qquad \frac{U^t y_o}{V^t x_o} \le 1 \qquad j = 1, \ldots, n \qquad (1)$$

$$u \ge o \quad v \ge o$$

Which is fractional model of CCR, but with changes of Charnes & Cooper (1962) this model is turned into the following linear model which is called CCR model with input nature.

$$\theta_o^* = Max \quad u^t y_o$$
$$v^t x_o = 1$$
$$u^t y_j - v^t x_j \le o \quad j = 1, \ldots, n \qquad (2)$$
$$(u, v) \ge o.$$

θ_o^* the obtained optimum amount from above – mentioned model, is the efficiency amount of DMU_0 which has been obtained through self – assessment method.

3. Cross efficiency

For obtaining cross efficiency of other DMUs we use optimum multiplications of the model and place vector of X_j , Y_j in it and then they are defined.

$$E_{jk} = \frac{\sum_{r=1}^{s} u_{rk}^* y_{rj}}{\sum_{i=1}^{m} v_{ik}^* x_{ij}} \qquad j = 1, \ldots, n$$

It is obvious that $\Theta_0^* = E_{kk}$ and also because $\sum_{i=1}^{m} v_{ik}^* x_{ik} = 1$ is one offeatur of this model , so $E_{rk} = \sum_{r=1}^{s} u_{rk}^* y_{rk}$. The intersection efficiency relating to DMU has been defined as follows: $E_j = \frac{1}{n}\sum_{k=1}^{n} E_{jk}$

4. propose model

It is assumed that all DMUs were evaluated by CCR model, so the efficiency of them were obtained, where θ_O^* is the efficiency of DMOU. Consider the following model:

$$Min \qquad \theta^u{}_o - \theta^l{}_0$$
$$s.t. \qquad \frac{UY_O}{VX_O} = \theta_O{}^*$$
$$\frac{UY_j}{VX_j} \le 1 \qquad j = 1, \ldots, n \qquad (3)$$
$$\theta^u{}_o = max\left\{\frac{UY_1}{VX_1}, \frac{UY_2}{VX_2}, \ldots, \frac{UY_n}{VX_n}\right\} \qquad (*)$$
$$\theta^l{}_o = min\left\{\frac{UY_1}{VX_1}, \frac{UY_2}{VX_2}, \ldots, \frac{UY_n}{VX_n}\right\} \qquad (**)$$
$$(U, V) \ge 0.$$

It should be mentioned that if $\frac{UY_o}{VX_0}$ exists in (*) and (*,*) constraints, $\theta^l{}_o = \theta_o^u$ and the value of objective function would be zero. Since the existence of DMU is not necessary, in (*) and (*,*) expression, $\frac{UY_o}{VX_0}$ would be removed. Also, (*) and (*,*) constraints in (3) model converted in the following in model (4).

$$\begin{aligned}
&\min \quad \theta^u{}_o - \theta^l{}_0 \\
&s.t. \quad \frac{UY_o}{VX_o} = \theta^*{}_o \\
&\frac{UY_j}{VX_j} \le 1 \qquad j = 1, \dots, n \\
&\theta^u{}_o \ge \frac{UY_j}{VX_j} \qquad j = 1, \dots, n, \quad j \ne 0 \qquad (4) \\
&\theta^u{}_o = \frac{UY_1}{VX_1}\ or\ \theta^u{}_o = \frac{UY_2}{VX_2}\ or \dots or \\
&\theta^u{}_o = \frac{UY_n}{VX_n} \qquad (*) \\
&\theta^l{}_o \le \frac{UY_j}{VX_j} \qquad j = 1, \dots, n, \quad j \ne 0 \\
&\theta^l{}_o = \frac{UY_1}{VX_1}\ or\ \theta^l{}_o = \frac{UY_2}{VX_2}\ or \dots or \\
&\theta^l{}_o = \frac{UY_n}{VX_n} \qquad (**) \\
&(U, V) \ge 0.
\end{aligned}$$

Theorem1: in model (4), (*) and (*,*) constraints are redundant in the optimum solution.

Proof:

Since the $\theta^u{}_o$ is minimized in objective function, so one of the equations in (*) constraint applies in optimum solution definitely, and because $\theta^l{}_o$ is maximized in objective function, then one of the equations in (**) constraint applies in optimum solution.

Therefore, according to Theorem, model (3-2) and following model have the same optimum solution.

$$\begin{aligned}
&\min \quad \theta^u{}_o - \theta^l{}_o \\
&s.t. \quad \frac{UY_o}{VX_o} = \theta^*{}_o \\
&\frac{UY_j}{VX_j} \le 1 \quad j = 1, \dots, n \qquad (*) \\
&\theta^u{}_o \ge \frac{UY_j}{VX_j} \quad j = 1, .., n, \quad j \ne 0 \qquad (5) \\
&\theta^l{}_o \le \frac{UY_j}{VX_j} \quad j = 1, .., n, \quad j \ne 0 \\
&(U, V) \ge 0.
\end{aligned}$$

Theorem2: The model (5) and the following one are equivalent.

$$\begin{aligned}
&\min \quad \theta^u{}_o - \theta^l{}_o \\
&s.t. \quad \frac{UY_o}{VX_o} = \theta^*{}_o \\
&\theta^u{}_o \ge \frac{UY_j}{VX_j} \quad j = 1, .., n, \quad j \ne 0 \qquad (6) \\
&\theta^l{}_o \le \frac{UY_j}{VX_j} \quad j = 1, .., n, \quad j \ne 0 \\
&\theta^u{}_o \le 1 \qquad (**) \\
&(U, V) \ge 0.
\end{aligned}$$

In fact, in model (5) the (*) constraint is replaced by (**) constraint in model (6).

Proof:

It is obvious that (*) and (*,*) constraints in model (6) cause to application of (*) constraint in model (5). So, by solving the model (6) the obtained optimum solution makes the difference of cross efficiency minimum. At least , the obtained weight results that one of the DMUs gets the highest efficiency score, and the other one has the minimum reduction.

5. Numerical example

This example was resented in several papers till now. we compose this example for illustrating our model. Information of the problem is shown in table (1).

The weights of each DMU after solving model is given by table (2).

So in this case because of obtaining several efficient DMUs, A method to rank all DMUs is necessary. We used our method and solve the model for each DMU and the unique weights were obtained that shown in table (3).

Table (1)

DMU	x_1	x_2	y_1	y_2	$DEA\ score$
1	1.5	0.2	1.4	0.35	1
2	4	0.7	1.4	2.1	1
3	3.2	1.2	4.2	1.05	1
4	5.2	2	2.8	4.2	1
5	3.5	1.2	1.9	2.5	0.9775
6	3.2	0.7	1.4	1.5	0.8674

Table (2)

	θ	v_1	v_2	u_1	u_2
DMU1	1	0	5.00	0.71	0
DMU2	1	0	1.43	0	0.48
DMU3	1	0.31	0	0.24	0
DMU4	1	0.19	0	0	0.24
DMU5	0.98	0.11	0.51	0.12	0.3
DMU6	0.87	0.15	0.72	0.16	0.43

Table (3)

	$\theta_u - \theta_l$	θ_l	θ_u	v_1	v_2	u_1	u_2
DMU1	0.14	0.86	1	0.2	0.8	0.2	0.49
DMU2	0.17	0.83	1	0.18	0.82	0.18	0.49
DMU3	0.2	0.8	1	0.32	0.68	0.31	0.51
DMU4	0.14	0.86	1	0.18	0.8	0.2	0.49
DMU5	0.17	0.83	1	0.18	0.82	0.19	0.49
DMU6	0.18	0.82	1	0.17	0.83	0.18	0.49

Theorem3. If the under evaluating DMU is inefficient, then for the optimum solution of model (6) always $\theta_j^{u^*} = 1$.

Proof. According to the constraints ${\theta^u}_o \leq 1$ and ${\theta^u}_o \geq \frac{UY_j}{VX_j}$:

$$\frac{UY_j}{VX_j} \leq 1 \text{ ، } j = 1, \ldots, n \text{ ، } j \neq o$$

And also because $\theta_0^* \leq 1$ then $\frac{UY_o}{VX_o} \leq 1$ and the constraint $(U, V) \geq 0$ causes each feasible solution of model (3-4) is a feasible solution for CCR model in evaluating DMU_o , and $\theta_0^* = \frac{UY_o}{VX_o}$, so each feasible solution of model (3-4) is an optimum solution in CCR model in evaluating DMU_o . Because at least one of constraints $\frac{UY_j}{VX_j} \leq 1$ is tight, then $\theta_j^{u^*} = 1$.

6. Conclusion

In this paper, A new method is introduced for ranking DMUs as a secondary goal. The model minimizes the differences between upper and lower bounds af solutions. So, all DMUs is ranked by this method. Also we present a simpler model for inefficient DMUs.

A numerical example is given in this paper, solved by the new method, which ranked all the DMUs. This method can be developed in other cases, for example : DMUs by interval data, negative data and etc.

References:

Adler, N., Friedman, L., Sinuany-Stern, Z,.2002. Review of methods in the data envelopment analysis context. European Journal of Operational Research. 140.249-265.

Allen, R., Athanassopoulos, A., Dyson, R.G., Thanassoulis, E., 1997. Weights restrictions and value judgements in data envelopment analysis: evolution, development and future directions. Annals of Operations Research 73, 13–34.

Andersen, P., & Petersen, N. C. (1993). A procedure for ranking efficient units in data envelopment analysis. Management Science, 39, 1261–1264.

Banker, R.D., Charnes, A., Cooper, W.W., 1984. Some models for estimating technical and scale inefficiencies in data envelopment analysis. Management Science 30, 1078-1092.

Charnes, A., Cooper, W.W., Rhodes, E., 1978. Measuring the efficiency of decision-making units. European Journal of Operational Research 2, 429-444.

Dimitris k., Despotis, Lampirini v., Stamati, Yiannis G. Smirlis,. 2009 data envelopment analysis with nonlinear virtual outputs and inputs. European Journal of Operational Research. 202. 604-613

S. Dimitrov, W. Sutton, Promoting symmetric weight selection in data envelopment analysis: a penalty function approach, European Journal of Operational Research, 200 (1)(2010) 281-288.

Doyle, J. R., & Green, R. H. (1994). Efficiency and cross-efficiency in DEA :
Derivations, meanings and uses. Journal of the Operational Research Society, 45, 567–578.

Farell, M.J., 1957. The measurement of productive efficiency. Jornal of the Royal Statistical Society 120, 253-290.

G.R. Jahanshahloo, F. Hosseinzadeh Lotfi, Y. Jafari, R. Maddahi, Selecting symmetric weights as a secondary goal in DEA cross-efficiency evaluation, Applied Mathematical Modelling, 35 (2011) 544–549.

Liang , L., Wu, J., Cook, W.D., Zhu, J., 2008. Alternative secondary goals in DEA cross_efficiency evaluation. International Journal of Production Economic 113(2), 1025-1030.

Rodder, W., Reucher,T,./ European Journal of Operational Research 212 (2011) 148-154.

Sexton, T. R., Silkman, R. H., & Hogan, A. J. (1986). Data envelopment analysis: Critique and extensions. In R. H. Silkman (Ed.), Measuring efficiency: An assessment of data envelopment analysis (pp. 73–105). San Francisco, CA: Jossey-Bass

Wang, Y. M., Luo, Y., & Liang, L. (2011). Common weights for fully ranking decision making units by regression analysis. Expert Systems with Applications (2011), doi:10.1016/j.eswa.2011.01.004

Wang, Y.M., Chin, K.S., Luo, Y., 2011.Cross-efficiency evaluation based on ideal and anti-ideal decision making units. Expert Systems with Applications 38 (8) 10312-10319.