A Novel High-Order Fuzzy Systems with Decomposition in to Zero-and First-Order Fuzzy Structures in Nonlinear Dynamic Systems

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Abstract– Fuzzy modeling is a relatively new system modeling method with a proven efficiency record in various fields. Although zero- and first-order fuzzy systems are common due to their simplicity, their linear structure faces challenges when modeling nonlinear systems with state-variable interaction. These challenges include an increase in the number of rules and the inability to stabilize highly nonlinear systems. One solution is to use high-order fuzzy systems, which have a nonlinear structure and can represent model input interactions. In previous research, high-order fuzzy modeling has been investigated for static and nonlinear systems based on data, but such modeling has not been applied to dynamic systems with nonlinear nature which is a model of i ndustrial processes. The present paper proposes a novel fuzzy structure inspired by the Taylor series expansion for dynamic systems with nonlinear state-space equations. This structure has a high degree of freedom in modeling complex nonlinear processes and can be adapted to the state-space equations of the system. The main novelty of this method is the conversion of a nonlinear high-order fuzzy structure into a set of first-order fuzzy structures. Another advantage is the ability to calculate the coefficients of the high-order fuzzy system from the Taylor series coefficients of the dynamic system's model. Fuzzy systems have made various applications possible in the field of approximation. The present paper also proves the approximation ability and convergence of the proposed structure and determines its convergence criteria.

Keywords: Nonlinear dynamic system modeling, high-order fuzzy systems, approximation, Taylor series expansion, stability

1. Introduction

Process control is a major field of automation and control. Nonlinearities and other complexities in industrial processes necessitate more advanced controllers. This is due to their role in reducing production costs, system stabilization time, and production waste. Intelligent control techniques require suitable models based on real system data so that both model accuracy can be improved and model stability can be demonstrated [1-3]. Among modeling methods, fuzzy models are a relatively new technique with a proven history of efficiency in various applications [4-7]. The main advantage of fuzzy systems is in the modeling and control of nonlinear systems. However, an increase in the complexity of these systems leads to an increase in the number of rules needed to reduce modeling errors. This problem is intensified with the increase in input interactions in such systems. Although zero- and first-order fuzzy systems have shown good modeling performance in numerous cases, they may sometimes face problems. Specifically, in systems where the training data originate from a complex nonlinear process and there is considerable interaction between the input variables, the number of considered fuzzy rules must be larger in order to achieve a satisfactory estimation accuracy. In this case, the cluster radius is assumed to be sufficiently small, and the data are clustered in regions where the system behavior is nearly linear, leading to a larger number of fuzzy rules. This issue, in turn, causes the fuzzy system parameters (the value of which must be found) to increase. As a result, determining these parameters and simulating the resulting system becomes more difficult and time-consuming [8-10]. In recent years, high-order fuzzy systems have been proposed to deal with this problem. A high-order fuzzy system refers to a Sugenofuzzy system where the function in the

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Received: 2023.08.12; Accepted: 2023.09.27

consequent of the rule has an order higher than 1 [11- 13]. The presence of these nonlinear terms in the high-order structure, which include products of the inputs, allows the Sugeno fuzzy system to retain its benefits while performing estimations with fewer rules but with the same accuracy as zero- and first-order systems. In other words, higher-order systems exhibit smaller estimation errors for a given number of rules[14]. In addition, a reasonable number of rules increases the interpretability and transparency of the fuzzy system. Also, by paying attention to the rules and the corresponding fuzzy sets, one can better understand the phenomenon being modeled while performing estimations [12-17].

Nonlinearities and other complexities in industrial processes necessitate more advanced controllers. This is due to their role in reducing production costs, system stabilization time, and production waste. Various fuzzy system training methods have been used in the literature with the aim of achieving a model that both has a high estimation accuracy and minimizes the complexities of computing the fuzzy model coefficients and membership functions [4-6,18-20].

Second-order (quadratic) fuzzy models for nonlinear systems were proposed by [9, 21, 14]. In [9, 22, 23], a second-order fuzzy model was used to predict time series, such as the Mackey-Glass series. The hierarchical representation of high-order fuzzy systems via lower-order systems was demonstrated in [21]. Also, the approximation accuracy of smooth fuzzy models was discussed in [24]. Another application of second- and higher-order systems was demonstrated in [25, 26, 14]. These studies included a fuzzy-based model predictive control system and an optimal controller design for maximum power tracking implemented on a solar electricity generation process using photovoltaic cells. The results were compared to those in other studies. Takagi-Sugeno-Kang (TSK) fuzzy systems were used in [27] for predicting short-term electricity prices. In [28], a fuzzy model predictive controller using kernel ridge regression based on the TSK fuzzy model (TS-KRR) was proposed for discrete-time nonlinear systems. This system approximates an unknown nonlinear system by training the fuzzy parameters of the Takagi-Sugeno (TS) model from the input-output data. Then, a generalized model predictive controller is created via the TS-KRR combination and is simulated on a continuous stirred tank reactor (CSTR). The results were compared with those of the adaptive TS-KRR model predictive controller. A discrete generalized fuzzy model predictive controller using the TS fuzzy model based on multi-kernel least-squares support vector regression (MKLSSVR) was introduced in [29]. The reliability of this method in the nonlinear CSTR system was investigated. A novel learning algorithm based on the unstable Kalman filter was proposed in [30] for identifying the coefficients of the TS fuzzy model. In [31], the nonlinear model of a flying object was converted to a fuzzy polynomial model using fuzzy logic. Based on this model, a polynomial fuzzy controller was designed in the discrete-time domain for tracking the reference model. In this method, the designed controller guarantees system stability within the considered speed range based on Lyaponuv's stability theory. Another application of high-order fuzzy models is in forecasting enrollments [22]. Observer-based tracking control was designed for a class of polynomial fuzzy systems without disturbance in [32] and with disturbance in [33].

Since stability is required in every nonlinear system before a controller can be designed, the proof and conditions of stability in high-order fuzzy systems are among the topics addressed by numerous studies. stability in nonlinear fuzzy systems was investigated in [34-38], where Lyapunov-based methods with second-order or piecewise-linear Lyapunov functions were used for stability analysis. In [39], a novel criterion based on the second Lyapunov method for the stability of high-order TSK dynamic fuzzy systems was proved using a theorem, and the challenge was resolved.

Past research has not presented any high-order fuzzy system with the mentioned specifications for nonlinear dynamic systems. Almost all previous studies have evaluated and modeled fuzzy structures on static and data-based systems with available input-output data [21, 31, 40]. Furthermore, researchers have utilized optimization and intelligent methods or a combination of the two for training high-order fuzzy models [18, 7, 20, 11, 4]. The present research aims to introduce a novel high-order fuzzy structure for modeling dynamic systems described with nonlinear state-space equations. Inspired by the Taylor series expansion, this structure has a high degree of freedom in modeling such processes and can be adapted to the state-space equations of the system.

Based on high-order fuzzy systems, a new structure for fuzzy modeling is proposed and analyzed in this paper. The proposed structure has a high degree of freedom for modeling complex nonlinear processes while being capable of adapting to the Taylor series expansion of the dynamic equation of the process. In this method, a high-order fuzzy system is converted to a set of zero- and first-order fuzzy structures so that the advantages of high-order fuzzy systems can be used in conjunction with the computational simplicity of low-order fuzzy systems. Another innovation of this work is adapting the Taylor series expansion of the differential equation of the system $\dot{X} = F(X, u)$to its high-order fuzzy model such that the two have corresponding terms. This series can be easily modeled using zero-order and first-order fuzzy systems. By adding the corresponding terms to second-order terms, a fuzzy structure closely resembling high-order structures can be created. Using this structure can reduce the number of fuzzy rules while increasing the modeling accuracy.

Therefore, this paper first addresses the nature of this fuzzy model and then proves its approximation ability using the Stone–Weierstrass theorem. The difference (state) equations of dynamic systems using the proposed model are addressed in the following section. Next, a method is developed for calculating the coefficients of the high-order fuzzy model from its Taylor series expansion. Finally, based on the stability analysis method used by [39], the stability of an inherently nonlinear system is simulated using a high-order fuzzy model. The proposed model can be applied to a class of nonlinear systems whose equations can be written in first-order state form and can be expanded using the Taylor series.

The paper is organized as follows: Section 2 will introduce necessary preliminaries and theorems for the present study. In sections 3 and 4 the theoretical results and Mathematical proofs of proposed work are discussed. The simulations results are given in section 5 to show the merits of the proposed approach. Finally, summarizing the presented study along with suggestions for Future research works will be in section 6

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2. Prerequisite Definitions and Theorems

If the "then" rules of a first-order fuzzy system consist of a set of discrete-time state equations, such as$x(k+1) = \widehat{A}x(k) + \widehat{B}u(k)$, it is called a dynamic fuzzy system [41]. The general form of the rules in these systems is as follows:

$$\text{if } x_1 \in D_1^l \,\&\, x_2 \in D_2^l \,\&\, \dots . \,\&\, x_n \in D_n^l \text{ then } x^l(k+1) = \widehat{A}^l x(k) + \widehat{B}^l u(k) \quad (1)$$

$$x(k) = \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_n(k) \end{bmatrix}, \quad u(k) = \begin{bmatrix} u_1(k) \\ u_2(k) \\ \vdots \\ u_m(k) \end{bmatrix} \quad (2)$$

where $l = 1,2, \dots, N$ is the rule number, and n is the number of state variables. Also, $x(k)$ is known as the vector of state variables, and $u(k)$ denotes the input vector of the state equations, where u_i is the i^{th} input vector, and m is the number of inputs. Moreover, D_i^l is the membership function of the open-loop system variables.

$\widehat{A}^l$ and $\widehat{B}^l$ represent $n \times n$ and$n \times m$ matrices respectively, where n is the number of state variables, and m is the number of inputs. The method used to calculate the matrices $\widehat{A}^l$ and $\widehat{B}^l$ is described in Section 3. If the system is closed-loop and the inputs are excited by a fuzzy state controller, such that the membership function of the l^{th} rule of the controller is S_i^l, the l^{th} rule of the closed-loop fuzzy system can be described as follows [41]:

$$\text{if } x_1 \in T_1^l \,\&\, x_2 \in T_2^l \,\&\, \dots . \,\&\, x_n \in T_n^l \text{ then } x^l(k+1) = A^l x(k) \quad (3)$$

T_i^l and A^l in this rule depend on the membership functions and parameters of the open-loop system and the controller. In other words, T_i^l is equal to D_i^l and S_i^l[41]. (A "fuzzy" and is implied here).Based on fuzzy inference rules, $x(k+1)$ can be written as follows:

$$x(k+1) = \frac{\sum_{l=1}^{N} x^l(k+1) v^l}{\sum_{l=1}^{N} v^l} = \frac{\sum_{l=1}^{N} A^l x(k) v^l}{\sum_{l=1}^{N} v^l} \quad (4)$$

where $x^l(k+1)$ is the output of the l^{th} rule, N is the total number of fuzzy rules, and v^l is the weight or effect of the l^{th} rule, which is determined according to the degree of membership of the state variables (x_i) in the fuzzy sets of the l^{th} rule:

$$v^l = \prod_{i=1}^{n} \mu_{T_i^l}(x_i) \quad (5)$$

Also, $\mu_{T_i^l}(x_i)$ is the membership function of the fuzzy set T_i^l. It is assumed that the controller and open-loop system both have l rules and that the weights are identical.

Based on the above definitions, the following fuzzy model is proposed.

2.2Fuzzy model proposed for High-order TSK dynamic nonlinear systems

The main idea behind this model was inspired by the work in [39], [21], and [14]. High-order fuzzy systems are a type of TSK fuzzy system where the consequent part of the rules consists of polynomials of the input variables and has an order higher than 1. These fuzzy systems are known as high-order Takagi-Sugeno-Kang (HOTSK) systems.

Definition 2.1 [39]: The general form of the l^{th} fuzzy rule of an HOTSK system of order R with n state variables is as follows:

$$\text{If } x_1 \text{ is } A_1^l \,\&\, x_2 \text{ is } A_2^l \,\&\, \cdots \,\&\, x_n \text{ is } A_n^l, \text{ then } y^l = \sum_{\substack{j_1+j_2+\cdots+j_n \le R \\ j_1, j_2, \cdots, j_n \ge 0}} \left(a_{j_1,j_2,\cdots,j_n}^l\right) x_1^{j_1} x_2^{j_2} \cdots x_n^{j_n} \quad (6)$$

In this equation, a polynomial of degree R is in the consequent of the rule, and l is the rule number. As demonstrated in [39], the above system can be represented by a number of zero- or first-order fuzzy systems. This concept is used in the simulation section of this paper for implementing the controller. Based on the above definition, dynamic HOTSK systems can be expressed as follows.

Definition 2.2: Assume $M_i(x)$ to be of the following form:

$$M_i(x) = x_1^{j1} x_2^{j2} \dots x_n^{jn} \tag{7}$$

$M_i(x)$ is a monomial with a maximum degree of R, where $R \geq j_1 + j_2 + \cdots j_n$.

Now, the l^{th} rule of a closed-loop dynamic HOTSK system is defined as follows:

if $x_1 \in T_1^l \,\&\, x_2 \in T_2^l \,\&\, \dots . \,\&\, x_n \in T_n^l$ then

$$x^l(k+1) = \sum_{i=1}^{Q} M_i(x) A_i^l x(k) \tag{8}$$

In this equation, Q is the number of rules in the "then" part of the fuzzy rules, and $M_i(x)$ is a monomial of degree $R-1$. The A_i^ls are $n \times n$ matrices and can be determined via various identification methods. In the present work, these matrices are computed using the Taylor series expansion of the nonlinear state equations around the equilibrium point of the system. In the simulation section of this paper, this expansion is performed around points the coordinates of which mostly belong to the fuzzy sets $T_1^l, T_2^l, \cdots, T_n^l$. Based on Eq. (4), the fuzzy inference of such a system results in the following:

$$x(k+1) = \frac{\sum_{l=1}^{N} \sum_{i=1}^{Q} M_i(x) A_i{}^l x(k) v^l}{\sum_{l=1}^{N} v^l} \tag{9}$$

As can be seen, the proposed representation (8) is a decomposition of the nonlinear state-space equations in the matrix $A_i{}^l$. As explained in [39], these matrices are key factors in determining the stability criterion of the proposed dynamic HOTSK system. The numerical example provided in the simulation section will demonstrate the applications of Eq. (8).

References [14, 42] have addressed model validation, several examples of quadratic high-order fuzzy model implementation, and a data-oriented nonlinear static system and have compared their methods to various techniques in terms of the number of rules, number of training iterations, and training duration. However, the present work targets nonlinear dynamic systems, and the coefficients of the proposed higher-order fuzzy model are calculated using the Taylor series expansions. Naturally, the cost of fuzzy computations is higher than other methods due to the complexity of fuzzy relationships.

2.3 Proof of the approximation ability of the proposed fuzzy model and its convergence criteria

This section proves the approximation ability of the proposed fuzzy model and examines its convergence criteria.

For simplicity, the symbol $F(x)$ is used to represent the structure proposed for the consequent part of the fuzzy rules, and its expansion is expressed as follows:

$$F(x) = \sum_{i=1}^{q} f_i(x) M_i(x), \quad q = \frac{(R+n)!}{R!n!} \tag{10}$$

where, as already mentioned, the $M_i(x)$'s represent monomials with a maximum degree of R, the $f_i(x)$'s denote a group of output obtained from a zero-order TSK fuzzy system with input variables x_i [42], and$n$ is the number of state variables (inputs to the fuzzy model). First, it must be proved that $F(x)$ can be a linear approximator and that conditions may be found where $F(x)$ can converge to a set of raw data or dynamic(8)quations of the nonlinear system $g(x)$.

In order to examine the approximation ability of the proposed structure, the Stone–Weierstrass theorem [43] is used. One must consider the following conditions for using this theorem:

1. In the $f_i(x)$'s, all the input membership functions must be considered to be Gaussian.
2. Fuzzification must be of the single-tone type.
3. The inference engine must be of the product type.

The following proves that $F(x)$ satisfies the above theorem. To this end, Points 1, 2, and 3 of the Stone–Weierstrass theorem are proved for it. In other words, if $Z$is considered a set of functions in the form of $F(x)$ for the space R^n, where n is the number of variables in the vector X (assuming F_a, F_b to be two members of Z with a degree of n),

$$F_a = \sum_{i=0}^{q} f_{a_i}(x) M_i(x), F_b = \sum_{i=0}^{q} f_{b_i}(x) M_i(x) \tag{11}$$

1. Z is an algebra since

A: Z is closed with respect to addition.

It must be shown that $F_a + F_b$ and $F_a . F_b$ are members of Z.

$$F_a + F_b = \sum_{i=0}^{q} f_{a_i}(x) M_i(x) + \sum_{i=0}^{q} f_{b_i}(x) M_i(x) = \sum_{i=0}^{n} [f_{a_i}(x) + f_{b_i}(x)] M_i(x) \tag{12}$$

Therefore, it is sufficient to demonstrate that $f_{a_i}(x) + f_{b_i}(x)$ can be expressed as the output of a zero-order fuzzy system, such as $f_{c_i}(x)$, where $f_{c_i}(x) = f_{a_i}(x) + f_{b_i}(x)$.

This has been shown in [41]. In addition,

$$F_a . F_b = \sum_{i=0}^{q} f_{a_i}(x) M_i(x) \; . \sum_{i=0}^{q} f_{b_i}(x) M_i(x) =$$

$$\sum_{i=0}^{n}[\sum_{j=0}^{n}[f_{a_i}(x)M_i(x)\, f_{b_j}(x)M_j(x)]] = \sum_{i=0}^{q}[\sum_{j=0}^{q}[\, f_{a_i}(x)f_{b_j}(x)M_i(x)M_j(x)]] \quad (13)$$

In this equation, $M_i(x)M_j(x)$ is the product of two monomials with a maximum degree of R. One may consider this product a monomial and write

$$M_i(x)M_j(x) = \widehat{M}_k(x) \quad (14)$$

Where $\widehat{M}_k(x)$ is a polynomial with a maximum degree of $2R$. Overall, considering the repetitive cases of the number of monomials generated in the form of $\widehat{M}_k(x)$, $2q = 2\frac{(2R+n)!}{2R!n!}$, and the number of non-repetitive terms is $\frac{n!(2R)!}{(n+2R)!}$.

Now, it must be proved that the product of two fuzzy systems $f_{a_j}(x).f_{b_j}(x)$ is itself a fuzzy system. This has been done in [41] and will not be repeated here. One may write $\hat{f}_k(x) = f_{a_i}(x)f_{b_j}(x)$

where $\hat{f}_k(x)$ is a new fuzzy system with the properties mentioned for $f_{a_i}(x)$ and $f_{b_j}(x)$, and the subscript k varies as stated for $\widehat{M}_k(x)$. Hence,

$$F_a.F_b = \sum_{k=1}^{2q} \hat{f}_k(x)\widehat{M}_k(x) \quad (15)$$

In this sum, a number of the $\widehat{M}_k(x)$'s are repeated, which does not affect the proof anyway.

One may conclude that if $F_a, F_b \in Z$, then $(F_a + F_b) \in Z$ and $F_a.F_b \in Z$. Regarding the scalar product, if one assumes A to be a real number, then $AF_a \in Z$,

$$AF_a = \sum_{i=0}^{q} Af_{a_i}(x)M_i(x) \quad (16)$$

where $Af_{a_i}(x)$ is also a fuzzy system with the same specifications as $f_{a_i}(x)$. The proof is provided in [41].

Hence, it was shown that Z is an algebra.

2. Z is a separator since

It is sufficient to provide an example where if $x, y \in R^n$ and $x \neq y$,

Then, $F \in Z$ and $F(x) \neq F(y)$.

If one assumes at least one non-zero element, such as x, exists in X and defines the following:

$F(x) = xf_1(x)$

then $F \in Z$. Now, if f_1 is defined according to [41], then $F(x) \neq F(y)$.

3. Z is not exactly zero at any point.

It is sufficient that one defines $F(x) = f_1(x)$ and the consequents of all the rules are greater than zero at $f_1(x)$. In this case, since

$$\forall X \in R^m, f_1(x) \neq 0 \Rightarrow F(x) \neq 0$$

one may conclude that the proposed structure $F(x) = \sum_{i=1}^{q} f_i(x)M_i(x)$ satisfies the Stone–Weierstrass theorem. Therefore, for every function continuous on R^n, such as $g(x)$, providing $X \in R^n$, one can find an$F(x)$ in the form of Eq. (11), the maximum difference of which from $g(x)$ is less than the arbitrary value ε. In other words,

$$\sup_{x \in U}|F(x) - g(x)| < \varepsilon$$

Therefore, $F(x)$ can converge to any function that is continuous on R^n. Hence, high-order fuzzy systems are approximators.

2.4 Representation of the difference equations (state equations) of dynamic systems by the proposed model

If the model of the system under study is a dynamic one and the system's behavior is described by one or more difference equations, such that the discrete-time dynamic equations of the system are as follows,

$$\begin{cases} x(k+1) = G(x(k), u(k)) \\ y(k) = H(x(k), u(k)) \end{cases} \quad (17)$$

where k is the calculation step number, $x(k)$ is the state variable vector, $u(k)$ is the system input vector at the k^{th} instant, G and H are continuous and differentiable vector functions dependent on $x(k), u(k)$, and $y(k)$ is the output of the system, then the Taylor expansion of the G and$H$ functions can be considered for modeling the difference form of the system using a high-order fuzzy model (assuming the values of the state variables to be available).

For this purpose, the following must be taken into account:

If $F(z): R^n \to R$ and $z \in R^n$ is a continuous and infinitely differentiable function and $a \in R^n$ is a point at which $F(a)$ is defined, the F^{th}-order Taylor series expansion of m about a can be expressed as follows [44]:

$$F(z) \cong \sum_{|\alpha| \leq m} \frac{D^{\alpha}(F(a))}{\alpha!}(z-a)^{\alpha} \quad (18)$$

If the following definitions are considered in the above expression:

$$|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n \quad (19)$$

$$\alpha! = \alpha_1!\,\alpha_2!\,\alpha_3! \ldots \alpha_n!$$

and assuming a and z to be vectors of the following form:

$$a = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}, z = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} \quad (20)$$

then $(z-a)^{\alpha} = (z-a_1)^{\alpha_1}(z-a_2)^{\alpha_2} \ldots (z-a_n)^{\alpha_n}$

where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are integers whose sum must be less than or equal to m: $|\alpha| \leq m$

Now, considering these definitions, one may assume that if the vector functions $G$and$H$ are written as follows:

$$G = \begin{bmatrix} G_1 \\ G_2 \\ \vdots \\ G_n \end{bmatrix}, H = \begin{bmatrix} H_1 \\ H_2 \\ \vdots \\ H_n \end{bmatrix} \quad (21)$$

then each function G_i or H_i can be expanded similarly to $F(z)$ up to the desired degree based on the Taylor expansion. Put more accurately, if

$$z = \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} \quad (22)$$

one may write

$$G(x(k), u(k)) = G(z) = \begin{bmatrix} G_1 \\ G_2 \\ \vdots \\ G_n \end{bmatrix}, G_i(z) \cong \sum_{\substack{|\alpha|<m \\ i=1,2,\ldots,n}} \frac{D^{\alpha}(G_i(a))}{\alpha!}(z-a)^{\alpha} \quad (23)$$

The point a here is the point at which the system is modeled and can be an operational point or an equilibrium point of the system. Expanding the above expression, one gets

$$G_i(z) \cong \sum_{|\alpha|<m} U(\alpha)V(z) \quad (24)$$

where the $V(z)$'s are monomials in terms of the components of the vector z, and the $U(\alpha)$'s are parameters in terms of α. Therefore, the components of the vector function $G(z)$ may be expressed as follows:

$$G_i(z) = U_1(\alpha)V_1(z) + U_2(\alpha)V_2(z) + \cdots \equiv \sum_{i=1}^{q} f_i(z)M_i(z) \quad (25)$$

In other words, if sufficient data is available from the $G_i(z)$'s and $H_i(z)$'s, one can approximate them as $\sum_{i=1}^{q} f_i(z)M_i(z)$ using the proposed model. In this expression, z consists of the inputs and the state variables, as defined in Eq. (22), and the f_i's and M_i's are the outputs of the zero-order fuzzy system and the monomials defined in Eq. (7).

3. Approximation of The State-Space Equations of a Nonlinear Dynamic System Using The 2nd Degree Taylor Expansion

One important method of linearizing nonlinear methods is to use the Taylor series expansion of the system's model about its equilibrium point. Considering its three main terms, this expansion can be written for a system of the form $\dot{X} = F(X, u)$ as follows:

$$f_i(Z_0 + \Delta Z) \cong f_i(Z_0) + \nabla f_i(Z_0)\Delta Z + \frac{1}{2}\Delta Z^T H_i \Delta Z \quad (26)$$

where the f_is are vectors of the function $F(X, u)$, and $\nabla f_i(Z_0)$ and H_i are the gradient and Hessian, respectively, of the function f_i about the point Z_0. The following explains how to calculate the high-order fuzzy model coefficients from the coefficients of the Taylor expansion of the system's model. Assume a nonlinear dynamic continuous system to be described by a set of state-space equations as follows:

$$\dot{X} = F(X, u) = \begin{bmatrix} f_1(X, u) \\ f_2(X, u) \\ \vdots \\ f_n(X, u) \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad (27)$$

If the state-space equations of the nonlinear dynamic system (27) are approximated as the following matrix using the 2nd-order Taylor expansion,

$$\dot{X} = F(X, u) \cong F_0 + A_0\Delta X + B_0\Delta u + \frac{1}{2}\left[\sum_{i=1}^{Q} \Delta x_i (A_i\Delta X + B_i\Delta u) + \Delta u(A_u\Delta X + B_u\Delta u)\right] \quad (28)$$

where the following equations are substituted to simplify the representation of the Taylor series expansion of $\dot{X} = F(X, u)$:

$$Z = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ u \end{bmatrix} = \begin{bmatrix} X \\ u \end{bmatrix},$$

$$\Delta Z^T = [\Delta x_1 \ \ \Delta x_2 \ \ldots \ \Delta x_n \ \ \Delta u] \quad (29)$$

$$\nabla f_i(Z) = \begin{bmatrix} \frac{\partial f_i}{\partial x_1} & \frac{\partial f_i}{\partial x_2} & \cdots & \frac{\partial f_i}{\partial x_n} & \frac{\partial f_i}{\partial u} \end{bmatrix} \quad (30)$$

$$H_i = \begin{bmatrix} \frac{\partial^2 f_i}{\partial {x_1}^2} & \frac{\partial^2 f_i}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f_i}{\partial x_1 \partial x_n} & \frac{\partial^2 f_i}{\partial x_1 \partial u} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \frac{\partial^2 f_i}{\partial x_n \partial x_1} & \frac{\partial^2 f_i}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f_i}{\partial {x_n}^2} & \frac{\partial^2 f_i}{\partial x_n \partial u} \\ \frac{\partial^2 f_i}{\partial u \partial x_1} & \frac{\partial^2 f_i}{\partial u \partial x_2} & \cdots & \frac{\partial^2 f_i}{\partial u \partial x_n} & \frac{\partial^2 f_i}{\partial u^2} \end{bmatrix} \quad (31)$$

then, the following set of equations holds in Eq. (28):

$$F_0 = F(X_0, u_0),\ \Delta X = \begin{bmatrix} \Delta x_1 \\ \vdots \\ \Delta x_n \end{bmatrix}, J_A(F) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}, J_B(F) = \begin{bmatrix} \frac{\partial f_1}{\partial u} \\ \vdots \\ \frac{\partial f_n}{\partial u} \end{bmatrix}$$

$$A_i = \frac{\partial J_A(F)}{\partial x_i}\bigg|_{\substack{X=X_0 \\ u=u_0}},\ A_u = \frac{\partial J_A(F)}{\partial u}\bigg|_{\substack{X=X_0 \\ u=u_0}},\ A_0 = J_A(F)|_{\substack{X=X_0 \\ u=u_0}}$$

$$B_i = \frac{\partial J_B(F)}{\partial x_i}\bigg|_{\substack{X=X_0 \\ u=u_0}},\ B_u = \frac{\partial J_B(F)}{\partial u}\bigg|_{\substack{X=X_0 \\ u=u_0}},\ B_0 = J_B(F)|_{\substack{X=X_0 \\ u=u_0}}$$

(32)

Proof: The main idea for proving Eq. (28) was derived from Eq. (26). If $F^T = [f_1 f_2 f_3 \ \ldots \ f_n]$, then using Eq. (26):

$$\begin{bmatrix} f_1(Z_0 + \Delta Z) \\ \vdots \\ f_n(Z_0 + \Delta Z) \end{bmatrix} \cong \begin{bmatrix} f_1(Z_0) \\ \vdots \\ f_n(Z_0) \end{bmatrix} + \begin{bmatrix} \nabla f_1(Z_0)\Delta Z \\ \vdots \\ \nabla f_n(Z_0)\Delta Z \end{bmatrix} + \frac{1}{2}\begin{bmatrix} \Delta Z^T H_1 \Delta Z \\ \vdots \\ \Delta Z^T H_n \Delta Z \end{bmatrix} \quad (33)$$

where Z_0 and ΔZ in Eq. (33) were obtained from Eq. (29). It is clear that

$$F_0 = F(X_0, u_0) = \begin{bmatrix} f_1(Z_0) \\ \vdots \\ f_n(Z_0) \end{bmatrix} \tag{34}$$

Considering Eq. (30) and assuming the following equations can be defined and substituted into the gradient of F:

$$\begin{bmatrix} \nabla f_1 \\ \vdots \\ \nabla f_n \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} & \frac{\partial f_1}{\partial u} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} & \frac{\partial f_n}{\partial u} \end{bmatrix} \triangleq J(F) = [J_A(F) \quad J_B(F)] \tag{35}$$

based on Eq. (32) and (29),

$$\begin{bmatrix} \nabla f_1(Z_0)\Delta Z \\ \vdots \\ \nabla f_n(Z_0)\Delta Z \end{bmatrix} = [J_A(F) \quad J_B(F)]\Delta Z =$$

$$[J_A(F) \quad J_B(F)] \begin{bmatrix} \Delta X \\ \Delta u \end{bmatrix} = A_0 \Delta X + B_0 \Delta u \tag{36}$$

to calculate the 2nd-degree terms of the Taylor expansion according to Eq. (31), one may write

$$Z^T H_i Z$$

$$= [x_1 \quad x_2 \quad \ldots \quad x_n \quad u] \begin{bmatrix} \frac{\partial^2 f_i}{\partial x_1{}^2} & \frac{\partial^2 f_i}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f_i}{\partial x_1 \partial x_n} & \frac{\partial^2 f_i}{\partial x_1 \partial u} \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ \frac{\partial^2 f_i}{\partial x_n \partial x_1} & \frac{\partial^2 f_i}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f_i}{\partial x_n{}^2} & \frac{\partial^2 f_i}{\partial x_n \partial u} \\ \frac{\partial^2 f_i}{\partial u \partial x_1} & \frac{\partial^2 f_i}{\partial u \partial x_2} & \cdots & \frac{\partial^2 f_i}{\partial u \partial x_n} & \frac{\partial^2 f_i}{\partial u^2} \end{bmatrix} Z$$

$$= [x_1 \quad x_2 \quad \ldots \quad x_n \quad u] \begin{bmatrix} \frac{\partial \nabla f_i}{\partial x_1} \\ \vdots \\ \frac{\partial \nabla f_i}{\partial x_n} \\ \frac{\partial \nabla f_i}{\partial u} \end{bmatrix} Z \tag{37}$$

The above equation is equal to

$$Z^T H_i Z = x_1 \left[\frac{\partial \nabla f_i}{\partial x_1}\right] Z + \cdots + x_n \left[\frac{\partial \nabla f_i}{\partial x_n}\right] Z + u \left[\frac{\partial \nabla f_i}{\partial u}\right] Z \tag{38}$$

Considering $Z^T H_1 Z Z^T H_n Z$, ..., and $Z^T H_2 Z$, to be the components of a vector,

$$\begin{bmatrix} Z^T H_1 Z \\ \vdots \\ Z^T H_n Z \end{bmatrix} = \begin{bmatrix} x_1 \left[\frac{\partial \nabla f_1}{\partial x_1}\right] Z + \cdots + x_n \left[\frac{\partial \nabla f_1}{\partial x_n}\right] Z + u \left[\frac{\partial \nabla f_1}{\partial u}\right] Z \\ x_1 \left[\frac{\partial \nabla f_2}{\partial x_1}\right] Z + \cdots + x_n \left[\frac{\partial \nabla f_2}{\partial x_n}\right] Z + u \left[\frac{\partial \nabla f_2}{\partial u}\right] Z \\ \vdots \\ x_1 \left[\frac{\partial \nabla f_n}{\partial x_1}\right] Z + \cdots + x_n \left[\frac{\partial \nabla f_n}{\partial x_n}\right] Z + u \left[\frac{\partial \nabla f_n}{\partial u}\right] Z \end{bmatrix} \tag{39}$$

the following can be derived from Eq. (35):

$$\begin{bmatrix} Z^T H_1 Z \\ \vdots \\ Z^T H_n Z \end{bmatrix} = \left[x_1 \frac{\partial J(F)}{\partial x_1} + \cdots x_n \frac{\partial J(F)}{\partial x_n} + u \frac{\partial J(F)}{\partial u}\right] Z =$$

$$x_1 \frac{\partial}{\partial x_1} [J_A(F) \quad J_B(F)] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ u \end{bmatrix} + \ldots x_n \frac{\partial}{\partial x_n} [J_A(F) \quad J_B(F)] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ u \end{bmatrix} +$$

$$u \frac{\partial}{\partial u} [J_A(F) \quad J_B(F)] \begin{bmatrix} x_1 \\ x_1 \\ \vdots \\ x_n \\ u \end{bmatrix} \tag{40}$$

Considering Eq. (29) and (26), the above equation can simply be rewritten as follows:

$$\begin{bmatrix} Z^T H_1 Z \\ \vdots \\ Z^T H_n Z \end{bmatrix} = x_1 \left[\left(\frac{\partial}{\partial x_1} J_A(F)\right) X + \left(\frac{\partial}{\partial x_1} J_B(F)\right) u\right] +$$

$$\cdots x_n \left[\left(\frac{\partial}{\partial x_n} J_A(F)\right) X + \left(\frac{\partial}{\partial x_n} J_B(F)\right) u\right] + u \left[\left(\frac{\partial}{\partial u} J_A(F)\right) X + \left(\frac{\partial}{\partial u} J_B(F)\right) u\right] \tag{41}$$

As can be seen, without loss of generality, Z can be replaced by ΔZ, X by ΔX, and u by Δu. Based on Eq. (32),

$$\begin{bmatrix} \Delta Z^T H_1 \Delta Z \\ \vdots \\ \Delta Z^T H_n \Delta Z \end{bmatrix} = \Delta x_1 [A_1 \Delta X + B_1 \Delta u] + \cdots \Delta x_n [A_n \Delta X + B_n \Delta u] + \Delta u [A_u \Delta X + B_u \Delta u] \tag{42}$$

Substituting Eq. (34), (36), and (42) in Eq. (26), one obtains Eq. (28), and the proof is completed.

The steps carried out above to calculate Eq. (28) are one of the results of this study. This theorem can be expressed via tensor representation as a Taylor series expansion containing terms of degree 3 or higher.

4. The Proposed High-Order Fuzzy Representation As a Combination of Zero-Order and First-Order Fuzzy

As seen from Eq. (26), the first and second terms of this series can be easily modeled using zero-order and first-order fuzzy structures, respectively. Hence, by adding terms corresponding to the third term from Eq. (26), one can create a fussy structure closely resembling a high-order structure. The following figure displays the proposed high-order fuzzy system. As seen in this figure, a high-order fuzzy system has been converted to several zero- and first-order fuzzy subsystems, such that the output signal of the high-order fuzzy system equals the sum of the outputs of the zero- and first-order systems.

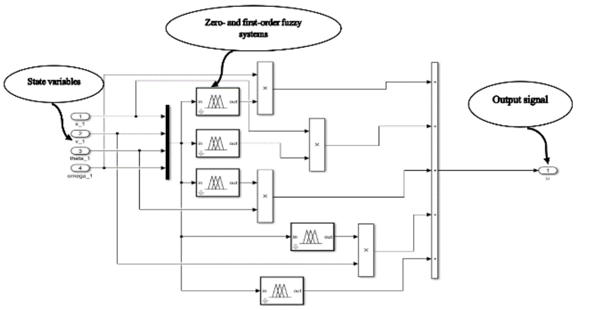

Fig.1. The proposed high-order fuzzy representation as a combination of zero-order and first-order fuzzy structures

5. Numerical Simulation

Two examples are presented to obtain a better understanding of Eq. (6), demonstrate the validity of high-order fuzzy models and their role in stabilizing nonlinear systems, and analyze the approximation accuracy of high-order fuzzy models.

5.1 Example 1: Mechanical model of a double pendulum inverted on a cart

The double pendulum inverted on a cart (DPIC) is a common dynamic system of current interest in the field of nonlinear control [45]. As shown in Fig. 2, this system consists of a linear rail, a cart moving on the rail, and two pendulums. The lower pendulum is hinged to the moving cart at one end and to the upper pendulum at the other end. Both pendulums can rotate freely around the axis perpendicular to the plane of the rail. Both pendulums must remain vertical under unstable conditions; hence, they will fall without an effective control effort. It is assumed that the cart can freely move left or right on the horizontal mount without any friction and that the force $u(t)$ is the only effective control force acting on the system.

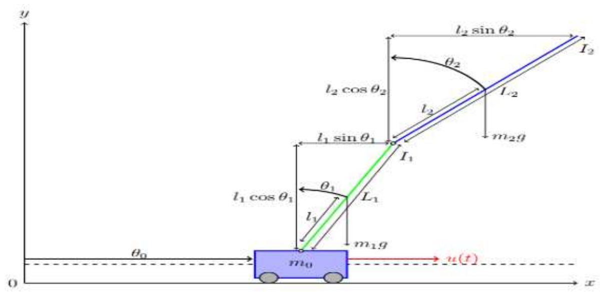

Fig. 2. Double pendulum inverted on a cart

The DPIC system is modeled in such a way that the cart is at the point $(0,0)$ in terms of Cartesian coordinates. m_0 is the mass of the cart, m_1 and m_2 are the masses of the first and second cart, respectively, θ_0 is the horizontal position of the cart, and θ_1 and θ_2 are the angles between the pendulums and the vertical. Also, l_1 and l_2 are the distances between the center of rotation of the pendulums and their respective centers of mass. Moreover, L_1 and L_2 are the lengths of the pendulums. $I_1$and I_2 are the moments of inertia of the pendulums about their centers of mass. In addition, g is the gravitational constant, and the force $u(t)$ is the control input to the system. Each pendulum can rotate a full 360°. When the DPIC is off balance, it loses its equilibrium and oscillates infinitely. With friction, the oscillations of the DPIC slow down gradually and the system converges to the lower stable point of the pendulum $\theta_1 = \theta_2 = \pi$.

Assuming no external forces at the hinges, the dynamic model governing the DPIC can be obtained using the Lagrange method as the following nonlinear 2nd-order differential equation:

$$D(\theta)\ddot{\theta} + C(\theta,\dot{\theta})\dot{\theta} + G(\theta) = Hu \tag{43}$$

Using [46], the DPIC system is simplified in the form of Eq. (45).The matrices, D, C, G, and H have been computed in [46].

The DPIC system has three degrees of freedom with one input and, hence, is more difficult to control. Six state variables are needed to describe its state-space model, including the position x and horizontal speed v of the cart, pendulum angles θ_1 and θ_2, and pendulum angular speeds ω_1 and ω_2. The state variables vector and the input in the DPIC are as follows:

$$X = \begin{bmatrix} x \\ \theta_1 \\ \theta_2 \\ \dot{x} \\ \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} x \\ \theta_1 \\ \theta_2 \\ v \\ \omega_1 \\ \omega_2 \end{bmatrix} = \begin{bmatrix} \theta_0 \\ \theta_1 \\ \theta_2 \\ \dot{\theta}_0 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}, \qquad U = \begin{bmatrix} u \\ 0 \\ 0 \end{bmatrix} \tag{44}$$

Hence, the state equations are determined as follows:

$$\dot{X} = \begin{bmatrix} \dot{x} \\ \dot{\theta}_1 \\ \dot{\theta}_2 \\ \ddot{x} \\ \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & & & \\ 0 & 0 & 0 & & -D^{-1}C & \\ 0 & 0 & 0 & & & \end{bmatrix} \begin{bmatrix} x \\ \theta_1 \\ \theta_2 \\ \dot{x} \\ \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ -D^{-1}G \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ D^{-1}H \end{bmatrix} u \tag{45}$$

where the coefficients matrices of the state variables and the input are 6×6 and 6×1, respectively.

The following examines the results of the simulations performed for controlling the deviation of the DPIC system using the proposed high-order fuzzy controller. This system was simulated in the m-file and Simulink environments of MATLAB. Five Gaussian membership functions were considered for each state variable in the fuzzy model, resulting in a total of 25 rules. In this example, a set of discrete state-space equations are calculated assuming $t_s = 0.005$. Although the calculation of the coefficient matrices of the HOTSK model, the proof of stability, and the calculation of the controller matrices (K_i^l) were implemented, they are not mentioned here due to the large number of obtained coefficients.

First, the stability of the pendulums when both of them deviate in the same direction is studied. At a deviation angle of +20°, only a high-order fuzzy controller can stabilize the double pendulum on the cart. In this case, the HOTSK control signal can stabilize the double pendulum with a smaller amplitude. In contrast, classical and zero- and first-order fuzzy controllers are unable to stabilize the system.

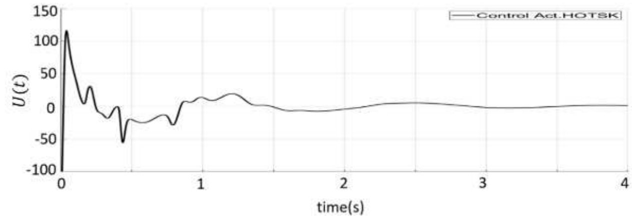

Fig. 5. The control effort of the proposed high-order fuzzy controller with a deviation of $\theta_1 = 20°, \theta_2 = 20°$

Since the cart moves to the right, the case with the deviation angles of the lower and upper pendulums in opposite directions (i.e., a positive angle for the lower pendulum and a negative angle for the upper pendulum) is one of the worst in terms of stability. As shown in Figs. 3 and 4, the inverted double pendulum with an initial deviation of $\theta_1 = 15°, \theta_2 = -15°$ can be stabilized only using a high-order fuzzy controller.

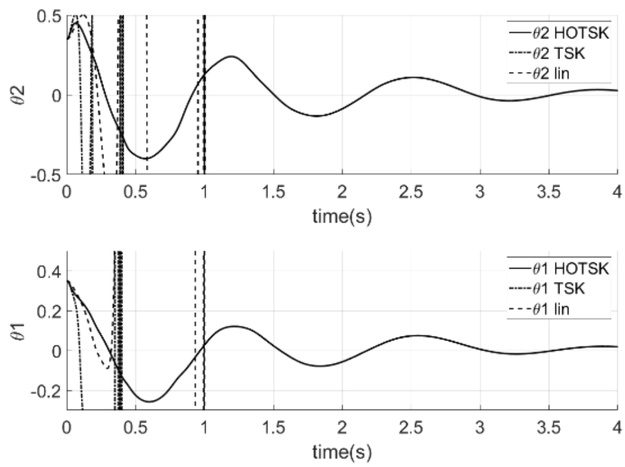

Fig. 3. A comparison of the deviation angles of the DPIC (with linear, zero- and first-order fuzzy, and high-order fuzzy controllers) with an initial deviation of $\theta_1 = 20°, \theta_2 = 20°$

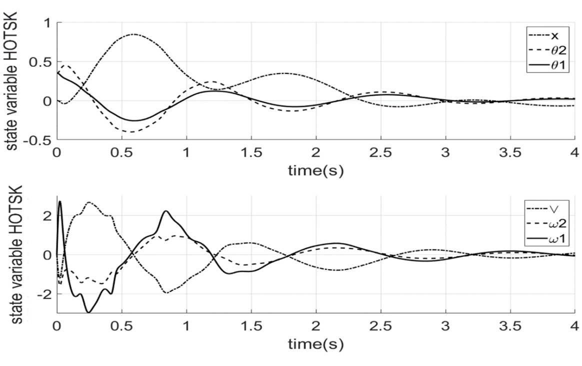

Fig. 4. State variables of the DPIC with a high-order fuzzy controller assuming $\theta_1 = 20°, \theta_2 = 20°$

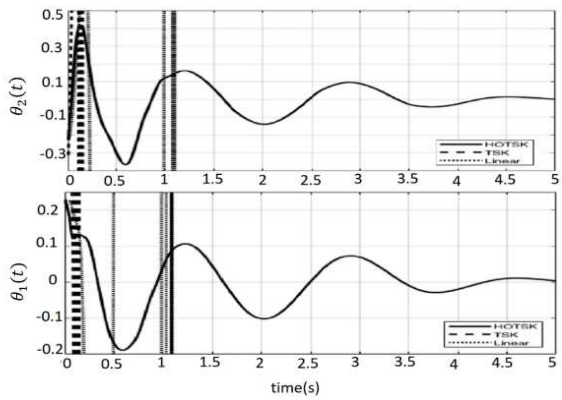

Fig. 6. A comparison of the deviation angles of the DPIC (with linear, zero- and first-order fuzzy, and high-order fuzzy controllers) with an initial deviation of $\theta_1 = 15°, \theta_2 = -15°$

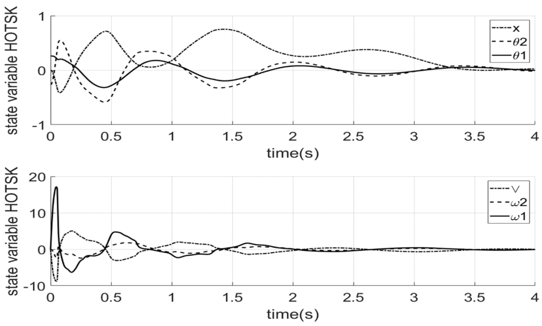

Fig. 7. State variables of the DPIC with a high-order fuzzy controller assuming $\theta_1 = 15°, \theta_2 = -15°$

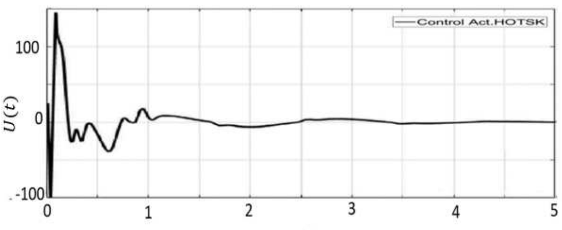

Fig. 8. Output control signal of the high-order fuzzy controller(control effort) with an initial deviation of $\theta_1 = 15°, \theta_2 = -15°$

Based on the above results, the DPIC nonlinear dynamic closed-loop system attains asymptotic stability only with a high-order fuzzy controller and an initial deviation of $\theta_1 = 15°, \theta_2 = -15°$.

5.2 Comparison of the high-order fuzzy, zero- and first-order fuzzy, and classical linear controllers

The initial deviation threshold of the inverted pendulum for stability and the computational burden of the proposed fuzzy method for zero- and first-order fuzzy, high-order fuzzy, and classical linear controllers were examined in three different Simulink files, with the results presented in Table 1:

Table 1. A comparison of the computation time and stability threshold of the deviation angle for different DPIC controllers DPIC

Controller	*Controller computation time(second)*	*Initial deviation stability threshold(degree)*
High-order fuzzy (HOTSK)	4.69 s	79°
Zero- and first-order fuzzy (TSK)	1.49 s	60°
Classical linear	0.245 s	45°

As expected, the burden and time of high-order fuzzy computations are more than the other methods due to the complexity of the high-order fuzzy relationships.

5.3 Example 2: Nonlinear two-variable Sinc function

In this example, the function of Eq. (46) is considered in order to investigate the approximation accuracy of high-order fuzzy models and compare it with those of first- and second-order fuzzy structures in modeling a nonlinear function. The zero- and first-order and quadratic fuzzy models of this system were obtained. The approximation error is presented in Table 2.

$$y = \frac{\sin(x_1)}{x_1} . \frac{\sin(x_2)}{x_2}, \ X \in [-10,10]:[-10,10] \quad (46)$$

In this example, the product of the two inputs x_1 and x_2 (i.e., the mutual effect of the inputs) is seen in the output.

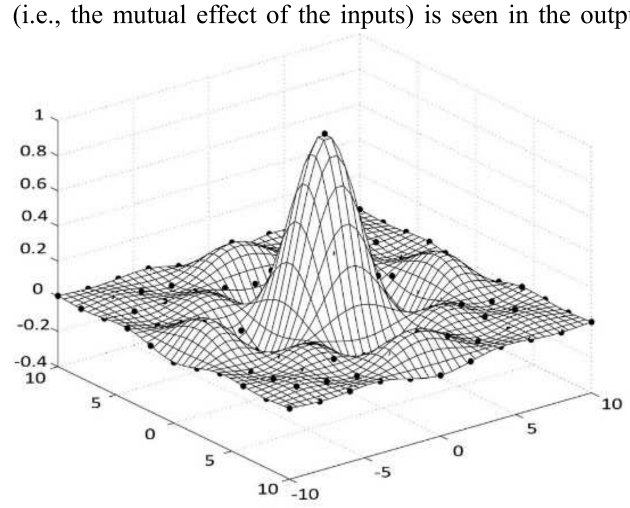

Fig. 9. Plot of the function in Example 2

Table 2. A comparison of the approximation accuracies of the fuzzy models of Example 2

≠	*Modeling method*	*Number of rules*	*Approximation accuracy*	
			RMSE	NDEI
1	Zero- and first-order fuzzy	16	0.0555	0.3795
2	Quadratic fuzzy	16	0.1086	0.7422
3	(Proposed high-order fuzzy)	16	0.0376	0.2577
4	(Proposed high-order fuzzy)	32	0.0137	0.0933

In this example, based on Table 2, the fuzzy rule base is considered complete for the presented models. In zero-order fuzzy, each input has 4 membership functions, resulting in a total of 16 rules. In second-order fuzzy with 4 inputs, 2 membership functions are considered for each, resulting in 16 rules. In the high-order fuzzy model with 5 inputs (x_1, x_2, $x_1.x_2$, x_1^2, and x_2^2), interaction effects between inputs are taken into account, and 2 membership functions are considered for each. Initially, without considering the term $x_1.x_2$ in the antecedentpart of fuzzy rules, which is similar to row 2, but due to its use in the consequent of rules, it provides better modeling accuracy. It has been modeled a second time with this term. In this case, there are 32 rules. Both in the antecedent and consequent parts of the rules, the term $x_1.x_2$ is considered. As anticipated, the modeling estimation error has been reduced. The function related to this example, as shown in Figure 9, is a non-smooth function and is inherently more challenging to approximate compared to smoother functions. A model with more rules and membership functions performs better. Finally, based on the table, the high-order fuzzy models have a smaller error compared to the first-order and quadratic models.

6. Conclusion

One of the most important applications of fuzzy systems is solving control problems. While a significant advantage of this structure is modeling and controlling nonlinear systems, as processes become more complex, the number of rules required to reduce modeling errors increases. This issue becomes more pronounced with increased significant input interactions. High-order fuzzy systems can provide precise approximations for modeling nonlinear dynamic systems. In this paper, the approximation ability of the high-order fuzzy systems was proved, and the coefficients of these models were presented. The high-order models were based on the representation of the high-order dynamic fuzzy systems derived from 2nd-degree Taylor series expansions, which led to the matrix decomposition of the state-space equations of the nonlinear system. The matrices obtained from this decomposition played a key role in determining the stability conditions discussed in [39]. The validity of the presented method in stabilizing a nonlinear system with high-order fuzzy controllers with the model coefficients obtained from the proposed method was studied in the last section. Despite the advantages of high-order fuzzy systems in modeling nonlinear systems, as mentioned previously, an increase in computational burden compared to first-order fuzzy systems due to the complexity of the fuzzy computations in the presented method is among the disadvantages of such models.

For future research, the authors recommend optimizing the introduced control system and determining the coefficients of the controller matrix. It is also suggested that nonlinear system identification methods be used to estimate the matrices A_0^l, A_i^l, A_u^l, B_0^l, B_i^l, and B_u^l. The performance of the proposed method under uncertainty and disturbance is another aspect deserving prospective analysis.

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