Constrained Controllability of Linear Discrete Time Systems

A sufficient condition based on Farkas' Lemma

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Abstract – This paper provides sufficient conditions for controllability of discrete time linear systems with input saturation. Controllability is a central notion in linear system theory, optimal quadratic regulators as well as model predictive control algorithms. The more realistic notion of constrained controllability has given less attention probably due to mathematical complications. Most of the existing works on constrained controllability studies the properties of reachable sets. However, there are only a few works which investigate the problem of whether a state is reachable or not. In this paper, a set of sufficient conditions are firstly given for constrained controllability of time varying linear systems in discrete time formulation. The given conditions are obtained using the Farkas' lemma for alternative inequalities. The obtained results improve existing literature by providing conditions for both null-controllability and controllability regardless of the system stability. A sufficient condition is then given for the special case of time invariant single input linear systems with diagonal Jordan canonical form. Numerical examples are given for clarification.

Keywords: Controllability, Input constraint, Saturation, Constrained control, Farkas' lemma.

1. Introduction

Controllability has been the focus of several studies since 1960's [1].This essential notion, characterizes the ability of a system to steer its state trajectory from a set of predefined initial states to a final state in finite time [1, 2]. The controllability property has been proven vital to state space feedback control schemes including pole placement design, some variations of model predictive control and linear quadratic regulator design. Controllability is of significant importance in time varying and nonlinear system theory as well. The controllability property depends on the definition of state variables [1], system internal interconnections [1], access of control input to the plant [3-6], sampling rates [4, 7], and the structure and characteristics of admissible inputs (in case of constrained control) or states [8].

While innumerous results have been published in this area, relatively less attention has been given to the more realistic case of constrained controllability in which the control input is subject to saturation constraints [2]. Unlike the similarity between un-constrained controllability of continuous time and discrete time systems, the existing results on constrained controllability of discrete time linear systems [8-20] is different from that of continuous time systems [21-29]. The constrained controllability property is characterized by a set of achievable states, a set of initial states, the system dynamics as well as the constraint description. Input constraint is the most common and probably the most restrictive (from a feedback control perspective) constraint to be dealt with. The problem of controllability with input constraint investigates whether there exists a sequence of control inputs fulfilling a specific constraint and at the same time, able to steer the state vector of the system from a given initial state to a predefined final state. This form of problem, has been given less attention compared with the case of un-constrained controllability. Another shortcoming is that much less attention is given to the constrained controllability of time varying systems. However, due to the growing use of networked implementations, time varying systems are becoming more important since many network models include time varying elements (for instance time varying delays or intermittent

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connections [3], bandwidth limitations [4-6] in shared communication media and multi rate sampling schemes[7], each of which changing a time invariant plant model to a time varying plant/ network model.

To review the existing literature on constrained controllability of time invariant linear systems, consider the following state space description;

$$x(k+1) = Ax(k) + Bu(k) \quad (1)$$

Vectors $u(k) \in R^{n_u}$ and $x(k) \in R^n$ define the control input and the state variable. Matrices $A_{n\times n}$ and $B_{n\times n_u}$ are the state and input distribution matrices. In [8] a necessary and sufficient condition is given for null-controllability of (1) with two control inputs ($u(k) = [u_1(k)u_2(k)]^T$). In [8] and many later works a set of null-controllable initial states is given for (1). In later works, necessary and sufficient conditions are established for local and global controllability [9-12] of (1) based on the system eigenvalues and/or eigenvectors. In [11], some necessary and sufficient conditions are given for local null-controllability (local controllability as defined therein) with constraints defined by convex sets (e.g. saturated input) and global controllability with constraints defined by convex cones (e.g. non-negative input). The null-controllability region for the local case which assumes bounded sets for the constraint has not been given by [11]. Reference [10] studied some properties of the controllable regions for the local controllability problem. In [11, 12], several properties of constrained finite time controllability of (1) with $B = I$ are established (I represents the identity matrix of appropriate dimensions). One of the results reported in [11] can be viewed as a generalization to that of [8] for multi-input systems (See also theorem 3 in [9]). Reference [13] studies the properties of reachability subspace of (1).An algorithm to calculate this region is given by [14] which may be viewed as complementary to the results of [9-11].

Reference [15] used a novel methodology leading to strong necessary and sufficient conditions for null-controllability of time varying linear systems. However, the provided conditions can't be directly related to the saturation constraints or system matrices (See equations 6-8 and theorem 2.4 therein). References [16-18] are focused on the computation and properties of null-controllable regions using an approach similar to that of [9-11]. Authors of [15] studied null controllability of single input systems in general (theorem 1 therein) and single input systems having only real positive eigenvalues (theorem 2 therein). The results of [11, 12] are based on the assumption that the constraint set is a closed polyhedral cone (See [11, 12] and [18]). This constraint set may be of practical interest in some cases (for example positive linear systems, in which only non-negative inputs are allowed) however, the more practical constraint model is that of [10] which assumes a bounded set containing zero in its interior (e.g. saturation constraint). Reference [10] arrives at the conclusion (See Theorem IV.5 in [18] or [10]) that system (1) with an arbitrary bounded constraint set is null-controllable if it is stable. This is not surprising since stability guarantees that the system state tends to the origin with zero-input. Reference [14] gives a similar result for stable (semi-stable according to [16]) systems, and derives null-controllable regions for anti-stable systems. Reference [19] investigated the constrained reachability problem for time invariant nonlinear discrete time systems with disturbances. The proposed computational methods are based on polyhedral algebra and computational geometry. The computational methods can be implemented on a software program. It is assumed that the system is piecewise affine and the constraints are polygonal. Reference [20] provides necessary and sufficient conditions for global constrained controllability of continuous time, time varying linear systems. This work also gives a necessary and sufficient condition for the existence of a control input sequence (satisfying the constraints) which steers the system to the origin from a specified initial state. The work of [20] was one of the pioneering works in the field. However evaluating the given conditions, is not straightforward. For instance, see theorem 2.3 (equation 2.5) or example 1 therein.

The works of [21-29] studied constrained controllability of continuous time systems. In [21] the notion of approximate controllability is investigated for time invariant sampled data linear systems. The provided results (see theorem 4.1 therein) consists of trivial conditions of non-pathologic sampling periods (see equation 4.3 therein), input distribution matrix having full rank (see 4.2 therein) and an equality constrains (4.1 therein) which is computationally hard to evaluate because one has to fist compute the matrices and then check the equation. In fact, there is no method to guarantee this condition hold. Reference [22] investigates a few important properties of the reachable and attainable sets of a linear time invariant, continuous time system. Theorems 3.1 through 3.3 therein contain interesting properties. However, as can be seen from examples 4.1 and 4.2 in [22], the provided results are relatively cumbersome especially for systems of higher dimensions. (See example 4.1, 4.2 and figures 1, 2 therein).

In [23-26] computational methods are presented for approximating the reachable sets of linear time varying and time invariant continuous time systems. Reference [27] uses the notion of approximate constrained controllability to address the problem of model predictive control for a class of nonlinear systems. The authors of [28] addresses the finite controllability problem of stochastic and deterministic linear systems while [29] studies the constrained controllability of flat systems.

In the present paper sufficient conditions are given for controllability and null-controllability of time varying systems when the control input vector is confined in a symmetric saturation constraint. The proposed conditions are different from previously given conditions and cover reachability and null-controllability of both stable and unstable systems. The result is then applied to the special case of single input time invariant linear systems with diagonal Jordan canonical form. The derived result is mathematically tractable and can be examined via simple algorithms. The rest of this paper is organized as follows. In the second section of this paper, the main results are given. The third section provides clarifying examples. The last section concludes the paper.

2. Constrained Controllability of Linear Systems

Consider a discrete time linear system described by the following state space description;

$$x(k+1) = A(k)x(k) + B(k)u(k) \qquad (2)$$

Vectors $u(k) \in R^{n_u}$ and $x(k) \in R^n$ are the control input and the state variable. Matrices $A_{n\times n}$ and $B_{n\times n_u}$ are the state and input distribution matrices. The following proposition states constrained controllability problem for the system described by (2).

Proposition 1.The final state x_f is controllable in l steps from the initial state x_0 by the system (2), with an input constraint set Ψ_l if and only if there exists a vector $U_l = [u^T(0) \ \dots \ u^T(l-1)]^T$ such that;

$$\begin{cases} \Phi_l U_l = x_f - A(l-1)A(l-2) \times \dots A(0)x_0 \\ \text{s.t. } U_l \in \Psi_l \end{cases} \qquad (3)$$

Where;

$$\Phi_l = [A(l-1) \times \dots A(1)B(0) \quad \dots \quad A(l-1)B(l-2) \quad B(l-1)] \qquad (4)$$

The constraint set Ψ_l defines how the control input is restrained. Two constraint sets of positive inputs and saturated inputs are commonly used in the literature. Due to its wide industrial occurrence, we assume saturation constraint (5) similar to [8, 11-15] as;

$$-u_c \le u(k) \le u_c\,; \quad k = 0,1,\dots;\ u_c > 0 \qquad (5)$$

In which $u_c \in R^{n_u}$ is a constant real-valued vector containing the saturation limit of each actuator as an entry and the inequalities are component wise in the same manner defined in [30] and references therein. Assuming (5), the constraint set shall be represented as;

$$\Psi_l = \{U_l \mid -U_{cl} \le U_l \le U_{cl};\ U_{cl} = [u_c^T \ \dots \ u_c^T]^T \in R^{n_u l \times 1}\} \qquad (6)$$

Remark 1. For a time invariant linear discrete time system described by (1), a similarity transformation (i.e. $\bar{x} = T^{-1}x$, $\bar{A} = T^{-1}AT$, $\bar{B} = T^{-1}B$, $\bar{\Phi}_l = T^{-1}\Phi_l$) do not affect constrained controllability problem (3). This will be used for the results provided for time invariant systems in the next section.

Lemma 1. [30]Suppose that M, N are matrices, row vectors v, w and z, column vectors q and p are of appropriate dimensions. Exactly one of the alternatives (7, 8) or (9, 10) holds;

$$\exists v \ge 0\,; \begin{cases} Mv = w & (7) \\ Nv \le z & (8) \end{cases}$$

$$\exists\, q \ge 0, p\,; \begin{cases} pM + qN \ge 0 & (9) \\ pw + qz < 0 & (10) \end{cases}$$

In which, the vector inequalities are component-wise.

Lemma 1 is a variation of the well-known Farkas' lemma of alternative inequalities [30]. Farkas' lemma addresses solvability of linear inequalities [30]. This lemma is an alternative inequality lemma which states that exactly one of two alternative inequalities has a solution. The Farkas' lemma and its generalizations have been used in linear and convex programming for several decades [31].

The Farkas' lemma can also be used to examine feasibility of optimization problems with linear constraints [31]. Several algebraic and geometric proofs have been mentioned for this lemma [30, 32-34].

Lemma 1 states that there exists a vector (namely;v) with non-negative entries which simultaneously fulfills (7) and (8). If such a vector does not exist, then the other alternative (9, 10) holds. Many problems can be stated in the form of the first (second) alternative while it is more convenient to examine solvability in the second (first) form. The first alternative form is indeed a system of linear equations with a set of linear constraints. The second alternative is a set of two vector inequalities.

Theorem 1. The final state $x_f$is not controllable in $l$steps from the initial state $x_0$by the system (2) with constraint (5), ifthere existrow vectors $q \geq 0$, p which fulfill the following;

$$\begin{cases} p\Phi_l + q \geq 0 \\ p(-x_f + A(l-1)A(l-2) \times \dots A(0)x_0 + \Phi_l U_c) + 2qU_c < 0 \end{cases} \tag{11}$$

Proof: Write the constrained controllability problem as;

$$\Phi_l U_l = x_f - A(l-1)A(l-2) \times \dots A(0)x_0 \quad s.t. \begin{cases} U_l \leq U_c \\ U_l \geq -U_c \end{cases} \tag{12}$$

In which$U_c \in R^{l \times n_u}$ is defined as$U_c = [u_c^T \quad \dots \quad u_c^T]^T$. Subtract both sides of the equality from$\Phi_l U_c$. Also subtract U_c from the inequality to obtain;

$$\Phi_l(U_c - U_l) = -x_f + A(l-1)A(l-2) \times \dots A(0)x_0 + \Phi_l U_c \quad s.t. \begin{cases} U_c - U_l \geq 0 \\ U_c - U_l \leq 2U_c \end{cases} \tag{13}$$

The problem (13) can be viewed as the first alternative problem of Lemma 1 shown by equations (7)-(8). Apply Lemma 1 to the problem after making the following identifications;

$$w \to -x_f + A(l-1)A(l-2) \times \dots A(0)x_0 + \Phi_l U_c$$

$$M \to \Phi_l, \qquad v \to U_c - U_l, \qquad N \to I, \qquad z \to 2U_c \tag{14}$$

Therefore (10) reads;

$$\exists\, q \geq 0, p\,; \begin{cases} p\Phi_l + q \geq 0 \\ -px_f + pA(l-1)A(l-2) \times \dots A(0)x_0 + p\Phi_l U_c + 2qU_c < 0 \end{cases} \tag{15}$$

Corollary 1. Assume that Φ_l is rank deficient. Define p^* as a row vector fulfilling$p^*\Phi_l = 0$. The final state $x_f$is not controllable in lsteps from $x_0$by the system (2) with constraint (6), if there exists a row vector $q \geq 0$ such that;

$$p^*(-x_f + A(l-1)A(l-2) \times \dots A(0)x_0) + 2qU_c < 0 \tag{16}$$

Proof: (16) is established by applying Theorem 1 and using the assumption$p^*\Phi_l = 0$. ■

Theorem 2. The final state $x_f$is controllable in $l$steps from the initial state $x_0$by the system (2) with constraint (6), if for any arbitrary row vectors $q \geq 0, p$, the statement$p\Phi_l + q \geq 0$ implies;

$$p\big(-x_f + A(l-1)A(l-2) \times \dots A(0)x_0\big) + qU_c \geq 0 \tag{17}$$

Proof: Recall (11);

$$p(-x_f + A(l-1)A(l-2) \times \dots A(0)x_0 + \Phi_l U_c) + 2qU_c \geq 0 \tag{18}$$

Subtract the positive term $p\Phi_l U_c + qU_c$to derive a sufficient condition as;

$$p(-x_f + A(l-1)A(l-2) \times \dots A(0)x_0) + qU_c \geq 0$$

This completes the proof.

■

For the purpose of the next theorem, consider the following single input time invariant linear system in diagonal Jordan form (i.e.$A = \mathrm{diag}\{\lambda_1, \dots, \lambda_n\}$);

$$x(k+1) = A_{n\times n}x(k) + b_{n\times 1}u(k); \quad -u_c \le u(k) \le u_c \tag{19}$$

Theorem 3. The final state $x_f$is controllable in $l$steps from the initial state $x_0$by (19), if;

$$H_l^{-T}¥ \le 0 \tag{20}$$

Where;

$$H_l = \begin{bmatrix} \lambda_1^{l-1}b_1 & \dots & \lambda_n^{l-1}b_n \\ \vdots & \ddots & \vdots \\ \lambda_1\, b_1 & \dots & \lambda_n\, b_n \\ b_1 & \dots & b_n \end{bmatrix}_{l\times n},$$

$$¥ = \begin{bmatrix} x_{f1} - \lambda_1^l x_{01} + \dfrac{\lambda_1^l - 1}{\lambda_1 - 1} b_1 u_c \\ \vdots \\ x_{fn} - \lambda_n^l x_{0n} + \dfrac{\lambda_n^l - 1}{\lambda_n - 1} b_n u_c \end{bmatrix} \tag{21}$$

In which $\lambda_1, \dots, \lambda_n$ are eigenvalues of$A$ and;

$$x_f = [x_{f1} \quad \cdots \quad x_{fn}]^T, \qquad x_0 = [x_{01} \quad \cdots \quad x_{0n}]^T$$

Proof: Define $\underline{q} = [\underline{q}_1 \quad \cdots \quad \underline{q}_l]^T \ge 0$ such that;

$$p\Phi_l + \underline{q} = 0 \tag{22}$$

This corresponds to;

$$\underline{q_i} = \begin{cases} -[p\Phi_l]_i & ; \quad [p\Phi_l]_i < 0 \\ 0 & ; \quad [p\Phi_l]_i \ge 0 \end{cases} \quad ; i = 1, \dots, l$$

We will have;

$$p\Phi_l + \underline{q} = 0 \Rightarrow \begin{cases} p_1\lambda_1^{l-1}b_1 + \dots + p_n\lambda_n^{l-1}b_n + \underline{q}_1 = 0 \\ \vdots \\ p_1\lambda_1 b_1 + \dots + p_n\lambda_n b_n + \underline{q}_{l-1} = 0 \\ p_1 b_1 + p_2 b_2 + \dots + p_n b_n + \underline{q}_l = 0 \end{cases} \tag{23}$$

Note that according to (23), It is verified that;

$$p\Phi_l = H_l p^T$$

Rewriting (17) yields;

$$p(-x_f + A^l x_0) + qU_c = -p_1x_{f1} - \dots - p_n x_{fn} + p_1\lambda_1^l x_{01} + \dots + p_n\lambda_n^l x_{0n} + \sum_{i=1}^{l} q_i u_c \tag{24}$$

Note that since $q, u_c > 0$;

$$\sum_{i=1}^{l} q_i u_c \ge \sum_{i=1}^{l} \underline{q}_i u_c \tag{25}$$

Substitute each $\underline{q}_i$from (23), and using (25), (24) will yield;

$$p(-x_f + A^l x_0) + qU_c \ge -p_1x_{f1} - \dots - p_n x_{fn} + p_1\lambda_1^l x_{01} + \dots + p_n\lambda_n^l x_{0n} - p_1\lambda_1^{l-1}b_1 - \dots - p_n\lambda_n^{l-1}b_n - \dots - p_1 b_1 - \dots - p_n b_n \tag{26}$$

Rearrange the right hand side of (26) as;

$$p_1\left(x_{f1} - \lambda_1^l x_{01} + b_1 u_c \sum_{i=0}^{l-1}\lambda_1^i\right) \dots + p_n\left(x_{fn} - \lambda_n^l x_{0n} + b_n u_c \sum_{i=0}^{l-1}\lambda_n^i\right) \tag{27}$$

Writing the summation in its close form, (28) will be resulted;

$$p_1\left(x_{f1} - \lambda_1^l x_{01} + \frac{b_1 u_c(\lambda_1^l - 1)}{\lambda_1 - 1}\right) \dots + p_n\left(x_{fn} - \lambda_n^l x_{0n} + \frac{b_1 u_c(\lambda_n^l - 1)}{\lambda_n - 1}\right) \tag{28}$$

Rearranging in matrix form, (26) becomes;

$$p\left(-x_f + A^l x_0\right) + qU_c \geq p¥ \quad (29)$$

Also note that;

$$pH_l^T + \underline{q} = 0 \Rightarrow p = -\underline{q}H_l^{-T} \quad (30)$$

In which H_l^{-T} represents the right hand side inverse of the transpose of H_l. Using (30), the right hand side of (29) will be non-negative, if;

$$-\underline{q}H_l^{-T}¥ \geq 0 \quad (31)$$

This establishes (20) since $\underline{q} \geq 0$. The proof is complete.

■

Remark 2. A diagonal Jordan form exists for system (19) if the system has either distinct real eigenvalues or repeated eigenvalues with linearly independent eigenvectors [35]. Also note that (20) requires that a right hand side inverse exists for H_l^T which requires $l \geq n$.

Theorem 3 enhances existing results in the literature in many aspects. It states constrained controllability conditions for a system regardless of its stability. It also deals with both null-controllability and reachability. The provided condition (20) is simple to evaluate.

3. Examples

Example 1. Consider a memoryless system (i.e. $A(k) = 0\,; k = 0, \ldots$) with a time varying input distribution vector $b(k)$, and a scalar input $u(k)$ described by the following equation and input constraint;

$$x(k+1) = b(k)u(k) \quad ; \quad -u_c \leq u(k) \leq u_c$$

It is inevitable to assume; $l = 1$ since the system is memoryless. Therefore, a sufficient condition for constrained controllability is derived as;

$$pb(k) + q \geq 0 \Rightarrow -px_f + qu_c \geq 0$$

Setting $x_f = 0$ (i.e. constrained null-controllability) results;

$$pb(k) + q \geq 0 \Rightarrow qu_c \geq 0$$

Note that there exist many solutions to the left hand side inequality. (One solution is $p = 0$ and $q > 0$ arbitrarily chosen). Therefore, the memoryless linear system is constrained null-controllable for any $u_c > 0$ and any x_0. This is trivial as $u(k) = 0$ will suffice for null controllability. It is also in conformity with the result of [18] (Theorem IV, equation (8b)) for time invariant systems (i.e. $b(k) = b; k > 0$ because (8b) in [18] is satisfied due to the fact that the memoryless system has no unstable eigenvalue.

Example 2. Consider a scalar time invariant linear system as;

$$x(k+1) = ax(k) + bu(k) \quad ; \; a, b > 0, \\ -u_c \leq u(k) \leq u_c$$

For any p, q fulfilling;

$$p[a^{l-1}b \quad \ldots \quad b] + q \geq 0$$

Which implies;

$$pa^{l-1}b + q_1 \geq 0 \\ \vdots \\ pb + q_l \geq 0$$

Condition (17) reads; (Note that it is assumed $a, b > 0$ and $q \geq 0$)

$$p\left(-x_f + a^l x_0\right) + qU_c \geq p\left(-x_f + a^l x_0\right) + \sum_{i=1}^{l} q_i\, u_c \\ \geq \frac{q_l x_f}{b} - \frac{q_l a^l x_0}{b} + \sum_{i=1}^{l} q_i\, u_c$$

The right hand side term equals;

$$\sum_{i=1}^{l-1} q_i\, u_c + \frac{x_f - a^l x_0 + bu_c}{b} q_l$$

Therefore, a sufficient condition for constrained controllability from initial state x_0 to the final state x_f would be;

$$\frac{x_f - a^l x_0 + bu_c}{b} \geq 0$$

Assigning $x_f = 0$ (null-controllability case) and $0 < a < 1$ (a stable system with a positive eigenvalue), this result conforms to that of [18] (where constrained null-controllability is proved for all stable systems) since $a^l x_0$ can be made arbitrary small by increasing the number of steps l until it fulfills $a^l x_0 \le b u_c$. Note that this condition is an alternative to that resulted from applying Theorem 3 to the scalar single input system.

Example 3. Assume the following system with zero initial conditions $x_0 = 0$ with $u_c = 1$;

$$x(k+1) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} u(k) \ ; \ \Phi_l = \begin{bmatrix} 0 & \dots & 0 & 1 \\ 2 \times (-1)^{l-1} & \dots & -2 & 1 \\ 0 & \dots & 0 & -1 \end{bmatrix}$$

For $l > 3$, inequality $p\Phi_l + q \ge 0$ is equivalent to the following;

$$-2p_2 + q_1 \ge 0\,, \qquad 2p_2 + q_2 \ge 0, \qquad p_1 + p_2 - p_3 + q_3 \ge 0$$

Therefore, $p = [1 \ \ 0 \ \ 1]$ with any $q \ge 0$ fulfill $p\Phi_l + q \ge 0$. A final state x_f is not controllable from zero initial state if it fulfills:

$$-x_{f1} - x_{f3} + 2(q_1 + q_2 + q_3) < 0$$

Since $q \ge 0$ may be chosen arbitrarily small, any final state fulfilling $-x_{f1} - x_{f3} < 0$ is not controllable from zero initial state. Also, by choosing $p = [-1 \ \ 0 \ \ -1]$ any final state fulfilling $-x_{f1} - x_{f3} > 0$ is not constrained controllable with $u_c = 1$ as well. Now assume $x_{f1} + x_{f3} = 0$. Inequality (17) reads;

$$-p_1 x_{f1} + p_3 x_{f1} - p_2 x_{f2} + q_1 + q_2 + q_3 \ge 0$$

Using the last inequality in (19) results;

$$-p_1 x_{f1} + p_3 x_{f1} - p_2 x_{f2} + q_1 + q_2 + q_3 \ge p_2(x_{f1} - x_{f2}) + q_1 + q_2$$

Multiply the first and second inequalities in (19) by positive scalars $\alpha, \beta > 0$, and add the two resulting inequalities to obtain;

$$\frac{2(\alpha - \beta)}{\alpha + \beta} p_2 + q_2 \ge 0$$

It is resulted that for $-2 < x_{f1} - x_{f2} < 2$, one has;

$$-p_1 x_{f1} + p_3 x_{f1} - p_2 x_{f2} + q_1 + q_2 + q_3 \ge p_2(x_{f1} - x_{f2}) + q_2 \ge 0$$

As a result, any final state with $x_{f1} + x_{f3} = 0$ and $-2 < x_{f1} - x_{f2} < 2$ is constrained controllable with $u_c = 1$, from zero initial state.

Example 4. This example examines Theorem 3. Consider the following system;

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad b = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \qquad H_l = \begin{bmatrix} 1 & 0 \\ \vdots & \vdots \\ 1 & 0 \\ 1 & 0 \\ 1 & 2 \end{bmatrix},$$

$$¥ = \begin{bmatrix} x_{f1} - x_{01} + 2u_c \\ x_{fn} + 2u_c \end{bmatrix}$$

A choice for H_l^{-T} would be;

$$H_l^{-T} = \begin{bmatrix} \frac{1}{l} & 0 \\ \vdots & \vdots \\ \frac{1}{l} & 0 \\ \frac{1}{l} & 0 \\ \frac{1}{l} & \frac{1}{2} \end{bmatrix}$$

Which yields;

$$H_l^{-T}¥ = \frac{1}{l} \begin{bmatrix} (x_{f1} - x_{01} + 2u_c) \\ \vdots \\ (x_{f1} - x_{01} + 2u_c) \\ (x_{f1} - x_{01} + 2u_c) + \frac{l}{2}(x_{f2} + 2u_c) \end{bmatrix}$$

Therefore, the following condition is achieved for constrained controllability;

$$x_{f1} - x_{01} + 2u_c \le 0\,, \qquad x_{f1} - x_{01} + (l+2)u_c + \frac{l}{2} x_{f2} \le 0$$

Example 5. Consider the following time varying linear system:

$$x(k+1) = a^k x(k) + \left(1 + (-1)^k\right) u(k) \quad ; \ -u_c \le u(k) \le u_c \ ; \ a > 0$$

It can be shown that this (time varying) system is stable if $-1 < a < 1$ and unstable if $|a| > 1$. The system is affected by the input at every other time sample. This occurs when the control input is transmitted to the actuator via a bandwidth limited communication medium or a switched channel without zero order hold as described in [4, 36]. Scalar ais a system parameter. Note that:

$$\Phi_l = [2a^{l(l+1)/2} \quad \dots \quad a(1+(-1)^{l-2}) \quad 1+(-1)^{l-1}]$$

This means:

$$\Phi_l = \begin{cases} [2a^{l(l+1)/2} \quad 0 \ \dots \quad 2a \quad 0] & ; l = 2,4,6,\dots \\ [2a^{l(l+1)/2} \quad 0 \ \dots \quad 0 \quad 2] & ; l = 1,3,5,\dots \end{cases}$$

Therefore:

$$p\Phi_l + q = \begin{cases} [2pa^{l(l+1)/2} + q \quad 0 \ \dots \quad 2pa + q \quad 0] & ; l = 2,4,6,\dots \\ [2pa^{l(l+1)/2} + q \quad 0 \ \dots \quad 0 \quad 2p + q] & ; l = 1,3,5,\dots \end{cases}$$

Inequality $p\Phi_l + q \geq 0$ is equivalent to:

$$p\Phi_l + q \geq 0 \Leftrightarrow \begin{cases} 2pa^{k(k+1)/2} > -q & ; k = 2,4,6,\dots l \\ 2pa^{k(k+1)/2} > -q & ; k = 1,3,5,\dots l \end{cases}$$

To apply Theorem 2, note that:

$$\begin{aligned} p(-x_f + A(l-1)A(l-2) \times \dots A(0)x_0) + qU_c \\ = p\left(-x_f + a^{l(l+1)/2}x_0\right) + qu_c \end{aligned}$$

If:

$$-x_f + a^{l(l+1)/2}x_0 > 0$$

Then from$2pa^{k(k+1)/2} > -q$, we deduce:

$$p\left(-x_f + a^{l(l+1)/2}x_0\right) + qu_c > \frac{q}{2a^{k(k+1)/2}}\left(x_f - a^{l(l+1)/2}x_0 + 2u_c\right) \ ; \ 0 < k < l$$

Since$q > 0$, a set of l-step constrained controllable initial /final states with even or odd lis:

$$x_f - a^{l(l+1)/2}x_0 + 2u_c > 0$$

Combining this condition with$-x_f + a^l x_0 > 0$, we obtain the following sufficient condition:

$$0 < -x_f + a^{l(l+1)/2}x_0 < 2u_c$$

For the specific case of null-controllability (i.e.$x_f = 0$) with$a < 1$, we can always fulfill the given inequality by increasingl; the number of steps. On the contrary, if$a > 1$, increasing the number of steps does not help fulfill this condition. This conforms to the intuition that for stable systems, null-controllability is always fulfilled for sufficiently large number of steps. This is proved in [18] for general linear time invariant systems.

Focusing on the problem of controllability from origin (i.e. $x_0 = 0$), the given condition only establishes constrained controllability for negative final states of specified magnitudes (i.e.$0 < -x_f < 2u_c$). This, however, does not rule out controllability of any positive final state. In other words, the given condition is not conclusive in this case.

To numerically examine this result, assume$u_c = 1$, $a \in (1.05,1.5)$ and $x(0) \in (-10,10)$ with the following control input:

$$u(k) = -\frac{1}{2}a^k x(k)$$

This control input steers the system state to the origin in one step, if constraint existed. However, in the constrained case, this control input may not be able to steer the system state to the origin or may need a few number of steps. Note that this choice of control input does not necessarily result in minimum number of steps. Table 1 shows the simulation results. For each value of the parametera, a range of initial states which can be steered to the origin by the saturated version of the given control input in no more than 100 steps. For$a \geq 1.35$, the only constrained controllable initial state is the origin itself. Note that, in Table 1, $u_c$is assumed as unity.

Table 1. Set of constrained null-controllable initial states with $u_c = 1$

a	1.05	1.1	1.15	1.2	1.25	1.3	1.35 ≤
$x(0)$	$(-4,4)$	$(-2,2)$	$(-1,1)$				0

4. Conclusion

In this paper, sufficient conditions are provided for reachability and controllability of time varying linear discrete time systems. The provided conditions are then applied to the case of single input, time invariant linear systems with diagonal Jordan form as a special case. Numerical examples are given to clarify the results.

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