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Fuzzy Ordinary and Fractional General Sigmoid Function Activated Neural Network Approximation

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Abstract. Here we research the univariate fuzzy ordinary and fractional quantitative approximation of fuzzy real valued functions on a compact interval by quasi-interpolation general sigmoid activation function relied on fuzzy neural network operators. These approximations are derived by establishing fuzzy Jackson type inequalities involving the fuzzy moduli of continuity of the function, or of the right and left Caputo fuzzy fractional derivatives of the involved function. The approximations are fuzzy pointwise and fuzzy uniform. The related feed-forward fuzzy neural networks are with one hidden layer. We study in particular the fuzzy integer derivative and just fuzzy continuous cases. Our fuzzy fractional approximation result using higher order fuzzy differentiation converges better than in the fuzzy just continuous case.

AMS Subject Classification 2020: 26A33; 26E50; 41A17; 41A25; 41A30; 41A36; 47S40.
Keywords and Phrases: General sigmoid activation function, Neural network fuzzy fractional approximation, Fuzzy quasi-interpolation operator, Fuzzy modulus of continuity, Fuzzy derivative and fuzzy fractional derivative.

1 Introduction

The author in [1] and [2], see chapters 2-5, was the first to derive quantitative neural network approximations to continuous functions with rates by very specifically defined neural network operators of Cardaliaguet-Euvrard and "Squashing" types, by employing the modulus of continuity of the engaged function or its high order derivative, and producing very tight Jackson type inequalities. He studied there both the univariate and multivariate cases. The defining of these operators "bell-shaped" and "squashing" functions are assumed to be of compact support.

The author inspired by [23], continued his studies on neural networks approximation by introducing and using the proper quasi-interpolation operators of sigmoidal and hyperbolic tangent type which resulted in [10], [13] - [22], by treating both the univariate and multivariate cases.

Continuation of the author's works ([17], [18] and [19], Chapter 20) is this article where fuzzy neural network approximation based on a general sigmoid activation function is taken at the fractional and ordinary levels resulting in higher rates of approximation. We involve the fuzzy ordinary derivatives and the right and left Caputo fuzzy fractional derivatives of the fuzzy function under approximation and we establish tight fuzzy Jackson type inequalities. An extensive background is given on fuzzyness, fractional calculus and neural networks, all needed to present our work.

Our fuzzy feed-forward neural networks (FFNNs) are with one hidden layer. About neural networks in general study [29], [32], [33].

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2 Fuzzy Fractional Mathematical Analysis Basics

(see also [19], pp. 432-444)

We need the following basic background

Definition 2.1. (see [36]) Let $\mu : \mathbb{R} \to [0,1]$ with the following properties:

(i) is normal, i.e., $\exists\ x_0 \in \mathbb{R}$; $\mu(x_0) = 1$.

(ii) $\mu(\lambda x + (1-\lambda) y) \geq \min\{\mu(x), \mu(y)\}$, $\forall\ x, y \in \mathbb{R}$, $\forall\ \lambda \in [0,1]$ (μ is called a convex fuzzy subset).

(iii) μ is upper semicontinuous on $\mathbb{R}$, i.e. $\forall\ x_0 \in \mathbb{R}$ and $\forall\ \varepsilon > 0$, $\exists$ neighborhood $V(x_0) : \mu(x) \leq \mu(x_0) + \varepsilon$, $\forall\ x \in V(x_0)$.

(iv) The set $\overline{\text{supp}(\mu)}$ is compact in $\mathbb{R}$ (where $\text{supp}(\mu) := \{x \in \mathbb{R} : \mu(x) > 0\}$).

We call μ a fuzzy real number. Denote the set of all μ with $\mathbb{R}_{\mathcal{F}}$.

E.g. $\chi_{\{x_0\}} \in \mathbb{R}_{\mathcal{F}}$, for any $x_0 \in \mathbb{R}$, where $\chi_{\{x_0\}}$ is the characteristic function at x_0.

For $0 < r \leq 1$ and $\mu \in \mathbb{R}_{\mathcal{F}}$ define

$$[\mu]^r := \{x \in \mathbb{R} : \mu(x) \geq r\}$$

and

$$[\mu]^0 := \overline{\{x \in \mathbb{R} : \mu(x) \geq 0\}}.$$

Then it is well known that for each $r \in [0,1]$, $[\mu]^r$ is a closed and bounded interval on $\mathbb{R}$ ([28]).

For $u, v \in \mathbb{R}_{\mathcal{F}}$ and $\lambda \in \mathbb{R}$, we define uniquely the sum $u \oplus v$ and the product $\lambda \odot u$ by

$$[u \oplus v]^r = [u]^r + [v]^r, \quad [\lambda \odot u]^r = \lambda [u]^r, \quad \forall\ r \in [0,1],$$

where

$[u]^r + [v]^r$ means the usual addition of two intervals (as substes of $\mathbb{R}$) and

$\lambda [u]^r$ means the usual product between a scalar and a subset of $\mathbb{R}$ (see, e.g. [36]).

Notice $1 \odot u = u$ and it holds

$$u \oplus v = v \oplus u, \ \lambda \odot u = u \odot \lambda.$$

If $0 \leq r_1 \leq r_2 \leq 1$ then

$$[u]^{r_2} \subseteq [u]^{r_1}.$$

Actually $[u]^r = \left[u_-^{(r)}, u_+^{(r)}\right]$, where $u_-^{(r)} \leq u_+^{(r)}$, $u_-^{(r)}$, $u_+^{(r)} \in \mathbb{R}$, $\forall\ r \in [0,1]$.

For $\lambda > 0$ one has $\lambda u_\pm^{(r)} = (\lambda \odot u)_\pm^{(r)}$, respectively.

Define $D : \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_{\mathcal{F}}$ by

$$D(u,v) := \sup_{r \in [0,1]} \max\left\{\left|u_-^{(r)} - v_-^{(r)}\right|, \left|u_+^{(r)} - v_+^{(r)}\right|\right\},$$

where

$$[v]^r = \left[v_-^{(r)}, v_+^{(r)}\right]; \ \ u, v \in \mathbb{R}_{\mathcal{F}}.$$

We have that D is a metric on $\mathbb{R}_{\mathcal{F}}$.

Then $(\mathbb{R}_{\mathcal{F}}, D)$ is a complete metric space, see [36], [37].

Here $\overset{*}{\sum}$ stands for fuzzy summation and $\widetilde{o} := \chi_{\{0\}} \in \mathbb{R}_{\mathcal{F}}$ is the neutral element with respect to $\oplus$, i.e.,

$$u \oplus \widetilde{0} = \widetilde{0} \oplus u = u, \ \ \forall\ u \in \mathbb{R}_{\mathcal{F}}.$$

Denote

$$D^*(f,g) = \sup_{x \in X \subseteq \mathbb{R}} D(f,g),$$

where $f, g : X \to \mathbb{R}_{\mathcal{F}}$.

We mention

Definition 2.2. Let $f : X \subseteq \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$, X interval, we define the (first) fuzzy modulus of continuity of f by

$$\omega_1^{(\mathcal{F})}(f,\delta)_X = \sup_{x,y\in X,\ |x-y|\le\delta} D(f(x), f(y)),\ \ \delta>0.$$

When $g : X \subseteq \mathbb{R} \to \mathbb{R}$, we define

$$\omega_1(g,\delta) = \omega_1(g,\delta)_X = \sup_{x,y\in X,\ |x-y|\le\delta} |g(x)-g(y)|.$$

We define by $C_{\mathcal{F}}^U(\mathbb{R})$ the space of fuzzy uniformly continuous functions from $\mathbb{R} \to \mathbb{R}_{\mathcal{F}}$, also $C_{\mathcal{F}}(\mathbb{R})$ is the space of fuzzy continuous functions on $\mathbb{R}$, and $C_b(\mathbb{R}, \mathbb{R}_{\mathcal{F}})$ is the fuzzy continuous and bounded functions.

We mention

Proposition 2.3. *([5]) Let $f \in C_{\mathcal{F}}^U(X)$. Then $\omega_1^{(\mathcal{F})}(f,\delta)_X < \infty$, for any $\delta > 0$.*

By [9], p. 129 we have that $C_{\mathcal{F}}^U([a,b]) = C_{\mathcal{F}}([a,b])$, fuzzy continuous functions on $[a,b] \subset \mathbb{R}$.

Proposition 2.4. *([5]) It holds*

$$\lim_{\delta\to 0}\omega_1^{(\mathcal{F})}(f,\delta)_X = \omega_1^{(\mathcal{F})}(f,0)_X = 0,$$

iff $f \in C_{\mathcal{F}}^U(X)$, where X is a compact interval.

Proposition 2.5. *([5]) Here $[f]^r = \left[f_-^{(r)}, f_+^{(r)}\right]$, $r \in [0,1]$. Let $f \in C_{\mathcal{F}}(\mathbb{R})$. Then $f_\pm^{(r)}$ are equicontinuous with respect to $r \in [0,1]$ over $\mathbb{R}$, respectively in $\pm$.*

Note 2.6. It is clear by Propositions 2.4, 2.5, that if $f \in C_{\mathcal{F}}^U(\mathbb{R})$, then $f_\pm^{(r)} \in C_U(\mathbb{R})$ (uniformly continuous on $\mathbb{R}$). Also if $f \in C_b(\mathbb{R}, \mathbb{R}_{\mathcal{F}})$ implies $f_\pm^{(r)} \in C_b(\mathbb{R})$ (continuous and bounded functions on $\mathbb{R}$).

Proposition 2.7. *Let $f : \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$. Assume that $\omega_1^{\mathcal{F}}(f,\delta)_X$, $\omega_1\left(f_-^{(r)},\delta\right)_X$, $\omega_1\left(f_+^{(r)},\delta\right)_X$ are finite for any $\delta > 0$, $r \in [0,1]$, where X any interval of $\mathbb{R}$.*

Then

$$\omega_1^{(\mathcal{F})}(f,\delta)_X = \sup_{r\in[0,1]} \max\left\{\omega_1\left(f_-^{(r)},\delta\right)_X, \omega_1\left(f_+^{(r)},\delta\right)_X\right\}.$$

Proof. Similar to Proposition 14.15, p. 246 of [9]. $\square$

We need

Remark 2.8. ([3]). Here $r \in [0,1]$, $x_i^{(r)}, y_i^{(r)} \in \mathbb{R}$, $i = 1, ..., m \in \mathbb{N}$. Suppose that

$$\sup_{r\in[0,1]} \max\left(x_i^{(r)}, y_i^{(r)}\right) \in \mathbb{R}, \text{ for } i = 1, ..., m.$$

Then one sees easily that

$$\sup_{r\in[0,1]} \max\left(\sum_{i=1}^m x_i^{(r)}, \sum_{i=1}^m y_i^{(r)}\right) \le \sum_{i=1}^m \sup_{r\in[0,1]} \max\left(x_i^{(r)}, y_i^{(r)}\right). \tag{1}$$

We need

Definition 2.9. Let $x, y \in \mathbb{R}_{\mathcal{F}}$. If there exists $z \in \mathbb{R}_{\mathcal{F}} : x = y \oplus z$, then we call z the H-difference on x and y, denoted $x - y$.

Definition 2.10. ([35]) Let $T := [x_0, x_0 + \beta] \subset \mathbb{R}$, with $\beta > 0$. A function $f : T \to \mathbb{R}_{\mathcal{F}}$ is H-differentiable at $x \in T$ if there exists an $f'(x) \in \mathbb{R}_{\mathcal{F}}$ such that the limits (with respect to D)

$$\lim_{h \to 0+} \frac{f(x+h) - f(x)}{h}, \quad \lim_{h \to 0+} \frac{f(x) - f(x-h)}{h} \tag{2}$$

exist and are equal to $f'(x)$.

We call f' the H-derivative or fuzzy derivative of f at x.

Above is assumed that the H-differences $f(x+h) - f(x)$, $f(x) - f(x-h)$ exists in $\mathbb{R}_{\mathcal{F}}$ in a neighborhood of x.

Higher order H-fuzzy derivatives are defined the obvious way, like in the real case.

We denote by $C^N_{\mathcal{F}}(\mathbb{R})$, $N \geq 1$, the space of all N-times continuously H-fuzzy differentiable functions from $\mathbb{R}$ into $\mathbb{R}_{\mathcal{F}}$, similarly is defined $C^N_{\mathcal{F}}([a,b])$, $[a,b] \subset \mathbb{R}$.

We mention

Theorem 2.11. *([30]) Let $f : \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$ be H-fuzzy differentiable. Let $t \in \mathbb{R}$, $0 \leq r \leq 1$. Clearly*

$$[f(t)]^r = \left[f(t)^{(r)}_-, f(t)^{(r)}_+ \right] \subseteq \mathbb{R}.$$

Then $(f(t))^{(r)}_{\pm}$ are differentiable and

$$[f'(t)]^r = \left[\left(f(t)^{(r)}_- \right)', \left(f(t)^{(r)}_+ \right)' \right].$$

I.e.

$$(f')^{(r)}_{\pm} = \left(f^{(r)}_{\pm} \right)', \quad \forall\, r \in [0,1].$$

Remark 2.12. ([4]) Let $f \in C^N_{\mathcal{F}}(\mathbb{R})$, $N \geq 1$. Then by Theorem 2.11 we obtain

$$\left[f^{(i)}(t) \right]^r = \left[\left(f(t)^{(r)}_- \right)^{(i)}, \left(f(t)^{(r)}_+ \right)^{(i)} \right],$$

for $i = 0, 1, 2, ..., N$, and in particular we have that

$$\left(f^{(i)} \right)^{(r)}_{\pm} = \left(f^{(r)}_{\pm} \right)^{(i)},$$

for any $r \in [0,1]$, all $i = 0, 1, 2, ..., N$.

Note 2.13. ([4]) Let $f \in C^N_{\mathcal{F}}(\mathbb{R})$, $N \geq 1$. Then by Theorem 2.11 we have $f^{(r)}_{\pm} \in C^N(\mathbb{R})$, for any $r \in [0,1]$.

Items 11-13 are valid also on $[a,b]$.

By [9], p. 131, if $f \in C_{\mathcal{F}}([a,b])$, then f is a fuzzy bounded function.

We need also a particular case of the Fuzzy Henstock integral ($\delta(x) = \frac{\delta}{2}$), see [36].

Definition 2.14. ([27], p. 644) Let $f : [a,b] \to \mathbb{R}_{\mathcal{F}}$. We say that f is Fuzzy-Riemann integrable to $I \in \mathbb{R}_{\mathcal{F}}$ if for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any division $P = \{[u,v]; \xi\}$ of $[a,b]$ with the norms $\Delta(P) < \delta$, we have

$$D\left(\sum_P^* (v-u) \odot f(\xi), I \right) < \varepsilon.$$

We write

$$I := (FR) \int_a^b f(x)\, dx. \tag{3}$$

We mention

Theorem 2.15. *([28]) Let $f : [a, b] \to \mathbb{R}_{\mathcal{F}}$ be fuzzy continuous. Then*

$$(FR)\int_a^b f(x)\,dx$$

exists and belongs to $\mathbb{R}_{\mathcal{F}}$, furthermore it holds

$$\left[(FR)\int_a^b f(x)\,dx\right]^r = \left[\int_a^b (f)_-^{(r)}(x)\,dx, \int_a^b (f)_+^{(r)}(x)\,dx\right],$$

$\forall\ r \in [0, 1]$.

For the definition of general fuzzy integral we follow [31] next.

Definition 2.16. Let (Ω, Σ, μ) be a complete σ-finite measure space. We call $F : \Omega \to R_{\mathcal{F}}$ measurable iff $\forall$ closed $B \subseteq \mathbb{R}$ the function $F^{-1}(B) : \Omega \to [0, 1]$ defined by

$$F^{-1}(B)(w) := \sup_{x \in B} F(w)(x), \text{ all } w \in \Omega$$

is measurable, see [31].

Theorem 2.17. *([31]) For $F : \Omega \to \mathbb{R}_{\mathcal{F}}$,*

$$F(w) = \left\{\left(F_-^{(r)}(w), F_+^{(r)}(w)\right) | 0 \leq r \leq 1\right\},$$

the following are equivalent

(1) F is measurable,

(2) $\forall\ r \in [0, 1]$, $F_-^{(r)}$, $F_+^{(r)}$ are measurable.

Following [31], given that for each $r \in [0, 1]$, $F_-^{(r)}$, $F_+^{(r)}$ are integrable we have that the parametrized representation

$$\left\{\left(\int_A F_-^{(r)} d\mu, \int_A F_+^{(r)} d\mu\right) | 0 \leq r \leq 1\right\} \tag{4}$$

is a fuzzy real number for each $A \in \Sigma$.

The last fact leads to

Definition 2.18. ([31]) A measurable function $F : \Omega \to \mathbb{R}_{\mathcal{F}}$,

$$F(w) = \left\{\left(F_-^{(r)}(w), F_+^{(r)}(w)\right) | 0 \leq r \leq 1\right\}$$

is integrable if for each $r \in [0, 1]$, $F_{\pm}^{(r)}$ is integrable, or equivalently, if $F_{\pm}^{(0)}$ is integrable.

In this case, the fuzzy integral of F over $A \in \Sigma$ is defined by

$$\int_A F d\mu := \left\{\left(\int_A F_-^{(r)} d\mu, \int_A F_+^{(r)} d\mu\right) | 0 \leq r \leq 1\right\}.$$

By [31], F is integrable iff $w \to \|F(w)\|_{\mathcal{F}}$ is real-valued integrable.

Here denote

$$\|u\|_{\mathcal{F}} := D\left(u, \widetilde{0}\right), \ \ \forall\ u \in \mathbb{R}_{\mathcal{F}}.$$

We need also

Theorem 2.19. *([31]) Let $F, G : \Omega \to \mathbb{R}_{\mathcal{F}}$ be integrable. Then*
(1) Let $a, b \in \mathbb{R}$, then $aF + bG$ is integrable and for each $A \in \Sigma$,

$$\int_A (aF + bG)\, d\mu = a \int_A F d\mu + b \int_A G d\mu;$$

(2) $D(F, G)$ is a real- valued integrable function and for each $A \in \Sigma$,

$$D\left(\int_A F d\mu, \int_A G d\mu\right) \leq \int_A D(F, G)\, d\mu.$$

In particular,

$$\left\|\int_A F d\mu\right\|_{\mathcal{F}} \leq \int_A \|F\|_{\mathcal{F}}\, d\mu.$$

Above μ could be the Lebesgue measure, with all the basic properties valid here too.
Basically here we have

$$\left[\int_A F d\mu\right]^r = \left[\int_A F_-^{(r)} d\mu, \int_A F_+^{(r)} d\mu\right], \tag{5}$$

i.e.

$$\left(\int_A F d\mu\right)_{\pm}^{(r)} = \int_A F_{\pm}^{(r)} d\mu, \quad \forall\, r \in [0, 1].$$

We need

Definition 2.20. Let $\nu \geq 0$, $n = \lceil \nu \rceil$ ($\lceil \cdot \rceil$ is the ceiling of the number), $f \in AC^n([a, b])$ (space of functions f with $f^{(n-1)} \in AC([a, b])$, absolutely continuous functions). We call left Caputo fractional derivative (see [24], pp. 49-52, [26], [34]) the function

$$D_{*a}^{\nu} f(x) = \frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} f^{(n)}(t)\, dt, \tag{6}$$

$\forall\, x \in [a, b]$, where Γ is the gamma function $\Gamma(\nu) = \int_0^{\infty} e^{-t} t^{\nu-1} dt$, $\nu > 0$.
Notice $D_{*a}^{\nu} f \in L_1([a, b])$ and $D_{*a}^{\nu} f$ exists a.e. on $[a, b]$.
We set $D_{*a}^{0} f(x) = f(x)$, $\forall\, x \in [a, b]$.

Lemma 2.21. *([8]) Let $\nu > 0$, $\nu \notin \mathbb{N}$, $n = \lceil \nu \rceil$, $f \in C^{n-1}([a, b])$ and $f^{(n)} \in L_{\infty}([a, b])$. Then $D_{*a}^{\nu} f(a) = 0$.*

Definition 2.22. (see also [6], [25], [26]) Let $f \in AC^m([a, b])$, $m = \lceil \beta \rceil$, $\beta > 0$. The right Caputo fractional derivative of order $\beta > 0$ is given by

$$D_{b-}^{\beta} f(x) = \frac{(-1)^m}{\Gamma(m-\beta)} \int_x^b (\zeta - x)^{m-\beta-1} f^{(m)}(\zeta)\, d\zeta, \tag{7}$$

$\forall\, x \in [a, b]$. We set $D_{b-}^{0} f(x) = f(x)$. Notice that $D_{b-}^{\beta} f \in L_1([a, b])$ and $D_{b-}^{\beta} f$ exists a.e. on $[a, b]$.

Lemma 2.23. *([8]) Let $f \in C^{m-1}([a, b])$, $f^{(m)} \in L_{\infty}([a, b])$, $m = \lceil \beta \rceil$, $\beta > 0$, $\beta \notin \mathbb{N}$. Then $D_{b-}^{\beta} f(b) = 0$.*

Convention 2.24. We assume that

$$D_{*x_0}^{\beta} f(x) = 0, \ \text{ for } x < x_0, \tag{8}$$

and

$$D_{x_0-}^{\beta} f(x) = 0, \ \text{ for } x > x_0, \tag{9}$$

for all $x, x_0 \in [a, b]$.

We mention

Proposition 2.25. *([8]) Let $f \in C^n([a,b])$, $n = \lceil \nu \rceil$, $\nu > 0$. Then $D_{*a}^{\nu} f(x)$ is continuous in $x \in [a,b]$.*

Also we have

Proposition 2.26. *([8]) Let $f \in C^m([a,b])$, $m = \lceil \beta \rceil$, $\beta > 0$. Then $D_{b-}^{\beta} f(x)$ is continuous in $x \in [a,b]$.*

We further mention

Proposition 2.27. *([8]) Let $f \in C^{m-1}([a,b])$, $f^{(m)} \in L_\infty([a,b])$, $m = \lceil \beta \rceil$, $\beta > 0$ and*

$$D_{*x_0}^{\beta} f(x) = \frac{1}{\Gamma(m-\beta)} \int_{x_0}^{x} (x-t)^{m-\beta-1} f^{(m)}(t)\, dt, \tag{10}$$

for all $x, x_0 \in [a,b] : x \geq x_0$.

Then $D_{*x_0}^{\beta} f(x)$ is continuous in x_0.

Proposition 2.28. *([8]) Let $f \in C^{m-1}([a,b])$, $f^{(m)} \in L_\infty([a,b])$, $m = \lceil \beta \rceil$, $\beta > 0$ and*

$$D_{x_0-}^{\beta} f(x) = \frac{(-1)^m}{\Gamma(m-\beta)} \int_{x}^{x_0} (\zeta - x)^{m-\beta-1} f^{(m)}(\zeta)\, d\zeta, \tag{11}$$

for all $x, x_0 \in [a,b] : x \leq x_0$.

Then $D_{x_0-}^{\beta} f(x)$ is continuous in x_0.

We need

Proposition 2.29. *([8]) Let $g \in C([a,b])$, $0 < c < 1$, $x, x_0 \in [a,b]$. Define*

$$L(x, x_0) = \int_{x_0}^{x} (x-t)^{c-1} g(t)\, dt, \quad \text{for } x \geq x_0, \tag{12}$$

and $L(x, x_0) = 0$, for $x < x_0$.

Then L is jointly continuous in (x, x_0) on $[a,b]^2$.

We mention

Proposition 2.30. *([8]) Let $g \in C([a,b])$, $0 < c < 1$, $x, x_0 \in [a,b]$. Define*

$$K(x, x_0) = \int_{x_0}^{x} (\zeta - x)^{c-1} g(\zeta)\, d\zeta, \quad \text{for } x \leq x_0, \tag{13}$$

and $K(x, x_0) = 0$, for $x > x_0$.

Then $K(x, x_0)$ is jointly continuous from $[a,b]^2$ into $\mathbb{R}$.

Based on Propositions 2.29, 2.30 we derive

Corollary 2.31. *([8]) Let $f \in C^m([a,b])$, $m = \lceil \beta \rceil$, $\beta > 0$, $\beta \notin \mathbb{N}$, $x, x_0 \in [a,b]$. Then $D_{*x_0}^{\beta} f(x)$, $D_{x_0-}^{\beta} f(x)$ are jointly continuous functions in (x, x_0) from $[a,b]^2$ into $\mathbb{R}$.*

We need

Theorem 2.32. *([8]) Let $f : [a,b]^2 \to \mathbb{R}$ be jointly continous. Consider*

$$G(x) = \omega_1 (f(\cdot, x), \delta)_{[x,b]}, \tag{14}$$

$\delta > 0$, $x \in [a,b]$.

Then G is continuous in $x \in [a,b]$.

Also it holds

Theorem 2.33. *([8]) Let $f : [a,b]^2 \to \mathbb{R}$ be jointly continous. Then*

$$H(x) = \omega_1 (f(\cdot, x), \delta)_{[a,x]}, \tag{15}$$

$x \in [a,b]$, is continuous in $x \in [a,b]$, $\delta > 0$.

So that for $f \in C^m([a,b])$, $m = \lceil \beta \rceil$, $\beta > 0$, $\beta \notin \mathbb{N}$, $x, x_0 \in [a,b]$, we have that $\omega_1 \left(D_{*x}^{\beta} f, h \right)_{[x,b]}$, $\omega_1 \left(D_{x-}^{\beta} f, h \right)_{[a,x]}$ are continuous functions in $x \in [a,b]$, $h > 0$ is fixed.

We make

Remark 2.34. ([8]) Let $f \in C^{n-1}([a,b])$, $f^{(n)} \in L_\infty([a,b])$, $n = \lceil \nu \rceil$, $\nu > 0$, $\nu \notin \mathbb{N}$. Then we have

$$\left| D_{*a}^{\nu} f(x) \right| \leq \frac{\left\| f^{(n)} \right\|_\infty}{\Gamma(n - \nu + 1)} (x-a)^{n-\nu}, \ \forall \, x \in [a,b]. \tag{16}$$

Thus we observe

$$\omega_1 \left(D_{*a}^{\nu} f, \delta \right) = \sup_{\substack{x,y \in [a,b] \\ |x-y| \leq \delta}} \left| D_{*a}^{\nu} f(x) - D_{*a}^{\nu} f(y) \right| \tag{17}$$

$$\leq \sup_{\substack{x,y \in [a,b] \\ |x-y| \leq \delta}} \left(\frac{\left\| f^{(n)} \right\|_\infty}{\Gamma(n-\nu+1)} (x-a)^{n-\nu} + \frac{\left\| f^{(n)} \right\|_\infty}{\Gamma(n-\nu+1)} (y-a)^{n-\nu} \right)$$

$$\leq \frac{2 \left\| f^{(n)} \right\|_\infty}{\Gamma(n-\nu+1)} (b-a)^{n-\nu}. \tag{18}$$

Consequently

$$\omega_1 \left(D_{*a}^{\nu} f, \delta \right) \leq \frac{2 \left\| f^{(n)} \right\|_\infty}{\Gamma(n-\nu+1)} (b-a)^{n-\nu}. \tag{19}$$

Similarly, let $f \in C^{m-1}([a,b])$, $f^{(m)} \in L_\infty([a,b])$, $m = \lceil \beta \rceil$, $\beta > 0$, $\beta \notin \mathbb{N}$, then

$$\omega_1 \left(D_{b-}^{\beta} f, \delta \right) \leq \frac{2 \left\| f^{(m)} \right\|_\infty}{\Gamma(m-\beta+1)} (b-a)^{m-\beta}. \tag{20}$$

So for $f \in C^{m-1}([a,b])$, $f^{(m)} \in L_\infty([a,b])$, $m = \lceil \beta \rceil$, $\beta > 0$, $\beta \notin \mathbb{N}$, we find

$$\sup_{x_0 \in [a,b]} \omega_1 \left(D_{*x_0}^{\beta} f, \delta \right)_{[x_0,b]} \leq \frac{2 \left\| f^{(m)} \right\|_\infty}{\Gamma(m-\beta+1)} (b-a)^{m-\beta}, \tag{21}$$

and

$$\sup_{x_0 \in [a,b]} \omega_1 \left(D_{x_0-}^{\beta} f, \delta \right)_{[a,x_0]} \leq \frac{2 \left\| f^{(m)} \right\|_\infty}{\Gamma(m-\beta+1)} (b-a)^{m-\beta}. \tag{22}$$

By Proposition 15.114, p. 388 of [7], we get here that $D_{*x_0}^{\beta} f \in C([x_0,b])$, and by [12] we obtain that $D_{x_0-}^{\beta} f \in C([a,x_0])$.

We need

Definition 2.35. ([11]) Let $f \in C_{\mathcal{F}}([a,b])$ (fuzzy continuous on $[a,b] \subset \mathbb{R}$), $\nu > 0$.

We define the Fuzzy Fractional left Riemann-Liouville operator as

$$J_a^{\nu} f(x) := \frac{1}{\Gamma(\nu)} \odot \int_a^x (x-t)^{\nu-1} \odot f(t)\, dt, \quad x \in [a,b], \tag{23}$$

$$J_a^0 f := f.$$

Also, we define the Fuzzy Fractional right Riemann-Liouville operator as

$$I_{b-}^{\nu} f(x) := \frac{1}{\Gamma(\nu)} \odot \int_x^b (t-x)^{\nu-1} \odot f(t)\, dt, \quad x \in [a,b], \tag{24}$$

$$I_{b-}^0 f := f.$$

We mention

Definition 2.36. ([11]) Let $f : [a,b] \to \mathbb{R}_{\mathcal{F}}$ is called fuzzy absolutely continuous iff $\forall\ \epsilon > 0$, $\exists\ \delta > 0$ for every finite, pairwise disjoint, family

$$(c_k, d_k)_{k=1}^n \subseteq (a,b) \quad \text{with} \quad \sum_{k=1}^n (d_k - c_k) < \delta$$

we get

$$\sum_{k=1}^n D(f(d_k), f(c_k)) < \epsilon. \tag{25}$$

We denote the related space of functions by $AC_{\mathcal{F}}([a,b])$.

If $f \in AC_{\mathcal{F}}([a,b])$, then $f \in C_{\mathcal{F}}([a,b])$.

It holds

Proposition 2.37. *([11]) $f \in AC_{\mathcal{F}}([a,b]) \Leftrightarrow f_{\pm}^{(r)} \in AEC([a,b])$, $\forall\ r \in [0,1]$ (absolutely equicontinuous).*

We need

Definition 2.38. ([11]) We define the Fuzzy Fractional left Caputo derivative, $x \in [a,b]$.

Let $f \in C_{\mathcal{F}}^n([a,b])$, $n = \lceil \nu \rceil$, $\nu > 0$ ($\lceil \cdot \rceil$ denotes the ceiling). We define

$$D_{*a}^{\nu\mathcal{F}} f(x) := \frac{1}{\Gamma(n-\nu)} \odot \int_a^x (x-t)^{n-\nu-1} \odot f^{(n)}(t)\, dt \tag{26}$$

$$= \left\{ \left(\frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} \left(f^{(n)}\right)_-^{(r)}(t)\, dt, \right.\right.$$

$$\left.\left. \frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} \left(f^{(n)}\right)_+^{(r)}(t)\, dt \right) | 0 \le r \le 1 \right\} =$$

$$= \left\{ \left(\frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} \left(f_-^{(r)}\right)^{(n)}(t)\, dt, \right.\right.$$

$$\left.\left. \frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} \left(f_+^{(r)}\right)^{(n)}(t)\, dt \right) | 0 \le r \le 1 \right\}. \tag{27}$$

So, we get

$$\left[D_{*a}^{\nu\mathcal{F}} f(x)\right]^r = \left[\left(\frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} \left(f_-^{(r)}\right)^{(n)} (t)\, dt,\right.\right.$$
$$\left.\left.\frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} \left(f_+^{(r)}\right)^{(n)} (t)\, dt\right)\right], \quad 0 \leq r \leq 1. \tag{28}$$

That is

$$\left(D_{*a}^{\nu\mathcal{F}} f(x)\right)_{\pm}^{(r)} = \frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} \left(f_{\pm}^{(r)}\right)^{(n)} (t)\, dt = \left(D_{*a}^{\nu} \left(f_{\pm}^{(r)}\right)\right)(x),$$

see [7], [24].

I.e. we get that

$$\left(D_{*a}^{\nu\mathcal{F}} f(x)\right)_{\pm}^{(r)} = \left(D_{*a}^{\nu} \left(f_{\pm}^{(r)}\right)\right)(x), \tag{29}$$

$\forall\, x \in [a,b]$, in short

$$\left(D_{*a}^{\nu\mathcal{F}} f\right)_{\pm}^{(r)} = D_{*a}^{\nu} \left(f_{\pm}^{(r)}\right), \quad \forall\, r \in [0,1]. \tag{30}$$

We need

Lemma 2.39. *([11]) $D_{*a}^{\nu\mathcal{F}} f(x)$ is fuzzy continuous in $x \in [a,b]$.*

We need

Definition 2.40. ([11]) We define the Fuzzy Fractional right Caputo derivative, $x \in [a,b]$.

Let $f \in C_{\mathcal{F}}^n([a,b])$, $n = \lceil \nu \rceil$, $\nu > 0$. We define

$$D_{b-}^{\nu\mathcal{F}} f(x) := \frac{(-1)^n}{\Gamma(n-\nu)} \odot \int_x^b (t-x)^{n-\nu-1} \odot f^{(n)}(t)\, dt$$
$$= \left\{\left(\frac{(-1)^n}{\Gamma(n-\nu)} \int_x^b (t-x)^{n-\nu-1} \left(f^{(n)}\right)_-^{(r)} (t)\, dt,\right.\right.$$
$$\left.\left.\frac{(-1)^n}{\Gamma(n-\nu)} \int_x^b (t-x)^{n-\nu-1} \left(f^{(n)}\right)_+^{(r)} (t)\, dt\right) | 0 \leq r \leq 1\right\} \tag{31}$$
$$= \left\{\left(\frac{(-1)^n}{\Gamma(n-\nu)} \int_x^b (t-x)^{n-\nu-1} \left(f_-^{(r)}\right)^{(n)} (t)\, dt,\right.\right.$$
$$\left.\left.\frac{(-1)^n}{\Gamma(n-\nu)} \int_x^b (t-x)^{n-\nu-1} \left(f_+^{(r)}\right)^{(n)} (t)\, dt\right) | 0 \leq r \leq 1\right\}.$$

We get

$$\left[D_{b-}^{\nu\mathcal{F}} f(x)\right]^r = \left[\left(\frac{(-1)^n}{\Gamma(n-\nu)} \int_x^b (t-x)^{n-\nu-1} \left(f_-^{(r)}\right)^{(n)} (t)\, dt,\right.\right.$$
$$\left.\left.\frac{(-1)^n}{\Gamma(n-\nu)} \int_x^b (t-x)^{n-\nu-1} \left(f_+^{(r)}\right)^{(n)} (t)\, dt\right)\right], \quad 0 \leq r \leq 1.$$

That is

$$\left(D_{b-}^{\nu\mathcal{F}} f(x)\right)_{\pm}^{(r)} = \frac{(-1)^n}{\Gamma(n-\nu)} \int_x^b (t-x)^{n-\nu-1} \left(f_{\pm}^{(r)}\right)^{(n)} (t)\, dt = \left(D_{b-}^{\nu} \left(f_{\pm}^{(r)}\right)\right)(x),$$

see [6].

I.e. we get that

$$\left(D_{b-}^{\nu\mathcal{F}}f\left(x\right)\right)_{\pm}^{(r)}=\left(D_{b-}^{\nu}\left(f_{\pm}^{(r)}\right)\right)\left(x\right), \tag{32}$$

$\forall\ x\in[a,b]$, in short

$$\left(D_{b-}^{\nu\mathcal{F}}f\right)_{\pm}^{(r)}=D_{b-}^{\nu}\left(f_{\pm}^{(r)}\right),\quad \forall\ r\in[0,1]. \tag{33}$$

Clearly,

$$D_{b-}^{\nu}\left(f_{-}^{(r)}\right)\leq D_{b-}^{\nu}\left(f_{+}^{(r)}\right),\quad \forall\ r\in[0,1].$$

We need

Lemma 2.41. *([11]) $D_{b-}^{\nu\mathcal{F}}f\left(x\right)$ is fuzzy continuous in $x\in[a,b]$.*

3 Real Neural Network Approximation

Here we follow [22].

Let $h:\mathbb{R}\to[-1,1]$ be a general sigmoid function, such that it is strictly increasing, $h\left(0\right)=0$, $h\left(-x\right)=-h\left(x\right)$, $h\left(+\infty\right)=1$, $h\left(-\infty\right)=-1$. Also h is strictly convex over $(-\infty,0]$ and strictly concave over $[0,+\infty)$, with $h^{(2)}\in C\left(\mathbb{R}\right)$.

We consider the activation function

$$\psi\left(x\right):=\frac{1}{4}\left(h\left(x+1\right)-h\left(x-1\right)\right),\quad x\in\mathbb{R}, \tag{34}$$

As in [21], p. 45, we get that $\psi\left(-x\right)=\psi\left(x\right)$, thus ψ is an even function. Since $x+1>x-1$, then $h\left(x+1\right)>h\left(x-1\right)$, and $\psi\left(x\right)>0$, all $x\in\mathbb{R}$.

We see that

$$\psi\left(0\right)=\frac{h\left(1\right)}{2}. \tag{35}$$

Let $x>1$, we have that

$$\psi'\left(x\right)=\frac{1}{4}\left(h'\left(x+1\right)-h'\left(x-1\right)\right)<0,$$

by h' being strictly decreasing over $[0,+\infty)$.

Let now $0<x<1$, then $1-x>0$ and $0<1-x<1+x$. It holds $h'\left(x-1\right)=h'\left(1-x\right)>h'\left(x+1\right)$, so that again $\psi'\left(x\right)<0$. Consequently ψ is stritly decreasing on $\left(0,+\infty\right)$.

Clearly, ψ is strictly increasing on $(-\infty,0)$, and $\psi'\left(0\right)=0$.

See that

$$\lim_{x\to+\infty}\psi\left(x\right)=\frac{1}{4}\left(h\left(+\infty\right)-h\left(+\infty\right)\right)=0, \tag{36}$$

and

$$\lim_{x\to-\infty}\psi\left(x\right)=\frac{1}{4}\left(h\left(-\infty\right)-h\left(-\infty\right)\right)=0. \tag{37}$$

That is the x-axis is the horizontal asymptote on ψ.

Conclusion, ψ is a bell symmetric function with maximum

$$\psi\left(0\right)=\frac{h\left(1\right)}{2}.$$

We need

Theorem 3.1. *([22]) We have that*

$$\sum_{i=-\infty}^{\infty} \psi(x-i) = 1, \quad \forall\, x \in \mathbb{R}. \tag{38}$$

Theorem 3.2. *([22]) It holds*

$$\int_{-\infty}^{\infty} \psi(x)\, dx = 1. \tag{39}$$

Thus $\psi(x)$ is a density function on $\mathbb{R}$.
We give

Theorem 3.3. *([22]) Let $0 < \alpha < 1$, and $n \in \mathbb{N}$ with $n^{1-\alpha} > 2$. It holds*

$$\sum_{\begin{cases} k = -\infty \\ : |nx-k| \geq n^{1-\alpha} \end{cases}}^{\infty} \psi(nx-k) < \frac{\left(1 - h\left(n^{1-\alpha}-2\right)\right)}{2}. \tag{40}$$

Notice that

$$\lim_{n \to +\infty} \frac{\left(1 - h\left(n^{1-\alpha}-2\right)\right)}{2} = 0.$$

Denote by $\lfloor \cdot \rfloor$ the integral part of the number and by $\lceil \cdot \rceil$ the ceiling of the number.
We further give

Theorem 3.4. *([22]) Let $x \in [a,b] \subset \mathbb{R}$ and $n \in \mathbb{N}$ so that $\lceil na \rceil \leq \lfloor nb \rfloor$. It holds*

$$\frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi(nx-k)} < \frac{1}{\psi(1)}, \quad \forall\, x \in [a,b]. \tag{41}$$

Remark 3.5. ([22]) i) We have that

$$\lim_{n \to \infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi(nx-k) \neq 1, \tag{42}$$

for at least some $x \in [a,b]$.
ii) For large enough $n \in \mathbb{N}$ we always obtain $\lceil na \rceil \leq \lfloor nb \rfloor$. Also $a \leq \frac{k}{n} \leq b$, iff $\lceil na \rceil \leq k \leq \lfloor nb \rfloor$.
In general, by Theorem 3.1, it holds

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi(nx-k) \leq 1. \tag{43}$$

We give

Definition 3.6. ([22]) Let $f \in C([a,b])$ and $n \in \mathbb{N} : \lceil na \rceil \leq \lfloor nb \rfloor$. We introduce and define the linear neural network operator

$$A_n(f,x) := \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) \psi(nx-k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi(nx-k)}, \quad x \in [a,b]. \tag{44}$$

Clearly here $A_n(f,x) \in C([a,b])$. We present results for the pointwise and uniform convergence of $A_n(f,x)$ to $f(x)$ with rates.

We first give

Theorem 3.7. *([22]) Let $f \in C([a,b])$, $0<\alpha<1$, $n \in \mathbb{N} : n^{1-\alpha} > 2$, $x \in [a,b]$. Then*

i)

$$|A_n(f,x) - f(x)| \leq \frac{1}{\psi(1)} \left[\omega_1\left(f, \frac{1}{n^\alpha}\right) + \left(1 - h\left(n^{1-\alpha} - 2\right)\right) \|f\|_\infty \right] =: \rho, \tag{45}$$

and

ii)

$$\|A_n(f) - f\|_\infty \leq \rho. \tag{46}$$

We notice $\lim_{n\to\infty} A_n(f) = f$, pointwise and uniformly.

The speed of convergence is $\max\left(\frac{1}{n^\alpha}, \left(1 - h\left(n^{1-\alpha} - 2\right)\right)\right)$.

In the next we discuss high order neural network approximation by using the smoothness of f.

Theorem 3.8. *([22]) Let $f \in C^N([a,b])$, $n, N \in \mathbb{N}$, $0<\alpha<1$, $x \in [a,b]$ and $n^{1-\alpha} > 2$. Then*

i)

$$|A_n(f,x) - f(x)| \leq \frac{1}{\psi(1)} \left\{ \sum_{j=1}^{N} \frac{\left\| f^{(j)}(x) \right\|}{j!} \left[\frac{1}{n^{\alpha j}} + \frac{\left(1 - h\left(n^{1-\alpha} - 2\right)\right)}{2} (b-a)^j \right] + \right. \tag{47}$$

$$\left. \left[\omega_1\left(f^{(N)}, \frac{1}{n^\alpha}\right) \frac{1}{n^{\alpha N} N!} + \frac{\left(1 - h\left(n^{1-\alpha} - 2\right)\right) \left\| f^{(N)} \right\|_\infty (b-a)^N}{N!} \right] \right\}$$

ii) assume further $f^{(j)}(x_0) = 0$, $j = 1, ..., N$, for some $x_0 \in [a,b]$, it holds

$$|A_n(f,x_0) - f(x_0)| \leq \frac{1}{\psi(1)}$$

$$\left\{ \omega_1\left(f^{(N)}, \frac{1}{n^\alpha}\right) \frac{1}{n^{\alpha N} N!} + \frac{\left(1 - h\left(n^{1-\alpha} - 2\right)\right) \left\| f^{(N)} \right\|_\infty (b-a)^N}{N!} \right\}, \tag{48}$$

and

iii)

$$\|A_n(f) - f\|_\infty \leq \frac{1}{\psi(1)} \left\{ \sum_{j=1}^{N} \frac{\left\| f^{(j)} \right\|_\infty}{j!} \left[\frac{1}{n^{\alpha j}} + \frac{\left(1 - h\left(n^{1-\alpha} - 2\right)\right)}{2} (b-a)^j \right] + \right.$$

$$\left. \left[\omega_1\left(f^{(N)}, \frac{1}{n^\alpha}\right) \frac{1}{n^{\alpha N} N!} + \frac{\left(1 - h\left(n^{1-\alpha} - 2\right)\right) \left\| f^{(N)} \right\|_\infty (b-a)^N}{N!} \right] \right\}. \tag{49}$$

Again we obtain $\lim_{n\to\infty} A_n(f) = f$, pointwise and uniformly.

We present the following fractional approximation result by neural networks.

Theorem 3.9. *([22]) Let $\alpha > 0$, $N = \lceil \alpha \rceil$, $\alpha \notin \mathbb{N}$, $f \in C^N([a,b])$, $0<\beta<1$, $x \in [a,b]$, $n \in \mathbb{N} : n^{1-\beta} > 2$. Then* *i)*

$$\left|A_n(f,x)-\sum_{j=1}^{N-1}\frac{f^{(j)}(x)}{j!}A_n\left((\cdot-x)^j\right)(x)-f(x)\right|\leq$$

$$\frac{(\psi(1))^{-1}}{\Gamma(\alpha+1)}\left\{\frac{\left(\omega_1\left(D_{x-}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[a,x]}+\omega_1\left(D_{*x}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[x,b]}\right)}{n^{\alpha\beta}}+\right.$$

$$\left.\left(\frac{1-h\left(n^{1-\beta}-2\right)}{2}\right)\left(\left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]}(x-a)^{\alpha}+\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}(b-x)^{\alpha}\right)\right\}, \tag{50}$$

ii) if $f^{(j)}(x)=0$, for $j=1,...,N-1$, we have

$$|A_n(f,x)-f(x)|\leq\frac{(\psi(1))^{-1}}{\Gamma(\alpha+1)}$$

$$\left\{\frac{\left(\omega_1\left(D_{x-}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[a,x]}+\omega_1\left(D_{*x}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[x,b]}\right)}{n^{\alpha\beta}}+\right.$$

$$\left.\left(\frac{1-h\left(n^{1-\beta}-2\right)}{2}\right)\left(\left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]}(x-a)^{\alpha}+\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}(b-x)^{\alpha}\right)\right\}, \tag{51}$$

iii)

$$|A_n(f,x)-f(x)|\leq(\psi(1))^{-1}$$

$$\left\{\sum_{j=1}^{N-1}\frac{\left\|f^{(j)}(x)\right\|}{j!}\left\{\frac{1}{n^{\beta j}}+(b-a)^j\left(\frac{1-h\left(n^{1-\beta}-2\right)}{2}\right)\right\}+\right.$$

$$\frac{1}{\Gamma(\alpha+1)}\left\{\frac{\left(\omega_1\left(D_{x-}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[a,x]}+\omega_1\left(D_{*x}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[x,b]}\right)}{n^{\alpha\beta}}+\right.$$

$$\left.\left.\left(\frac{1-h\left(n^{1-\beta}-2\right)}{2}\right)\left(\left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]}(x-a)^{\alpha}+\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}(b-x)^{\alpha}\right)\right\}\right\}, \tag{52}$$

$\forall\ x\in[a,b]$,

and

iv)

$$\|A_nf-f\|_{\infty}\leq(\psi(1))^{-1}$$

$$\left\{\sum_{j=1}^{N-1}\frac{\left\|f^{(j)}\right\|_{\infty}}{j!}\left\{\frac{1}{n^{\beta j}}+(b-a)^j\left(\frac{1-h\left(n^{1-\beta}-2\right)}{2}\right)\right\}+\right.$$

$$\frac{1}{\Gamma(\alpha+1)}\left\{\frac{\left(\sup_{x\in[a,b]}\omega_1\left(D_{x-}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[a,x]}+\sup_{x\in[a,b]}\omega_1\left(D_{*x}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[x,b]}\right)}{n^{\alpha\beta}}+\right.$$

$$\left(\frac{1-h\left(n^{1-\beta}-2\right)}{2}\right)(b-a)^{\alpha}\left(\sup_{x\in[a,b]}\left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]}+\sup_{x\in[a,b]}\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}\right)\Bigg\}\Bigg\}. \tag{53}$$

Above, when $N=1$ the sum $\sum_{j=1}^{N-1}\cdot=0$.

As we see here we obtain fractionally type pointwise and uniform convergence with rates of $A_n \to I$ the unit operator, as $n \to \infty$.

4 Main Results: Approximation by General Fuzzy Neural Network Operators

Let $f \in C_{\mathcal{F}}([a,b])$ (fuzzy continuous functions on $[a,b] \subset \mathbb{R}$), $n \in \mathbb{N}$. We define the following Fuzzy Quasi-Interpolation Neural Network operator

$$A_n^{\mathcal{F}}(f,x)=\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor *} f\left(\frac{k}{n}\right)\odot\frac{\psi(nx-k)}{\sum\limits_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\psi(nx-k)}, \tag{54}$$

$\forall\, x \in [a,b]$, see also (44).

The fuzzy sum in (54) is finite.

Let $r \in [0,1]$, we observe that

$$\left[A_n^{\mathcal{F}}(f,x)\right]^r=\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\left[f\left(\frac{k}{n}\right)\right]^r\left(\frac{\psi(nx-k)}{\sum\limits_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\psi(nx-k)}\right)=$$

$$\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\left[f_-^{(r)}\left(\frac{k}{n}\right),f_+^{(r)}\left(\frac{k}{n}\right)\right]\left(\frac{\psi(nx-k)}{\sum\limits_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\psi(nx-k)}\right)=$$

$$\left[\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}f_-^{(r)}\left(\frac{k}{n}\right)\left(\frac{\psi(nx-k)}{\sum\limits_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\psi(nx-k)}\right),\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}f_+^{(r)}\left(\frac{k}{n}\right)\left(\frac{\psi(nx-k)}{\sum\limits_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\psi(nx-k)}\right)\right] \tag{55}$$

$$=\left[A_n\left(f_-^{(r)},x\right),\ A_n\left(f_+^{(r)},x\right)\right].$$

We have proved that

$$\left(A_n^{\mathcal{F}}(f,x)\right)_{\pm}^{(r)}=\ A_n\left(f_{\pm}^{(r)},x\right), \tag{56}$$

respectively, $\forall\, r \in [0,1]$, $\forall\, x \in [a,b]$.

Therefore we get

$$D\left(A_n^{\mathcal{F}}(f,x),f(x)\right)=$$

$$\sup_{r\in[0,1]}\max\left\{\left|A_n\left(f_-^{(r)},x\right)-f_-^{(r)}(x)\right|,\left|A_n\left(f_+^{(r)},x\right)-f_+^{(r)}(x)\right|\right\}, \tag{57}$$

$\forall\ x \in [a,b]$.

We present our first fuzzy neural network approximation result.

Theorem 4.1. *Let $f \in C_{\mathcal{F}}([a,b])$, $0<\alpha<1$, $x \in [a,b]$, $n \in \mathbb{N}$ with $n^{1-\alpha} > 2$. Then*

1)

$$D\left(A_n^{\mathcal{F}}(f,x), f(x)\right) \leq \frac{1}{\psi(1)}\left[\omega_1^{(\mathcal{F})}\left(f, \frac{1}{n^\alpha}\right) + \left(1 - h\left(n^{1-\alpha}-2\right)\right) D^*\left(f, \widetilde{o}\right)\right] =: T_n, \tag{58}$$

and

2)

$$D^*\left(A_n^{\mathcal{F}}(f), f\right) \leq T_n. \tag{59}$$

We notice that $\lim_{n\to\infty}\left(A_n^{\mathcal{F}}(f)\right)(x) \overset{D}{\to} f(x)$, $\lim_{n\to\infty} A_n^{\mathcal{F}}(f) \overset{D^*}{\to} f$, pointwise and uniformly.

Proof. We have that $f_{\pm}^{(r)} \in C([a,b])$, $\forall\ r \in [0,1]$. Hence by (45), we obtain

$$\left|A_n\left(f_{\pm}^{(r)}, x\right) - f_{\pm}^{(r)}(x)\right| \leq \frac{1}{\psi(1)}\left[\omega_1\left(f_{\pm}^{(r)}, \frac{1}{n^\alpha}\right) + \left(1 - h\left(n^{1-\alpha}-2\right)\right)\left\|f_{\pm}^{(r)}\right\|_\infty\right] \tag{60}$$

(by Proposition 2.7 and $\left\|f_{\pm}^{(r)}\right\|_\infty \leq D^*(f,\widetilde{o})$)

$$\leq \frac{1}{\psi(1)}\left[\omega_1^{(\mathcal{F})}\left(f, \frac{1}{n^\alpha}\right) + \left(1 - h\left(n^{1-\alpha}-2\right)\right) D^*\left(f, \widetilde{o}\right)\right]. \tag{61}$$

Taking into account (57) the theorem is proved. $\square$

We also give

Theorem 4.2. *Let $f \in C_{\mathcal{F}}^N([a,b])$, $N \in \mathbb{N}$, $0<\alpha<1$, $x \in [a,b]$, $n \in \mathbb{N}$ with $n^{1-\alpha} > 2$. Then*

1)

$$D\left(A_n^{\mathcal{F}}(f,x), f(x)\right) \leq \frac{1}{\psi(1)}$$

$$\left\{\sum_{j_*=1}^{N} \frac{D\left(f^{(j_*)}(x), \widetilde{o}\right)}{j_*!}\left[\frac{1}{n^{\alpha j_*}} + \left(\frac{1-h\left(n^{1-\alpha}-2\right)}{2}\right)(b-a)^{j_*}\right] + \right.$$

$$\left.\left[\omega_1^{(\mathcal{F})}\left(f^{(N)}, \frac{1}{n^\alpha}\right)\frac{1}{n^{\alpha N} N!} + \left(1 - h\left(n^{1-\alpha}-2\right)\right) D^*\left(f^{(N)}, \widetilde{o}\right)\frac{(b-a)^N}{N!}\right]\right\}, \tag{62}$$

2) assume further that $f^{(j_*)}(x_0) = \widetilde{o}$, $j_* = 1, ..., N$, for some $x_0 \in [a,b]$, it holds

$$D\left(A_n^{\mathcal{F}}(f,x_0), f(x_0)\right) \leq$$

$$\frac{1}{\psi(1)}\left[\omega_1^{(\mathcal{F})}\left(f^{(N)}, \frac{1}{n^\alpha}\right)\frac{1}{n^{\alpha N} N!} + \left(1 - h\left(n^{1-\alpha}-2\right)\right) D^*\left(f^{(N)}, \widetilde{o}\right)\frac{(b-a)^N}{N!}\right], \tag{63}$$

notice here the extremely high rate of convergence $n^{-(N+1)\alpha}$,

3)

$$D^*\left(A_n^{\mathcal{F}}(f), f\right) \leq \frac{1}{\psi(1)}$$

$$\left\{\sum_{j_*=1}^{N}\frac{D^*\left(f^{(j_*)},\widetilde{o}\right)}{j_*!}\left[\frac{1}{n^{\alpha j_*}}+\left(\frac{1-h\left(n^{1-\alpha}-2\right)}{2}\right)(b-a)^{j_*}\right]+\right.$$

$$\left.\left[\omega_1^{(\mathcal{F})}\left(f^{(N)},\frac{1}{n^{\alpha}}\right)\frac{1}{n^{\alpha N}N!}+\left(1-h\left(n^{1-\alpha}-2\right)\right)D^*\left(f^{(N)},\widetilde{o}\right)\frac{(b-a)^N}{N!}\right]\right\}. \tag{64}$$

Proof. Since $f\in C_{\mathcal{F}}^{N}([a,b])$, $N\geq 1$, we have that $f_{\pm}^{(r)}\in C^N([a,b])$, $\forall\, r\in[0,1]$. Using (47), we get

$$\left|A_n\left(f_{\pm}^{(r)},x\right)-f_{\pm}^{(r)}(x)\right|\leq\frac{1}{\psi(1)} \tag{65}$$

$$\left\{\sum_{j_*=1}^{N}\frac{\left|\left(f_{\pm}^{(r)}\right)^{(j_*)}(x)\right|}{j_*!}\left[\frac{1}{n^{\alpha j_*}}+\left(\frac{1-h\left(n^{1-\alpha}-2\right)}{2}\right)(b-a)^{j_*}\right]+\right.$$

$$\left.\left[\omega_1\left(\left(f_{\pm}^{(r)}\right)^{(N)},\frac{1}{n^{\alpha}}\right)\frac{1}{n^{\alpha N}N!}+\left(1-h\left(n^{1-\alpha}-2\right)\right)\left\|\left(f_{\pm}^{(r)}\right)^{(N)}\right\|_{\infty}\frac{(b-a)^N}{N!}\right]\right\} \tag{66}$$

(by Remark 2.12)

$$=\frac{1}{\psi(1)}\left\{\sum_{j_*=1}^{N}\frac{\left|\left(f^{(j_*)}\right)_{\pm}^{(r)}(x)\right|}{j_*!}\left[\frac{1}{n^{\alpha j_*}}+\left(\frac{1-h\left(n^{1-\alpha}-2\right)}{2}\right)(b-a)^{j_*}\right]+\right.$$

$$\left.\left[\omega_1\left(\left(f^{(N)}\right)_{\pm}^{(r)},\frac{1}{n^{\alpha}}\right)\frac{1}{n^{\alpha N}N!}+\left(1-h\left(n^{1-\alpha}-2\right)\right)\left\|\left(f^{(N)}\right)_{\pm}^{(r)}\right\|_{\infty}\frac{(b-a)^N}{N!}\right]\right\}\leq$$

$$\frac{1}{\psi(1)}\left\{\sum_{j_*=1}^{N}\frac{D\left(f^{(j_*)}(x),\widetilde{o}\right)}{j_*!}\left[\frac{1}{n^{\alpha j_*}}+\left(\frac{1-h\left(n^{1-\alpha}-2\right)}{2}\right)(b-a)^{j_*}\right]+\right.$$

$$\left.\left[\omega_1^{(\mathcal{F})}\left(f^{(N)},\frac{1}{n^{\alpha}}\right)\frac{1}{n^{\alpha N}N!}+\left(1-h\left(n^{1-\alpha}-2\right)\right)D^*\left(f^{(N)},\widetilde{o}\right)\frac{(b-a)^N}{N!}\right]\right\}, \tag{67}$$

by Proposition 2.7, $\left\|\left(f^{(N)}\right)_{\pm}^{(r)}\right\|_{\infty}\leq D^*\left(f^{(N)},\widetilde{o}\right)$ and apply (57).

The theorem is proved. $\square$

Next we present

Theorem 4.3. *Let $\alpha>0$, $N=\lceil\alpha\rceil$, $\alpha\notin\mathbb{N}$, $f\in C_{\mathcal{F}}^{N}([a,b])$, $0<\beta<1$, $x\in[a,b]$, $n\in\mathbb{N}$, $n^{1-\beta}>2$. Then*
i)

$$D\left(A_n^{\mathcal{F}}(f,x),f(x)\right)\leq\frac{1}{\psi(1)}$$

$$\left\{\sum_{j_*=1}^{N-1}\frac{D\left(f^{(j_*)}(x),\widetilde{o}\right)}{j_*!}\left[\frac{1}{n^{\beta j_*}}+\left(\frac{1-h\left(n^{1-\beta}-2\right)}{2}\right)(b-a)^{j_*}\right]+\right.$$

$$\frac{1}{\Gamma(\alpha+1)}\left\{\frac{\left[\omega_1^{(\mathcal{F})}\left(\left(D_{x-}^{\alpha\mathcal{F}}f\right),\frac{1}{n^{\beta}}\right)_{[a,x]}+\omega_1^{(\mathcal{F})}\left(\left(D_{*x}^{\alpha\mathcal{F}}f\right),\frac{1}{n^{\beta}}\right)_{[x,b]}\right]}{n^{\alpha\beta}}+\right. \tag{68}$$

$$\left(\frac{1-h\left(n^{1-\beta}-2\right)}{2}\right)\left[D^{*}\left(\left(D_{x-}^{\alpha\mathcal{F}}f\right),\widetilde{o}\right)_{[a,x]}(x-a)^{\alpha}+D^{*}\left(\left(D_{*x}^{\alpha\mathcal{F}}f\right),\widetilde{o}\right)_{[x,b]}(b-x)^{\alpha}\right]\Bigg\}\Bigg\},$$

ii) if $f^{(j)}(x_0)=0,\ j=1,...,N-1,$ *for some* $x_0\in[a,b]$, *we have*

$$D\left(A_n^{\mathcal{F}}(f,x_0),f(x_0)\right)\leq$$

$$\frac{(\psi(1))^{-1}}{\Gamma(\alpha+1)}\left\{\frac{\left[\omega_1^{(\mathcal{F})}\left(\left(D_{x_0-}^{\alpha\mathcal{F}}f\right),\frac{1}{n^{\beta}}\right)_{[a,x_0]}+\omega_1^{(\mathcal{F})}\left(\left(D_{*x_0}^{\alpha\mathcal{F}}f\right),\frac{1}{n^{\beta}}\right)_{[x_0,b]}\right]}{n^{\alpha\beta}}+\right. \tag{69}$$

$$\left.\left(\frac{1-h\left(n^{1-\beta}-2\right)}{2}\right)\left[D^{*}\left(\left(D_{x_0-}^{\alpha\mathcal{F}}f\right),\widetilde{o}\right)_{[a,x_0]}(x_0-a)^{\alpha}+D^{*}\left(\left(D_{*x_0}^{\alpha\mathcal{F}}f\right),\widetilde{o}\right)_{[x_0,b]}(b-x_0)^{\alpha}\right]\right\},$$

when $\alpha>1$ *notice here the extremely high rate of convergence at* $n^{-(\alpha+1)\beta}$,

and

iii)

$$D^{*}\left(A_n^{\mathcal{F}}(f),f\right)\leq$$

$$\frac{1}{\psi(1)}\left\{\sum_{j_*=1}^{N-1}\frac{D^{*}\left(f^{(j_*)},\widetilde{o}\right)}{j_*!}\left[\frac{1}{n^{\beta j_*}}+\left(\frac{1-h\left(n^{1-\beta}-2\right)}{2}\right)(b-a)^{j_*}\right]+\right.$$

$$\frac{1}{\Gamma(\alpha+1)}\left\{\frac{\left[\sup_{x\in[a,b]}\omega_1^{(\mathcal{F})}\left(\left(D_{x-}^{\alpha\mathcal{F}}f\right),\frac{1}{n^{\beta}}\right)_{[a,x]}+\sup_{x\in[a,b]}\omega_1^{(\mathcal{F})}\left(\left(D_{*x}^{\alpha\mathcal{F}}f\right),\frac{1}{n^{\beta}}\right)_{[x,b]}\right]}{n^{\alpha\beta}}+\right. \tag{70}$$

$$\left(\frac{1-h\left(n^{1-\beta}-2\right)}{2}\right)(b-a)^{\alpha}\left[\sup_{x\in[a,b]}D^{*}\left(\left(D_{x-}^{\alpha\mathcal{F}}f\right),\widetilde{o}\right)_{[a,x]}+\sup_{x\in[a,b]}D^{*}\left(\left(D_{*x}^{\alpha\mathcal{F}}f\right),\widetilde{o}\right)_{[x,b]}\right]\Bigg\}\Bigg\},$$

above, when $N=1$ *the sum* $\sum_{j=1}^{N-1}\cdot=0$.

As we see here we obtain fractionally the fuzzy pointwise and uniform convergence with rates of $A_n^{\mathcal{F}}\to I$ *the unit operator, as* $n\to\infty$.

Proof. Here $f_{\pm}^{(r)}\in C^{N}([a,b])$, $\forall\ r\in[0,1]$, and $D_{x-}^{\alpha\mathcal{F}}f$, $D_{*x}^{\alpha\mathcal{F}}f$ are fuzzy continuous over $[a,b]$, $\forall\ x\in[a,b]$, so that $\left(D_{x-}^{\alpha\mathcal{F}}f\right)_{\pm}^{(r)}$, $\left(D_{*x}^{\alpha\mathcal{F}}f\right)_{\pm}^{(r)}\in C([a,b])$, $\forall\ r\in[0,1]$, $\forall\ x\in[a,b]$.

We observe by (52), $\forall\ x\in[a,b]$, that (respectively in $\pm$)

$$\left|A_n\left(f_{\pm}^{(r)},x\right)-f_{\pm}^{(r)}(x)\right|\leq\frac{1}{\psi(1)}$$

$$\left\{\sum_{j_*=1}^{N-1}\frac{\left|\left(f_{\pm}^{(r)}\right)^{(j_*)}(x)\right|}{j_*!}\left\{\frac{1}{n^{\beta j_*}}+\left(\frac{1-h\left(n^{1-\beta}-2\right)}{2}\right)(b-a)^{j_*}\right\}+\right. \tag{71}$$

$$\frac{1}{\Gamma(\alpha+1)}\left\{\frac{\left(\omega_1\left(D_{x-}^{\alpha}\left(f_{\pm}^{(r)}\right),\frac{1}{n^{\beta}}\right)_{[a,x]}+\omega_1\left(D_{*x}^{\alpha}\left(f_{\pm}^{(r)}\right),\frac{1}{n^{\beta}}\right)_{[x,b]}\right)}{n^{\alpha\beta}}+\right.$$

$$\left(\frac{1-h\left(n^{1-\beta}-2\right)}{2}\right)\left(\left\|D_{x-}^{\alpha}\left(f_{\pm}^{(r)}\right)\right\|_{\infty,[a,x]}(x-a)^{\alpha}+\left\|D_{*x}^{\alpha}\left(f_{\pm}^{(r)}\right)\right\|_{\infty,[x,b]}(b-x)^{\alpha}\right)\Bigg\}\Bigg\}=$$

(by Remark 2.12, (30), (33))

$$\frac{1}{\psi(1)}\left\{\sum_{j_*=1}^{N-1}\frac{\left|\left(f^{(j_*)}(x)\right)_{\pm}^{(r)}\right|}{j_*!}\left\{\frac{1}{n^{\beta j_*}}+\left(\frac{1-h\left(n^{1-\beta}-2\right)}{2}\right)(b-a)^{j_*}\right\}+\right.$$

$$\frac{1}{\Gamma(\alpha+1)}\left\{\frac{\left(\omega_1\left(\left(D_{x-}^{\alpha\mathcal{F}}f\right)_{\pm}^{(r)},\frac{1}{n^{\beta}}\right)_{[a,x]}+\omega_1\left(\left(D_{*x}^{\alpha\mathcal{F}}f\right)_{\pm}^{(r)},\frac{1}{n^{\beta}}\right)_{[x,b]}\right)}{n^{\alpha\beta}}+\right. \tag{72}$$

$$\left(\frac{1-h\left(n^{1-\beta}-2\right)}{2}\right)\left(\left\|\left(D_{x-}^{\alpha\mathcal{F}}f\right)_{\pm}^{(r)}\right\|_{\infty,[a,x]}(x-a)^{\alpha}+\left\|\left(D_{*x}^{\alpha\mathcal{F}}f\right)_{\pm}^{(r)}\right\|_{\infty,[x,b]}(b-x)^{\alpha}\right)\Bigg\}\Bigg\}\leq$$

$$\frac{1}{\psi(1)}\left\{\sum_{j_*=1}^{N-1}\frac{D\left(f^{(j_*)}(x),\widetilde{o}\right)}{j_*!}\left\{\frac{1}{n^{\beta j_*}}+\left(\frac{1-h\left(n^{1-\beta}-2\right)}{2}\right)(b-a)^{j_*}\right\}+\right.$$

$$\frac{1}{\Gamma(\alpha+1)}\left\{\frac{\left[\omega_1^{(\mathcal{F})}\left(\left(D_{x-}^{\alpha\mathcal{F}}f\right),\frac{1}{n^{\beta}}\right)_{[a,x]}+\omega_1^{(\mathcal{F})}\left(\left(D_{*x}^{\alpha\mathcal{F}}f\right),\frac{1}{n^{\beta}}\right)_{[x,b]}\right]}{n^{\alpha\beta}}+\right. \tag{73}$$

$$\left(\frac{1-h\left(n^{1-\beta}-2\right)}{2}\right)\left[D^*\left(\left(D_{x-}^{\alpha\mathcal{F}}f\right),\widetilde{o}\right)_{[a,x]}(x-a)^{\alpha}+D^*\left(\left(D_{*x}^{\alpha\mathcal{F}}f\right),\widetilde{o}\right)_{[x,b]}(b-x)^{\alpha}\right]\Bigg\}\Bigg\},$$

along with (57) proving all inequalities of theorem.

Here we notice that

$$\left(D_{x-}^{\alpha\mathcal{F}}f\right)_{\pm}^{(r)}(t)=\left(D_{x-}^{\alpha}\left(f_{\pm}^{(r)}\right)\right)(t)$$

$$=\frac{(-1)^N}{\Gamma(N-\alpha)}\int_t^x(s-t)^{N-\alpha-1}\left(f_{\pm}^{(r)}\right)^{(N)}(s)\,ds,$$

where $a\leq t\leq x$.

Hence

$$\left|\left(D_{x-}^{\alpha\mathcal{F}}f\right)_{\pm}^{(r)}(t)\right|\leq\frac{1}{\Gamma(N-\alpha)}\int_t^x(s-t)^{N-\alpha-1}\left|\left(f_{\pm}^{(r)}\right)^{(N)}(s)\right|ds$$

$$\leq\frac{\left\|\left(f^{(N)}\right)_{\pm}^{(r)}\right\|_{\infty}}{\Gamma(N-\alpha+1)}(b-a)^{N-\alpha}\leq\frac{D^*\left(f^{(N)},\widetilde{o}\right)}{\Gamma(N-\alpha+1)}(b-a)^{N-\alpha}.$$

So we have

$$\left|\left(D_{x-}^{\alpha\mathcal{F}}f\right)_{\pm}^{(r)}(t)\right|\leq\frac{D^*\left(f^{(N)},\widetilde{o}\right)}{\Gamma(N-\alpha+1)}(b-a)^{N-\alpha},$$

all $a\leq t\leq x$.

And it holds

$$\left\|\left(D_{x-}^{\alpha\mathcal{F}}f\right)_{\pm}^{(r)}\right\|_{\infty,[a,x]}\leq\frac{D^*\left(f^{(N)},\widetilde{o}\right)}{\Gamma(N-\alpha+1)}(b-a)^{N-\alpha}, \tag{74}$$

that is

$$D^*\left(\left(D_{x-}^{\alpha\mathcal{F}}f\right),\widetilde{o}\right)_{[a,x]}\leq\frac{D^*\left(f^{(N)},\widetilde{o}\right)}{\Gamma\left(N-\alpha+1\right)}\left(b-a\right)^{N-\alpha},$$

and

$$\sup_{x\in[a,b]}D^*\left(\left(D_{x-}^{\alpha\mathcal{F}}f\right),\widetilde{o}\right)_{[a,x]}\leq\frac{D^*\left(f^{(N)},\widetilde{o}\right)}{\Gamma\left(N-\alpha+1\right)}\left(b-a\right)^{N-\alpha}<\infty. \tag{75}$$

Similarly we have

$$\left(D_{*x}^{\alpha\mathcal{F}}f\right)_{\pm}^{(r)}(t)=\left(D_{*x}^{\alpha}\left(f_{\pm}^{(r)}\right)\right)(t)$$

$$=\frac{1}{\Gamma\left(N-\alpha\right)}\int_x^t(t-s)^{N-\alpha-1}\left(f_{\pm}^{(r)}\right)^{(N)}(s)\,ds,$$

where $x\leq t\leq b$.

Hence

$$\left|\left(D_{*x}^{\alpha\mathcal{F}}f\right)_{\pm}^{(r)}(t)\right|\leq\frac{1}{\Gamma\left(N-\alpha\right)}\int_x^t(t-s)^{N-\alpha-1}\left|\left(f^{(N)}\right)_{\pm}^{(r)}(s)\right|ds\leq$$

$$\frac{\left\|\left(f^{(N)}\right)_{\pm}^{(r)}\right\|_{\infty}}{\Gamma\left(N-\alpha+1\right)}\left(b-a\right)^{N-\alpha}\leq\frac{D^*\left(f^{(N)},\widetilde{o}\right)}{\Gamma\left(N-\alpha+1\right)}\left(b-a\right)^{N-\alpha},$$

$x\leq t\leq b$.

So we have

$$\left\|\left(D_{*x}^{\alpha\mathcal{F}}f\right)_{\pm}^{(r)}\right\|_{\infty,[x,b]}\leq\frac{D^*\left(f^{(N)},\widetilde{o}\right)}{\Gamma\left(N-\alpha+1\right)}\left(b-a\right)^{N-\alpha}, \tag{76}$$

that is

$$D^*\left(\left(D_{*x}^{\alpha\mathcal{F}}f\right),\widetilde{o}\right)_{[x,b]}\leq\frac{D^*\left(f^{(N)},\widetilde{o}\right)}{\Gamma\left(N-\alpha+1\right)}\left(b-a\right)^{N-\alpha},$$

and

$$\sup_{x\in[a,b]}D^*\left(\left(D_{*x}^{\alpha\mathcal{F}}f\right),\widetilde{o}\right)_{[x,b]}\leq\frac{D^*\left(f^{(N)},\widetilde{o}\right)}{\Gamma\left(N-\alpha+1\right)}\left(b-a\right)^{N-\alpha}<+\infty. \tag{77}$$

Furthermore we notice

$$\omega_1^{(\mathcal{F})}\left(\left(D_{x-}^{\alpha\mathcal{F}}f\right),\frac{1}{n^{\beta}}\right)_{[a,x]}=\sup_{\substack{s,t\in[a,x]\\|s-t|\leq\frac{1}{n^{\beta}}}}D\left(\left(D_{x-}^{\alpha\mathcal{F}}f\right)(s),\left(D_{x-}^{\alpha\mathcal{F}}f\right)(t)\right)\leq$$

$$\sup_{\substack{s,t\in[a,x]\\|s-t|\leq\frac{1}{n^{\beta}}}}\left\{D\left(\left(D_{x-}^{\alpha\mathcal{F}}f\right)(s),\widetilde{o}\right)+D\left(\left(D_{x-}^{\alpha\mathcal{F}}f\right)(t),\widetilde{o}\right)\right\}\leq 2D^*\left(\left(D_{x-}^{\alpha\mathcal{F}}f\right),\widetilde{o}\right)_{[a,x]}$$

$$\leq\frac{2D^*\left(f^{(N)},\widetilde{o}\right)}{\Gamma\left(N-\alpha+1\right)}\left(b-a\right)^{N-\alpha}.$$

Therefore it holds

$$\sup_{x\in[a,b]}\omega_1^{(\mathcal{F})}\left(\left(D_{x-}^{\alpha\mathcal{F}}f\right),\frac{1}{n^{\beta}}\right)_{[a,x]}\leq\frac{2D^*\left(f^{(N)},\widetilde{o}\right)}{\Gamma\left(N-\alpha+1\right)}\left(b-a\right)^{N-\alpha}<+\infty. \tag{78}$$

Similarly we observe

$$\omega_1^{(\mathcal{F})}\left(\left(D_{*x}^{\alpha\mathcal{F}}f\right),\frac{1}{n^\beta}\right)_{[x,b]}=\sup_{\substack{s,t\in[x,b]\\|s-t|\le\frac{1}{n^\beta}}}D\left(\left(D_{*x}^{\alpha\mathcal{F}}f\right)(s),\left(D_{*x}^{\alpha\mathcal{F}}f\right)(t)\right)\le$$

$$2D^*\left(\left(D_{*x}^{\alpha\mathcal{F}}f\right),\widetilde{o}\right)_{[x,b]}\le\frac{2D^*\left(f^{(N)},\widetilde{o}\right)}{\Gamma(N-\alpha+1)}(b-a)^{N-\alpha}.$$

Consequently it holds

$$\sup_{x\in[a,b]}\omega_1^{(\mathcal{F})}\left(\left(D_{*x}^{\alpha\mathcal{F}}f\right),\frac{1}{n^\beta}\right)_{[x,b]}\le\frac{2D^*\left(f^{(N)},\widetilde{o}\right)}{\Gamma(N-\alpha+1)}(b-a)^{N-\alpha}<+\infty. \tag{79}$$

So everything in the statements of the theorem makes sense.

The proof of the theorem is now completed. □

Corollary 4.4. *(to Theorem 4.3, $N=1$ case) Let $0<\alpha,\beta<1$, $f\in C^1_{\mathcal{F}}([a,b])$, $n\in\mathbb{N}$, $n^{1-\beta}>2$. Then*

$$D^*\left(A_n^{\mathcal{F}}(f),f\right)\le$$

$$\frac{(\psi(1))^{-1}}{\Gamma(\alpha+1)}\left\{\frac{\left[\sup_{x\in[a,b]}\omega_1^{(\mathcal{F})}\left(\left(D_{x-}^{\alpha\mathcal{F}}f\right),\frac{1}{n^\beta}\right)_{[a,x]}+\sup_{x\in[a,b]}\omega_1^{(\mathcal{F})}\left(\left(D_{*x}^{\alpha\mathcal{F}}f\right),\frac{1}{n^\beta}\right)_{[x,b]}\right]}{n^{\alpha\beta}}+\right. \tag{80}$$

$$\left.\left(\frac{1-h\left(n^{1-\beta}-2\right)}{2}\right)(b-a)^\alpha\left[\sup_{x\in[a,b]}D^*\left(\left(D_{x-}^{\alpha\mathcal{F}}f\right),\widetilde{o}\right)_{[a,x]}+\sup_{x\in[a,b]}D^*\left(\left(D_{*x}^{\alpha\mathcal{F}}f\right),\widetilde{o}\right)_{[x,b]}\right]\right\}.$$

Proof. By (70). □

Finally we specialize to $\alpha=\frac{1}{2}$.

Corollary 4.5. *(to Theorem 4.3) Let $0<\beta<1$, $f\in C^1_{\mathcal{F}}([a,b])$, $n\in\mathbb{N}$, $n^{1-\beta}>2$. Then*

$$D^*\left(A_n^{\mathcal{F}}(f),f\right)\le$$

$$\frac{2(\psi(1))^{-1}}{\sqrt{\pi}}\left\{\frac{\left[\sup_{x\in[a,b]}\omega_1^{(\mathcal{F})}\left(\left(D_{x-}^{\frac{1}{2}\mathcal{F}}f\right),\frac{1}{n^\beta}\right)_{[a,x]}+\sup_{x\in[a,b]}\omega_1^{(\mathcal{F})}\left(\left(D_{*x}^{\frac{1}{2}\mathcal{F}}f\right),\frac{1}{n^\beta}\right)_{[x,b]}\right]}{n^{\frac{\beta}{2}}}+\right. \tag{81}$$

$$\left.\left(\frac{1-h\left(n^{1-\beta}-2\right)}{2}\right)\sqrt{b-a}\left[\sup_{x\in[a,b]}D^*\left(\left(D_{x-}^{\frac{1}{2}\mathcal{F}}f\right),\widetilde{o}\right)_{[a,x]}+\sup_{x\in[a,b]}D^*\left(\left(D_{*x}^{\frac{1}{2}\mathcal{F}}f\right),\widetilde{o}\right)_{[x,b]}\right]\right\}.$$

Proof. By (80). □

5 Conclusion

We have extended to the fuzzy setting all the main approximation theorems of Section 3.

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