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On The Spectrum of Countable MV-algebras

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Abstract. In this paper we consider MV-algebras and their prime spectrum. We show that there is an uncountable MV-algebra that has the same spectrum as the free MV-algebra over one element, that is, the MV-algebra $Free_1$ of McNaughton functions from $[0,1]$ to $[0,1]$, the continuous, piecewise linear functions with integer coefficients. The construction is heavily based on Mundici equivalence between MV-algebras and lattice ordered abelian groups with the strong unit. Also, we heavily use the fact that two MV-algebras have the same spectrum if and only if their lattice of principal ideals is isomorphic. As an intermediate step we consider the MV-algebra A_1 of continuous, piecewise linear functions with rational coefficients. It is known that A_1 contains $Free_1$, and that A_1 and $Free_1$ are equispectral. However, A_1 is in some sense easy to work with than $Free_1$. Now, A_1 is still countable. To build an equispectral uncountable MV-algebra A_2, we consider certain "almost rational" functions on $[0,1]$, which are rational in every initial segment of $[0,1]$, but which can have an irrational limit in 1.

We exploit heavily, via Mundici equivalence, the properties of divisible lattice ordered abelian groups, which have an additional structure of vector spaces over the rational field.

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1 Introduction

MV-algebras are the algebraic counterpart of Łukasiewicz fuzzy logic in the same sense as Boolean algebras are the counterpart of classical logic. An important topological invariant of MV-algebras is the prime spectrum. However, unlike the particular case of Boolean algebras, whose prime spectrum is a complete invariant by Stone duality, see [11], there are different MV-algebras with the same spectrum. A simple example is given by the Boolean algebra $\{0,1\}$ and the real interval $[0,1]$. These two MV-algebras are not isomorphic (one has two elements, and the other has the power of the continuum) but their prime spectrum is the one point topological space.

As usual we denote the set of natural numbers by $\mathbb{N} = \{0,1,2,\ldots\}$, the set of positive integers $\mathbb{N}^+ = \{1,2,3,\ldots,\}$, the integer ring by $\mathbb{Z}$, the rational field by $\mathbb{Q}$, and the real field by $\mathbb{R}$.

MV-algebras are axiomatized in the following way (see [5] for a basic treatment and [9] for a more advanced text). They are algebraic structures of the form $(A,\oplus,0,\neg,1)$ where

- $(A,\oplus,0)$ is a commutative monoid;
- $\neg\neg x = x$;
- $\neg 0 = 1$;

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- $x \oplus 1 = 1$;
- $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$ (Mangani axiom).

We will denote an MV-algebra by A.

The most important (and motivating) example of MV-algebra is $[0,1]$, where $x \oplus y = min(x+y, 1)$ and $\neg x = 1 - x$. We have:

Lemma 1.1. *(see [6]) The MV-algebra* $[0,1]$ *generates the variety of MV-algebras.*

For $n \in \mathbb{N}^+$ and $x \in A$ we denote $nx = x \oplus x \ldots \oplus x$ where $\oplus$ occurs $n-1$ times. We let also $0x = 0$.

Recall that a lattice is a partially ordered set $(L, \leq)$ where for every $x, y \in L$ there is the supremum (least upper bound) $x \vee y$ and the infimum (greatest lower bound) $x \wedge y$. We will denote a lattice by L.

In any MV-algebra we have the partial order such that $x \leq y$ if and only if there is z such that $y = x \oplus z$. This order is a (distributive) lattice where $x \vee y = \neg(\neg x \oplus y) \oplus y$ and $x \wedge y = \neg(\neg x \vee \neg y)$.

2 Abelian ℓ-groups

A kind of algebraic structure close to MV-algebras are abelian ℓ-groups, see [5]. An abelian ℓ group is an abelian group with a lattice structure such that $x \leq y$ implies $x + z \leq y + z$. A strong unit of an ℓ-group G is an element $u \geq 0$ such that for every $x \in G$ there is a positive integer $n \in \mathbb{N}$ such that $x \leq nu$.

We will denote an abelian ℓ-group by G.

For $x \in G$ and $m \in \mathbb{Z}$ we denote by mx the usual multiplication by m.

We collect some useful ℓ-group rules of commutation (or anticommutation):

Lemma 2.1. *For every* a, b, c *in an abelian* ℓ*-group* G *and for every* $m, m' \in \mathbb{Z}$ *we have*

$$\begin{aligned}
a + (b \wedge c) &= (a + b) \wedge (a + c) \\
a + (b \vee c) &= (a + b) \vee (a + c) \\
0a &= 0 \\
m(a \wedge b) &= ma \wedge mb, m > 0 \\
m(a \vee b) &= ma \vee mb, m > 0 \\
m(a \wedge b) &= ma \vee mb, m < 0 \\
m(a \vee b) &= ma \wedge mb, m < 0 \\
m(a + b) &= ma + mb \\
m(m'a) &= (mm')a
\end{aligned} \tag{1}$$

Proof. It is known that the variety of abelian ℓ-groups is generated by $\mathbb{Z}$, see [1]. So it is enough to prove these assertions in $\mathbb{Z}$, which is easy. $\square$

We find it useful to recall divisible abelian ℓ-groups. An abelian ℓ-group G is called divisible if for every $x \in G$ and for every $n \in \mathbb{N}^+$ there is $y \in G$ such that $ny = x$. Abelian ℓ-groups are torsion free, see [4], so when y exists, it is unique, and it will be denoted by x/n.

A divisible abelian ℓ-group has a natural structure of a vector space over $\mathbb{Q}$, that is, if $m \in \mathbb{Z}$ and $n \in \mathbb{N}^+$, we denote $(m/n)x = (mx)/n$.

Now for divisible abelian ℓ-groups we not only have all properties of ℓ-groups, but some properties concerning multiplication by elements of $\mathbb{Q}$:

Lemma 2.2. *For every a, b, c in a divisible abelian ℓ-group G and for every $q, q' \in \mathbb{Q}$ we have*

$$\begin{aligned} q(a \wedge b) &= qa \wedge qb, q > 0 \\ q(a \vee b) &= qa \vee qb, q > 0 \\ q(a \wedge b) &= qa \vee qb, q < 0 \\ q(a \vee b) &= qa \wedge qb, q < 0 \\ q(a + b) &= qa + qb \\ q(q'a) &= (qq')a \end{aligned} \tag{2}$$

Proof. The idea is to write $q = m/n$ with $m \in \mathbb{Z}$ and $n \in \mathbb{N}^+$ and reduce to the integer case, which is treated in the Lemma 2.1. □

We derive from Lemmas 2.1 and 2.2 a lemma on subgroups:

Lemma 2.3. *Let G be a divisible abelian ℓ-group and S a subset of G. The divisible abelian ℓ-group generated by S is given by the lattice combinations of rational linear combinations of elements of S.*

Proof. Let T be the set of the lattice combinations of rational linear combinations of elements of S. It is enough to show that T is a divisible ℓ-group.

First, $0 \in T$ trivially, and by definition, T is closed under the lattice operations $\wedge$ and $\vee$.

To prove that T is closed under sum, we have to show that if $t, t' \in T$ then $t + t' \in T$. To this aim, if u is a ℓ-group term, let us denote by $n(u)$ the number of lattice operations in u.

Closure under sum can be proved by induction on $n(t + t')$. In fact, if $n(t + t') = 0$ then the thesis is clear. If $n(t + t') > 0$ then at least one of t, t' begins with $\circ$, where $\circ$ is one of $\wedge, \vee$. By symmetry we can suppose $t' = u \circ v$. Now $t + t' = t + (u \circ v) = (t + u) \circ (t + v)$ by Lemma 2.1. But $t + u$ and $t + v$ have less lattice operators than $t + t'$, so we can apply the inductive hypothesis and find $t + u \in T$ and $t + v \in T$, so $t + t' = (t + u) \circ (t + v) \in T$. This completes the inductive proof.

Finally, T is closed under multiplication by any $q \in \mathbb{Q}$ since the latter commutes (or anticommutes) with lattice operations and rational linear combinations, in the sense of Lemma 2.2. In particular taking $q = -1$, T is closed under inverse, so is a group, and taking $q = 1/n$, $n \in \mathbb{N}^+$, T is divisible. □

Mundici in [10] discovered an equivalence Γ between the category of MV-algebras and the category of abelian ℓ-groups with strong unit. Namely, $\Gamma(G, u)$ is the MV-algebra with universe $\{x \in G \mid 0 \leq x \leq u\}$ and operations $x \oplus y = (x + y) \wedge u$ and $\neg x = u - x$. The motivation of Mundici was the study of AF C^*-algebras in quantum mechanics. The equivalence Γ will be very useful in this work.

3 Ideals in MV-algebras and lattices

An ideal of an MV-algebra A is a subset I of A which is a monoid and is closed downwards. Ideals in ℓ-groups also exist, but they will not be used in this work.

An ideal of a lattice L is a subset I of L closed under $\vee$ and closed downwards.

An ideal I is called principal if it is generated by one element. In MV-algebras or lattices, every finitely generated ideal is principal.

If $a \in A$, we denote by $id_A(a)$ the principal ideal generated by a in A. So, $id_A(a)$ is the set of all $b \in A$ such that $b \leq na$ for some $n \in \mathbb{N}$.

An ideal P of an MV-algebra A is called prime if $P \neq A$ and $x \wedge y \in P$ implies $x \in P$ or $y \in P$. The same holds for prime ideals of a lattice L.

Following [13], we denote by $Id_c(A)$ the lattice of principal ideals of an MV-algebra A, where c stands for compact, since principal ideals are exactly the compact elements of the lattice of all ideals of A.

The lattice $Id_c(A)$ when A is an MV-algebra can also be characterized in the following way:

Lemma 3.1. *(see [3] for the introduction of the Belluce lattice) Let A be an MV-algebra. The lattice $Id_c(A)$ is isomorphic to the Belluce lattice $\beta(A)$, which is A modulo the equivalence of lying in the same prime ideals, where $[x] \wedge [y] = [x \wedge y]$ and $[x] \vee [y] = [x \vee y]$, and $[x]$ denotes the equivalence class of x.*

Proof. It is enough to show that, for every $x, y \in A$, we have $x \in id_A(y)$ if and only if every prime ideal containing y contains x.

Suppose $x \in id_A(y)$. Then $x \leq ny$ for some $n \in \mathbb{N}$. Let P prime with $y \in P$. Since P is an ideal, $ny \in P$ and $x \in P$.

Conversely, suppose $x \notin id_A(y)$. Let S be the set of ideals I of A such that $x \notin I$ and $y \in I$. Then S is nonempty since $id_A(y) \in S$. Every chain in S has an upper bound since S is closed under union. So, by Zorn's Lemma, S has a maximal element P. It is enough to show that P is prime.

First, P is a proper ideal since $x \notin P$.

Moreover, suppose $a \wedge b \in P$ but $a, b \notin P$. Then $P \cup \{a\}$ and $P \cup \{b\}$ generate an ideal containing x. So, $x \leq p_1 \vee n_1 a$ and $x \leq p_2 \vee n_2 b$ with $p_1, p_2 \in P$ and $n_1, n_2 \in \mathbb{N}$. We can write $n_1 + n_2 = n$ and $p_1 \vee p_2 = p$. Then $x \leq p \vee na$ and $x \leq p \vee nb$. Taking the infimum we have

$$x \leq (p \vee na) \wedge (p \vee nb) = p \vee (na \wedge nb) = p \vee n(a \wedge b) \tag{3}$$

(this can be proved by Lemma 1.1).

But $p \in P$ and $a \wedge b \in P$, so $x \in P$, contrary to the fact that $x \notin P$. □

The prime spectrum $Spec(A)$ is the topological space of the prime ideals of A where the basic opens are $O(a) = \{P \in Spec(A) | a \notin P\}$, where $a \in A$. This topology is called the Zariski topology. The prime spectrum of a lattice L, $Spec(L)$, is defined in the same way.

Spectra of MV-algebras have been characterized in [7] in terms of their compact open sets. They are particular spectral spaces. So, we can take advantage of Stone duality between spectral spaces and bounded distributive lattices, see [12]. If X is a spectral space, its Stone dual is the lattice $\overset{\circ}{K}(X)$ of compact open sets of X. Conversely, if L is a bounded distributive lattice, then $Spec(L)$ is the prime spectrum of L with the Zariski topology.

Moreover, the following are known:

Proposition 3.2. *For every MV-algebra A, $Spec(Id_c(A))$ is homeomorphic to $Spec(A)$.*

Proof. This follows from [3] and Lemma 3.1. □

Proposition 3.3. *For every MV-algebra A, the lattice $\overset{\circ}{K}(Spec(A))$ is isomorphic to $Id_c(A)$.*

Proof. By the previous proposition $\overset{\circ}{K}(Spec(A)) = \overset{\circ}{K}(Spec(Id_c(A)))$ and this is $Id_c(A)$ by the Stone duality of [12]. □

Summing up, it follows:

Lemma 3.4. *Let A, B be two MV-algebras. Then the topological spaces $Spec(A)$ and $Spec(B)$ are homeomorphic if and only if the lattices $Id_c(A)$ and $Id_c(B)$ are isomorphic.*

Proof. Suppose $Id_c(A) = Id_c(B)$. Then $Spec(Id_c(A)) = Spec(Id_c(B))$. So by Proposition 3.2, $Spec(A) = Spec(B)$.

Conversely, suppose $Spec(A) = Spec(B)$. Then $\overset{\circ}{K}(Spec(A)) = \overset{\circ}{K}(Spec(B))$ and by Proposition 3.3 we conclude $Id_c(A) = Id_c(B)$. □

4 Piecewise-F functions and free MV-algebras

In the next section we will build an MV-algebra consisting of one-dimensional functions with real values. So, in this section we introduce some notation which is ad hoc for the study of this kind of functions.

Given $a, b \in \mathbb{R}$ we use the standard notations for intervals like $[a, b] = \{x | a \leq x \leq b\}$ and $]a, b[= \{x | a < x < b\}$ and $[a, b[= \{x | a \leq x < b\}$.

Given a real valued function f, let $zeros(f)$ denote the set of zeros of f.

Let F be a class of functions. Let $a, b \in \mathbb{Q}$.

A continuous function $f : [a, b] \to \mathbb{R}$ is piecewise-F if there is $n \in \mathbb{N}$ and there are n rationals $c_1 = a < c_2 < \ldots < c_n = b$ such that every restriction $f|_{[c_i, c_{i+1}]}$ is in F.

More generally, we say that a continuous function $f : [a, b] \to \mathbb{R}$ is almost (piecewise) F if for every closed rational interval $J = [a, c]$ with $a < c < b$, the restriction $f|_J$ is piecewise F. Note that J is an initial segment of $[a, b]$.

Since MV-algebras are axiomatized by equations, they form a variety, and there are free MV-algebras $Free_k$ over any cardinal k. For $k = 1$ we have:

Theorem 4.1. *(see [5]) The MV-algebra $Free_1$ is given by one variable McNaughton functions, that is, the piecewise-$AFFINT$ functions from $[0, 1]$ to $[0, 1]$, where $AFFINT$ is the set of affine linear functions with integer coefficients.*

More generally one can consider the family $AFFRAT$ of affine linear functions with *rational* coefficients, or $AFFREAL$ with real coefficients (we do not need real coefficients in this work, but they are important in the context of Riesz MV-algebras, see [8]).

It is convenient to say that a segment in the Cartesian plane is rational if it has two rational extremes (this implies that the slope is rational unless the segment is vertical).

Lemma 4.2. *The graph of a piecewise $AFFRAT$ function from $[a, b]$ to $\mathbb{R}$ is a finite union of rational segments.*

Corollary 4.3. *Let f be a piecewise $AFFRAT$ function from $[a, b]$ to $\mathbb{R}$. The intersection of the graph of f with a rational segment is a finite union of rational points and rational segments.*

Proof. The intersection of two rational segments, if nonempty, is either a rational point or a rational segment (by analytic geometry). Now the thesis follows from the previous lemma by subdividing the graph of f into finitely many rational segments. $\square$

5 On the spectrum of $Free_1$

We have seen that there are different MV-algebras with homeomorphic spectrum.

We call equispectral two MV-algebras with homeomorphic spectrum.

We recall from a submitted work:

Proposition 5.1. *(see [2]) Let A be an MV-algebra. Then the equispectrality class of A is a set (in Zermelo-Fraenkel set theory) and it has at least cardinality $2^{\aleph_0}$.*

We conjecture that every equispectrality class has at least cardinality $2^{2^{\aleph_0}}$. This happens, for instance, for the class of $A = \{0, 1\}$.

The conjecture implies that every equispectrality class contains an uncountable MV-algebra. This question is relevant in view of the results of [10], where countable MV-algebras play a major role, since they are put in correspondence with certain AF-C^*-algebras in view of applications to quantum mechanics.

In this paper we focus on the equispectrality class of the MV-algebra $Free_1$. We will prove:

Theorem 5.2. *There is an uncountable MV-algebra equispectral with $Free_1$.*

Proof.

It is enough to find an uncountable MV-algebra with the same principal ideal lattice as $Free_1$. We will build two MV-algebras A_1 and A_2; A_1 is already known, whereas A_2 (to our knowledge) is new and is the witness of the theorem. We have $Free_1 \subseteq A_1 \subseteq A_2$.

$Free_1$, A_1 and A_2 are MV-algebras of continuous functions from $[0,1]$ to itself.

Let G_1 be the divisible ℓ-group of piecewise $AFFRAT$ functions from $[0,1]$ to $\mathbb{R}$, where $AFFRAT$ is the set of affine linear functions with *rational* coefficients.

Let $A_1 = \Gamma(G_1, 1)$. Note that A_1 is an MV-algebra consisting of all functions of the form $trunc(g)$ for $g \in G_1$, where $trunc$ is the truncation operator:

$$trunc(g) = (g \vee 0) \wedge 1. \tag{4}$$

Equivalently, A_1 is the MV-algebra of piecewise $AFFRAT$ functions from $[0,1]$ to $[0,1]$.

We can say that the elements of A_1 are generalized McNaughton functions, where integer coefficients are replaced by rational coefficients. In particular $Free_1$ is a MV-subalgebra of A_1.

It is known:

Lemma 5.3. *Two elements of A_1 generate the same ideal if and only if they have the same zeros.*

Proof. Let $f, g \in A_1$. If $id_{A_1}(f) = id_{A_1}(g)$ then $f \leq ng$ and $g \leq nf$ for some $n \in \mathbb{N}$, so they have the same zeros.

Conversely, suppose f, g have the same zeros.

Let us call maximal nonzero interval any maximal open interval $]a, b[\subseteq [0,1]$ where f, g have no zero. Note that a and b can be zeros of f, or 0, or 1.

Let $]a, b[$ be a maximal nonzero interval.

If c is sufficiently close to a, then f and g are linear in $]a, c[$. Likewise if d is sufficiently close to b, then f and g are linear in $]d, b[$. So, f/g is bounded in $]a, c[\cup]d, b[$. Moreover, $g \neq 0$ in $[c, d]$; so, by continuity, g is bounded from below in $[c, d]$, and since f is bounded in $[c, d]$, f/g is bounded in $[c, d]$. Summing up, f/g is bounded in $]a, b[$, and taking the supremum over all maximal nonzero intervals $]a, b[$, f/g is bounded in $[0,1]$ whenever $g \neq 0$. So, there is $n \in \mathbb{N}$ such that $f \leq ng$ whenever $g \neq 0$, hence $f \leq ng$ everywhere in $[0,1]$ since $zeros(f) = zeros(g)$.

By symmetry, there is also $n' \in \mathbb{N}$ such that $g \leq n'f$. So $id_{A_1}(f) = id_{A_1}(g)$. □

It is also known:

Lemma 5.4. *The MV-algebras $Free_1$ and A_1 are equispectral.*

Proof. By Lemma 3.4 it is enough to show that $Free_1$ and A_1 have the same lattice of principal ideals.

Given a function $f \in A_1$, let us consider the MV-algebraic sum nf, by a sufficiently large integer $n \in \mathbb{N}$. Then rational slopes of f become integer in nf, and the intersection of every piece of the function with the y axis is an integer point. So in nf, every piece is affine linear with integer coefficients (or constantly 1, which is trivially an affine function with integer coefficients), so nf is piecewise $AFFINT$, that is, $nf \in Free_1$.

Moreover, clearly the zeros of f and nf are the same. So, by Lemma 5.3, the ideals generated in A_1 by f and nf are the same. □

The previous lemma allows us to use A_1 rather than $Free_1$, which is more convenient for our purposes.

Let us begin our construction of A_2.

Let B be a vector space basis of $\mathbb{R}$ over $\mathbb{Q}$ such that $1 \in B$. Note that B has size $2^{\aleph_0}$.

For every $b \in B, b \neq 1$, let $(q_n(b))_{n \in \mathbb{N}}$ be a sequence of rationals converging to b and we define $u_b : [0,1] \to \mathbb{R}$ as the function such that

- u_b is continuous
- $u_b(1-1/n) = q_n(b)$
- u_b is linear between $1-1/n$ and $1-1/n+1$.

Note that u_b is almost $AFFRAT$ in $[0,1]$ and $u_b(1) = b$.

Let G_2 be the divisible ℓ-group generated by $G_1 \cup \{u_b, b \in B, b \neq 1\}$ in the function space $\mathbb{R}^{[0,1]}$ of all functions from $[0,1]$ to $\mathbb{R}$. Note that every element of G_2 is a continuous, almost $AFFRAT$ function from $[0,1]$ to $\mathbb{R}$.

Let $A_2 = \Gamma(G_2, 1)$ be the corresponding MV-algebra.

Note that A_2 is an MV-algebra consisting of the functions $trunc(g)$ for $g \in G_2$.

Since $G_1 \subseteq G_2$ we have $A_1 \subseteq A_2$. Moreover:

Corollary 5.5. *Two elements f, g of A_1 generate the same ideal in A_2 if and only if they have the same zeros.*

Proof. By Lemma 5.3, $zeros(f) = zeros(g)$ if and only if $f \leq ng$ and $g \leq nf$ for some $n \in \mathbb{N}$, that is, f an g generate the same ideal in A_2. □

Note that the proof of the previous corollary works only for $f, g \in A_1$.

The elements of G_2 have the following local representation as a sum:

Lemma 5.6. *Locally in 1, every element g_2 of G_2 has the form*

$$g_2 = g_1 + \Sigma_{b\neq 1} q_b u_b, \tag{5}$$

where $g_1 \in G_1$, Σ is a finite sum, $b \in B$, and $q_b \in \mathbb{Q}$. (Note that no lattice operations are necessary).

Proof. Let f, g be continuous functions from $[0,1]$ to $\mathbb{R}$. If $f(1) < g(1)$, then $f \wedge g = f$ locally in 1. Likewise if $f(1) > g(1)$, then $f \vee g = f$ locally in 1.

So, let g_2 be any element of G_2. By Lemma 2.3, g_2 is a lattice combination of rational linear combinations of $G_1 \cup \{u_b, b \in B, b \neq 1\}$.

So, g_2 is locally in 1 a lattice combination of a family $(f_i)_{i\in I}$ of rational linear combinations of $G_1 \cup \{u_b, b \in B, b \neq 1\}$ such that $f_i(1) = g_2(1)$ for every $i \in I$.

Since B is a basis of $\mathbb{R}$ over $\mathbb{Q}$, and $f_i(1) = g_2(1)$ is independent of i, every f_i is a sum of a rational linear combination of the u_b's independent of i, say $u_B = \Sigma_{b\neq 1} q_b u_b$, and a part $g_i \in G_1$ possibly dependent on i. So $f_i = u_B + g_i$ for every $i \in I$.

Hence, by Lemma 2.2, g_2 is, locally in 1, the sum of u_B and an element g_1 which is a lattice combination of rational linear combinations of elements of G_1, so $g_1 \in G_1$ since G_1 is a divisible ℓ-group. □

Lemma 5.7. *Let $g_2 \in G_2$, with*

$$g_2 = g_1 + \Sigma_{b\neq 1} q_b u_b \tag{6}$$

locally in 1 as in the previous lemma.

If $q_b \neq 0$ for some b, then $g_2(1)$ is irrational.

Proof. Evaluating the equation (6) in 1 we derive

$$g_2(1) = g_1(1) + \Sigma_{b\neq 1} q_b u_b(1) = g_1(1) + \Sigma_{b\neq 1} q_b b.$$

Since each b is in the base B of $\mathbb{R}$ and $q_b \in \mathbb{Q}$, it follows $\Sigma_{b\neq 1} q_b b$ is irrational; and since $g_1(1)$ is rational, $g_2(1)$ is irrational. □

Let $f \in A_2$. We want to show that there is $f_1 \in A_1$ such that $id_{A_2}(f) = id_{A_2}(f_1)$. The key idea is to "sandwich" f between two elements of A_1 whenever possible.

Suppose $f(1) \in \{0,1\}$. Then $f \in A_1$. In fact, locally in 1,

$$f = trunc(g_1 + \Sigma), \tag{7}$$

where $g_1 \in G_1$ and $\Sigma = \Sigma_{b \neq 1} q_b u_b$. Let us distinguish the possible values of $(g_1 + \Sigma)(1)$.

If $(g_1 + \Sigma)(1) > 1$ then f is an element of A_1 up to some rational abscissa less then 1, followed by a segment constant 1, so $f \in A_1$.

If $(g_1 + \Sigma)(1) = 1$ then $\Sigma = 0$ otherwise $\Sigma(1)$ should be irrational by Lemma 5.7, so $f = trunc(g_1) \in A_1$.

The case $0 < (g_1 + \Sigma)(1) < 1$ is impossible, otherwise by truncation we have $0 < f(1) < 1$, which is false by hypothesis.

If $(g_1 + \Sigma)(1) = 0$ then $\Sigma = 0$ otherwise $\Sigma(1)$ should be irrational by Lemma 5.7, so $f = trunc(g_1) \in A_1$.

Finally if $(g_1 + \Sigma)(1) < 0$ then f is an element of A_1 up to some rational abscissa less than 1, followed by a segment constant 0, so $f \in A_1$.

Otherwise suppose $f(1) \notin \{0,1\}$. We build two functions $h_1, h_2 \in A_1$ which "sandwich" f.

Let $p_f < 1$ be a rational such that f is nonzero in $[p_f, 1]$ (p_f exists since f is continuous and $f(1) \neq 0$).

Let q_f be a rational with $0 < q_f < f(1)$. Consider the segment s_1 joining $(1, q_f)$ and $(p_f, f(p_f))$. Note that the slope of s_1 is finite and negative, and that the extremes of s_1 are rational, so s_1 is a rational segment. Let us write $y = s_1(x)$ when $(x, y) \in s_1$.

Let $(y_1, f(y_1))$ be the rightmost intersection of s_1 and the graph of f. This intersection point exists since $(p_f, f(p_f))$ is in s_1 and is also in the graph of f. Moreover the point $(y_1, f(y_1))$ has rational coordinates, by Corollary 4.3. The function $h_1 : [0,1] \to [0,1]$ is defined by

$$h_1(x) = \begin{cases} f(x), x < y_1 \\ s_1(x), x \geq y_1. \end{cases} \tag{8}$$

Note that $h_1 \in A_1$, $h_1 \leq f$, and f and h_1 have the same zeros.

Likewise consider the segment s_2 joining $(1,1)$ and $(p_f, 0)$. The segment s_2 has finite positive slope and has two rational extremes, so it is rational. Let us write $y = s_2(x)$ when $(x, y) \in s_2$.

Let $(y_2, f(y_2))$ be the leftmost intersection of s_2 and the graph of f. This intersection exists by continuity since $(1,1)$ is above the graph of f and $(p_f, 0)$ is below the graph of f. Moreover the point $(y_2, f(y_2))$ has rational coordinates, by Corollary 4.3.

The function $h_2 : [0,1] \to [0,1]$ is defined by

$$h_2(x) = \begin{cases} f(x), x < y_2 \\ s_2(x), x \geq y_2. \end{cases} \tag{9}$$

Note that $h_2 \in A_1$, $f \leq h_2$, and f and h_2 have the same zeros.

Summing up, we have $h_1 \leq f \leq h_2$ and $zeros(h_1) = zeros(f) = zeros(h_2)$; so by Corollary 5.5, we have $id_{A_2}(h_1) = id_{A_2}(h_2)$; and since $h_1 \leq f \leq h_2$ we have $id_{A_2}(h_1) = id_{A_2}(h_2) = id_{A_2}(f)$. □

6 Conclusion

We hope to extend theorem 5.2 to every countable MV-algebra. Also we hope to solve the problem whether all equispectrality classes have size exactly $2^{2^{\aleph_0}}$.

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