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Fuzzy Subgroups and Digraphs Induced by Fuzzy Subgroups

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Abstract. Given a fuzzy subgroup μ of a group G, $x \rhd_u y$ if and only if $\mu(xy) < \mu(yx)$ defines a directed relation with an associated digraph $(G, \rhd_u)$. We consider (μ, ν)-homomorphisms $\varphi : (G, \mu) \to (H, \nu)$ where μ and ν are fuzzy subgroups of G and H respectively and the preservation of properties of the digraphs $(G, \rhd_u)$ several of which are also noted here, e.g., $(G, \rhd_u)$ is an anti-chain if and only if μ is a fuzzy normal subgroup of the group G.

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Keywords and Phrases: Fuzzy subgroup, μ-product relation, Fuzzy normal, Digraph, (μ, ν)-homomorphism.

1 Introduction

In this paper, we show that given a fuzzy subgroup μ of a group G, letting $x \rhd_u y$ if and only if $\mu(xy) < \mu(yx)$ defines a directed relation with an associated digraph $(G, \rhd_u)$ whose properties are related to both μ and the underlying group G. The associated digraph has a multitude of natural invariants associated with it, e.g., the adjacency matrix and its eigenvalues, the adjacency algebra and its dimension over the field of rationals, the radius, the diameter, and any other of the "standard" structures derived from such graphs. One can thus proceed to make a deeper study of the subject than we do here, where we mostly indicate some elementary properties of $(G, \rhd_u)$ as they relate to μ itself. Included in the fact that $(G, \rhd_u)$ is an anti-chain if and only if μ is a fuzzy normal subgroup of the group G. Furthermore we explore the consequences of homomorphisms induced on the digraphs $(G, \rhd_u)$ and $(H, \rhd_v)$ by (μ, ν)-homomorphisms $\varphi : (G, \mu) \to (H, \nu)$ to some extent including the effects on the (shortest) distance functions for these graphs, noting that distances shrink in general. For general references on fuzzy group theory we refer to [3, 5, 6].

2 Preliminaries

Rosenfeld [12] has defined fuzzy subgroupoid and fuzzy subgroups in the following way.

Definition 2.1. ([3]) *Let G be a group. A fuzzy set μ of G is said to be a fuzzy subgroup of G, if for all x, y in G,*

(i) $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$,

(ii) $\mu(x^{-1}) \geq \mu(x)$.

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The following properties of fuzzy subgroups of a group G have been noted by many authors [2, 4, 7].

Proposition 2.2. *Let μ be any fuzzy subgroup of a group G with identity e. Then the following statements are true:*

(i) $\mu(x^{-1}) = \mu(x) \leq \mu(e)$ *for all* $x \in G$,

(ii) $\mu(xy) = \mu(y)$ *for all* $y \in G \Longleftrightarrow \mu(x) = \mu(e)$, *where* $x \in G$,

(iii) *if* $\mu(x) < \mu(y)$ *for some* $x, y \in G$, *then* $\mu(xy) = \mu(x) = \mu(yx)$.

Proposition 2.3. ([3]) *Let G be a group and $A \subseteq G$. Then A is a subgroup of G if and only if the characteristic function χ_A of A is a fuzzy subgroup of G.*

Neggers and Kim in [8, 9, 10, 11] studied some relations between posets and several algebraic structures, e.g., semigroups, BCK-algebras, and associative algebras.

3 Fuzzy subgroups and digraphs

Given a fuzzy subgroup of a group $(G, \cdot)$, let

$$x \rhd_u y \Longleftrightarrow \mu(x \cdot y) < \mu(y \cdot x)$$

denote the *μ-product relation* associated with fuzzy subgroup μ of G. This relation can be viewed as a digraph on G induced by the fuzzy subgroup μ.

Proposition 3.1. *Let G be a group and H be a subgroup of G. If χ_H is the characteristic function of H, then H is a normal subgroup of G if and only if the relation $\rhd_{\chi_H}$ is trivial.*

Proof. Assume that H is not a normal subgroup of G and let $x \in G$. Then $xyx^{-1} \notin H$ for some $y \in H$. If $u := yx^{-1}$ then $ux = (yx^{-1})x = y \in H$ and $xu = x(yx^{-1}) \notin H$ and hence $\chi_H(xu) = 0 < 1 = \chi_H(ux)$, i.e., $x \rhd_{\chi_H} u$. This means that $\rhd_{\chi_H}$ is not a trivial relation, a contradiction. Conversely, assume $\rhd_{\chi_H}$ is not a trivial relation. Then $x \rhd_{\chi_H} y$ for some $x, y \in G$, and hence $\chi_H(xy) = 0, \chi_H(yx) = 1$. Thus $xy \notin H$, $yx \in H$. Since $H \rhd G$, $xy = x(yx)x^{-1} \in H$, a contradiction. □

Notice that a fuzzy subgroup μ of a group G is said to be *fuzzy normal* ([4]), if $\mu(xy) = \mu(yx)$ for all $x, y \in G$. This means precisely that *the fuzzy subgroup μ of G is fuzzy normal provided the relation $\rhd_u$ is trivial.* Thus we may consider the digraph naturally associated with $(G, \rhd_u)$ as a "measure" of the "amount" the fuzzy subgroup μ of G strays from being a fuzzy normal subgroup. If $x \rhd_u y$ then $\mu(xy) < \mu(yx)$, and thus by Proposition 2.2 *(iii)* it follows that $\mu(x) < \mu(y)$ and $\mu(y) < \mu(x)$ are both impossible, so that $\mu(x) = \mu(y)$. We conclude that:

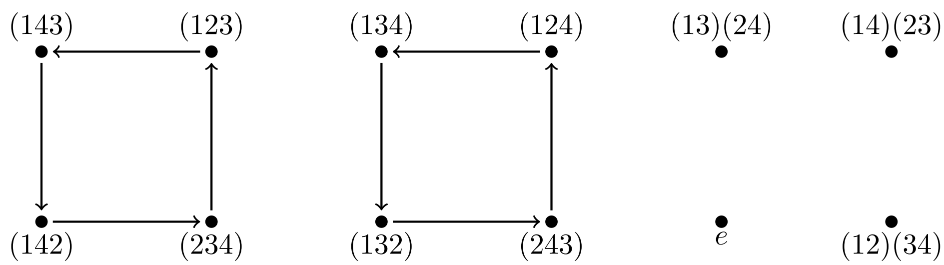
Proposition 3.2. *If μ is a fuzzy subgroup of a group G, then μ is constant on each component of the digraph $(G, \rhd_u)$.*

Example 3.3. Let $G := \{e, a, a^2, a^3, b, ab, a^2b, a^3b\}$ be the octic group, where $a^4 = e = b^2$ and $ba = a^{-1}b$. If we define a fuzzy subset $\mu : G \to [0, 1]$ by $\mu(e) > \mu(a^2) > \mu(a) = \mu(a^3) > \mu(b) = \mu(ab) = \mu(a^2b) = \mu(a^3b)$, then μ is a fuzzy subgroup of G ([3]). Since there are no $x, y \in G$ such that $\mu(xy) < \mu(yx)$, the digraph $(G, \rhd_u)$ is an anti-chain.

Example 3.4. Consider the alternating group

$A_4 := \{e, (12)(34), (13)(24), (14)(23), (123), (132), (142), (124), (234), (243), (134), (143)\}$.

Define a fuzzy subset μ on A_4 by $\mu(e) = 1$, $\mu((12)(34)) = 1/2$, $\mu((14)(23)) = \mu((13)(24)) = 1/3$, $\mu((ijk)) = 0$, where $i, j, k \in \{1, 2, 3, 4\}$. Then μ is a fuzzy subgroup of A_4 ([1]). It is easy to check that $(234) \rhd_u (123), (123) \rhd_u (143)$, $(142) \rhd_u (234)$, $(143) \rhd_u (142)$, $(132) \rhd_u (243), (134) \rhd_u (132), (243) \rhd_u (124)$ and $(124) \rhd_u (134)$. From this relation we get the following diagram:



$(A_4, \rhd_u)$

If x is an isolated point of the digraph $(G, \rhd_u)$, then $d^-(x) = d^+(x) = 0$, i.e., the in-degree and the out-degree are both equal to 0. Thus, $\mu(xy) = \mu(yx)$ for all $y \in G$, and although this does not mean that x is in the center $Z(G)$ of G, it follows that x has properties "somewhat like those in the center". Thus, let $Z_\mu(G)$ denote the collection of all isolated points of the digraph $(G, \rhd_u)$. Then it follows that $Z_\mu(G)$ contains $Z(G)$ and also that:

Theorem 3.5. *Let G be a group with identity e. If μ is a fuzzy subgroup of G, then $Z_\mu(G)$ is a subgroup of G.*

Proof. Clearly, $e \in Z_\mu(G)$. Let $x, y \in Z_\mu(G)$. For any $z \in G$, $\mu(z(xy)) = \mu((zx)y) = \mu(y(zx)) = \mu((yz)x)$. Since $x \in Z_\mu(G)$, $\mu((yz)x) = \mu(x(yz)) = \mu((xy)z)$. It follows that $xy \in Z_\mu(G)$.

Let $x \in Z_\mu(G)$. Given $y \in G$, by Proposition 2.2(i), we obtain $\mu(x^{-1}y) = \mu((x^{-1}y)^{-1}) = \mu(y^{-1}x) = \mu(xy^{-1}) = \mu((xy^{-1})^{-1}) = \mu(yx^{-1})$. Hence $x^{-1} \in Z_\mu(G)$. This proves the theorem. □

Theorem 3.6. *A fuzzy subgroup μ of a group G is fuzzy normal if and only if $G = Z_\mu(G)$.*

Proof. If μ is a fuzzy normal subgroup of G, then $\mu(xy) = \mu(yx)$ for all x, y, whence $\rhd_u$ is trivial and $(G, \rhd_u)$ is an anti-chain. Since $Z_\mu(G)$ is precisely the collection of all isolated points of $(G, \rhd_u)$, we obtain that if $Z_\mu(G) = G$. Assume $G = Z_\mu(G)$. Then every element x of G is an isolated point of $(G, \rhd_u)$, i.e., $x \rhd_u y$ does not hold for any $y \in G$. It follows that $\mu(xy) = \mu(yx)$ for all $y \in G$. Hence μ is fuzzy normal. □

Let G be a group, and let $F(G)$ be the set of all fuzzy subgroups of G. Then we pose the following conjecture:

Conjecture. $Z(G) = \cap_{\mu \in F(G)} Z_\mu(G)$.

Given a digraph $(G, \rhd_\mu)$, let $|G| = n < \infty$. Define a polynomial $P((G, \rhd_\mu); z) = \sum_{i=0}^{n-1} |G|_i z^i$, where $|G|_i = |\{x_0 \rhd x_1 \rhd \cdots \rhd x_i\}|$ is the number of vertices of length $i \geq 1$ and $|G|_0 = |Z_\mu(G)|$. We call $P((G, \rhd_u); z)$ the *directed polynomial* of the directed graph $(G, \rhd_u)$.

Example 3.7. The directed graph $(A_4, \rhd_u)$ of Example 3.4 has the directed polynomial $2z^4 + 4$.

4 (μ, ν)-homomorphisms for fuzzy subgroups

We denote (G, μ) the group G and a fuzzy subgroup $\mu : G \to [0, 1]$. Let (G, μ) and (H, ν) be fuzzy subgroups μ and ν of G and H respectively. A map $\varphi : G \to H$ is said to be a (μ, ν)-*homomorphism* if, for all $x, y \in G$,

(*i*) $\mu(x) < \mu(y)$ implies $\nu(\varphi(x)) < \nu(\varphi(y))$,

(*ii*) $\mu(x) = \mu(y)$ implies $\nu(\varphi(x)) = \nu(\varphi(y))$,

(*iii*) $\nu(\varphi(xy)) = \nu(\varphi(x)\varphi(y))$.

Proposition 4.1. *Let G, H be groups and let $\mu := \chi_S$ be a characteristic function of $S(\subseteq G)$ and let $\nu := \chi_T$ be a characteristic function of $T(\subseteq H)$. If $\varphi : G \to H$ is a (μ, ν)-homomorphism, then (i) $\varphi(S) \subseteq T$; (ii) $\varphi(G \setminus S) \subseteq H \setminus T$.*

Proof. If $\mu(x) < \mu(y)$, then $\mu(x) = 0$ and $\mu(y) = 1$, i.e., $x \notin S$ and $y \in S$. Since φ is a (μ, ν)-homomorphism, we obtain $\nu(\varphi(x)) < \nu(\varphi(y))$. It follows that $\nu(\varphi(x)) = 0$ and $\nu(\varphi((y)) = 1$, i.e., $\varphi(x) \notin T, \varphi(y) \in T$, which proves the proposition. □

Let μ be a fuzzy subset of a group G and let $\rhd_u$ be the μ-product relation on G and let $x, y \in G$. We denote an edge $x \to y$ if $x \rhd_u y$. Then $(G, \to) = (G, \rhd_u)$ is a digraph.

Given a digraph D, we denote the set of all vertices of D by $V(D)$, and denote the set of all edges of D by $A(D)$. Let D, H be digraphs. A map $\varphi : V(D) \to V(H)$ is called a *graph homomorphism* if it preserves edges, i.e., if $x \to y \in A(D)$ then $\varphi(x) \to \varphi(y) \in A(H)$.

Proposition 4.2. *If $\varphi : G \to H$ is a (μ, ν)-homomorphism, then $x \rhd_u y$ implies $\varphi(x) \rhd_v \varphi(y)$, i.e., φ induces a graph homomorphism $\widetilde{\varphi} : (G, \rhd_u) \to (H, \rhd_v)$.*

Proof. If $x \rhd_u y$, then $\mu(xy) < \mu(yx)$. Since φ is a (μ, ν)-homomorphism, we obtain $\nu(\varphi(x)\varphi(y)) = \nu(\varphi(xy)) < \nu(\varphi(yx)) = \nu(\varphi(y)\varphi(x))$ and therefore $\varphi(x) \rhd_v \varphi(y)$. □

Proposition 4.3. *If $\varphi : (G, \mu) \to (H, \nu)$ is both a (μ, ν)-homomorphism and a group homomorphism, and $\psi : (H, \nu) \to (K, \gamma)$ is a (ν, γ)-homomorphism, then $\psi \circ \varphi : (G, \mu) \to (K, \gamma)$ is a (μ, ψ)-homomorphism.*

Proof. Straightforward. □

If $d(x, y)$ represents the shortest distance from vertices x to y in $(G, \rhd_u)$ and if $\varphi : (G, \mu) \to (H, \nu)$ is an onto (μ, ν)-homomorphism, then the shortest path in $(G, \rhd_u)$ from x to y maps to a path in (H, ν) from $\varphi(x)$ to $\varphi(y)$ which map or may not be shortest. As a consequence, we find that $d(\varphi(x), \varphi(y)) \leq d(x, y)$. Thus, various "distance-related parameters", diameter, radius, etc. are shrunk by this process.

A (μ, ν)-homomorphism $\varphi : (G, \mu) \to (H, \nu)$ is said to be a *d-isometry* if for all $x, y \in G$, $d(x, y) = d(\varphi(x), \varphi(y))$. A (μ, ν)-homomorphism $\varphi : (G, \mu) \to (H, \nu)$ is said to be an (μ, ν)-*isomorphism* if φ is a bijective function.

Theorem 4.4. *If $\varphi : (G, \mu) \to (H, \nu)$ is a (μ, ν)-isomorphism, then $\varphi^{-1} : (H, \nu) \to (G, \mu)$ is a (ν, μ)-isomorphism.*

Proof. (*i*) Let $\nu(\alpha) < \nu(\beta)$. Since φ is a bijective function, there are $a, b \in G$ such that $\varphi(a) = \alpha, \varphi(b) = \beta$. Assume $\mu(\varphi^{-1}(\alpha)) \geq \mu(\varphi^{-1}(\beta))$. If $\mu(\varphi^{-1}(\alpha)) = \mu(\varphi^{-1}(\beta))$, then $\nu(\varphi(\varphi^{-1}(\alpha))) = \nu(\varphi(\varphi^{-1}(\beta)))$, i.e., $\nu(\alpha) = \nu(\beta)$, a contradiction. If $\mu(\varphi^{-1}(\alpha)) > \mu(\varphi^{-1}(\beta))$, then $\nu(\varphi(\varphi^{-1}(\alpha))) > \nu(\varphi(\varphi^{-1}(\beta)))$, i.e., $\nu(\beta) < \nu(\alpha)$, a contradiction. Hence we obtain $\mu(\varphi^{-1}(\alpha)) < \mu(\varphi^{-1}(\beta))$.
(*ii*) Assume that there are $\alpha, \beta \in H$ such that $\nu(\alpha) = \nu(\beta), \mu(\varphi^{-1}(\alpha)) \neq \mu(\varphi^{-1}(\beta))$. If we let $\alpha := \varphi(a), \beta := \varphi(b)$, then $\mu(a) \neq \mu(b)$. If $\mu(a) < \mu(b)$, then $\nu(\varphi(a)) < \nu(\varphi(b))$, since φ is a (μ, ν)-homomorphism. It follows

that $\nu(\varphi(a)) < \nu(\varphi(b))$, i.e., $\nu(\alpha) < \nu(\beta)$, a contradiction. If $\mu(b) < \mu(a)$, then it leads to a contraction that $\nu(\beta) < \nu(\alpha)$, a contradiction.
(iii) Assume that there are $p, q \in H$ such that $\mu(\varphi^{-1}(pq)) \neq \mu(\varphi^{-1}(p)\varphi^{-1}(q))$. If we let $\varphi(a) = p, \varphi(b) = q$, then $\mu(\varphi^{-1}(pq)) \neq \mu(ab)$. If $\mu(\varphi^{-1}(pq)) \not< \mu(ab)$, then $\nu(\varphi(\varphi^{-1}(pq)) < \nu(\varphi(ab)) = \nu(\varphi(a)\varphi(b))$, since φ is a (μ, ν)-homomorphism. It follows that $\nu(pq) < \nu(\varphi(a)\varphi(b)) = \nu(pq)$, a contradiction. The case $\nu(\varphi(\varphi^{-1}(pq)) > \nu(\varphi(ab))$ also leads to a contradiction. This proves that $\varphi^{-1} : (H, \nu) \to (G, \mu)$ is a (ν, μ)-homomorphism. □

Corollary 4.5. *If $\varphi : (G, \mu) \to (H, \nu)$ is a (μ, ν)-isomorphism, then φ is a d-isometry.*

Proof. It follows from Theorem 4.4 that

$$d(x, y) \geq d(\varphi(x), \varphi(y)) \geq d(\varphi^{-1}(\varphi(x)), \varphi^{-1}(\varphi(y)) = d(x, y),$$

for all $x, y \in G$. □

5 Conclusions

In this paper we defined a directed relation with an associated digraph $(G, \rhd_u)$ for any fuzzy subgroup μ of a group G, and obtained that if μ is a fuzzy subgroup of G, then the collection of all isolated points of the digraph $(G, \rhd_u)$ forms a subgroup of G. By introducing the notion of (μ, ν)-homomorphism, we discussed graph homomorphisms of digraphs. In the consequence of research, intuitionistic fuzzy theory, hesitant fuzzy theory and soft set theory can be applied to the fuzzy subgroups and digraphs also.

Conflict of Interest: The authors declare no conflict of interest.

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