Transactions on Fuzzy Sets and Systems
URL: https://tfss.journals.iau.ir/
Online ISSN: 2821-0131
Vol.1, No.1, (2022), 88-105
DOI: http://doi.org/10.30495/tfss.2022.690402

Article Type: Original Research Article (Invited Article)

Representations on Raised Very Thin H_v-fields

Thomas Vougiouklis

Abstract. The hyperstructures have applications in mathematics and other sciences such as biology, physics, linguistics, sociology, to mention but a few. For this, mainly, the largest class of the hyperstructures, the H_v-structures, is used, which satisfy the *weak axioms* where the non-empty intersection replaces the equality and they are straightly related to fuzzy set theory. The *fundamental relations* connect the H_v-structures with the classical ones, moreover, they reveal new concepts as the *H_v-fields*. H_v-numbers are called the elements of an H_v-field and they are used in representation theory. We introduce the *raised finite H_v-fields*, and present some results and examples on 2×2 representations on them.

AMS Subject Classification 2020: 20N20; 16Y99
Keywords and Phrases: Hyperstructure, Hope, H_v-structure, H_v-group, H_v-ring, H_v-field

1 Introduction

The hyperstructures called H_v-structures, introduced in 1990 [14] and [15] by Vougiouklis, satisfy the *weak axioms* where the non-empty intersection replaces the equality. The h/v-structures are a generalization of H_v-structures, where a *reproductivity of classes*, is valid instead of the reproductivity of elements [18] and [21]. Some basic definitions:

Algebraic hyperstructure $(H, \cdot)$, is a set H equipped with a *hyperoperation* (abbreviated by ***hope***):

$$\cdot : H \times H \to P(H) - \{\varnothing\}.$$

Denote
WASS the weak associativity: $(xy)z \cap x(yz) \neq \varnothing,\ \forall x, y, z \in H$
and
COW the weak commutativity: $xy \cap yx \neq \varnothing,\ \forall x, y \in H$.
The $(H, \cdot)$ is called H_v-semigroup if it is WASS, it is called ***H_v*-group** if it is reproductive H_v-semigroup: $xH = Hx = H,\ \forall x \in H$.
Motivation. The quotient of a group by any invariant subgroup, is a group. The quotient of a group by any subgroup is a hypergroup, Marty 1934. The quotient of a group by any partition H_v-group, Vougiouklis 1990.

In an H_v-semigroup $(H, \cdot)$, the powers are defined by

$$h^1 = \{h\},\ h^2 = h \cdot h,\ \ldots,\ h^n = h^{\circ} h^{\circ} \ldots h^{\circ},$$

Corresponding author: Thomas Vougiouklis, Email: tvougiou@eled.duth.gr, ORCID: 0000-0002-4076-8853
Received: 21 February 2022; Revised: 8 April 2022; Accepted: 9 April 2022; Published Online: 7 May 2022.

How to cite: T. Vougiouklis, Representations on Raised Very Thin H_v-fields, *Trans. Fuzzy Sets Syst.*, 1(1) (2022), 88-105.

where $(^\circ)$ is the n-ary circle hope: take the union of hyperproducts n times, with all possible patterns of parentheses on them. An $(H, \cdot)$ is cyclic of period s if there is a generator h and the minimum s, such that

$$H = h^1 \cup h^2 \cup \ldots \cup h^s.$$

Analogously, the cyclicity for the infinite period is defined. If there are h and s, the minimum one, such that $H = h^s$, then we say that the $(H, \cdot)$, is a single-power cyclic of period s.

A hyperstructure $(R, +, \cdot)$ is called $\boldsymbol{H_v}$**-ring** if $(+)$ and $(\cdot)$ are WASS, the reproduction axiom is valid for $(+)$, and $(\cdot)$ is weak distributive to $(+)$:

$$x(y+z) \cap (xy+xz) \neq \varnothing\ ,\ (x+y)z \cap (xz+yz) \neq \varnothing\ ,\ \forall x, y, z \in R.$$

Let $(R, +, \cdot)$ be an H_v-ring, a COW H_v-group $(M, +)$ is called $\boldsymbol{H_v}$**-module** over R, if there is an external hope

$$\cdot : R \times M \to P(M) - \{\varnothing\} : (a, x) \mapsto ax$$

such that, $\forall a, b \in R$ and $\forall x, y \in M$, we have

$$a(x+y) \cap (ax+ay) \neq \varnothing, \quad (a+b)x \cap (ax+bx) \neq \varnothing, \quad (ab)x \cap a(bx) \neq \varnothing.$$

In the case of an H_v-field F, which is defined later, instead of an H_v-ring R, then the H_v-vector space is defined.

For more definitions and applications on H_v-structures one can see in books and papers as [1], [3], [6], [15] and [16].

Let $(H, \cdot)$ and $(H, *)$ be H_v-semigroups, then the hope $(\cdot)$ is **smaller** than $(*)$, and $(*)$ greater than $(\cdot)$, iff there exists an automorphism

$$f \in \mathrm{Aut}(H, *) \text{ such that } \ xy \subset f(x * y),\ \forall x, y \in H.$$

We say that $(H, *)$ contains $(H, \cdot)$. If $(H, \cdot)$ is a classical structure then it is the basic structure, and $(H, *)$ is H_b-structure.

Minimal is called an H_v-group if it contains no other H_v-group on the same set. We extend this definition to any H_v-structures with more hopes.

The little theorem. *Greater hopes than the ones which are WASS or COW, are WASS or COW, respectively.*

The little theorem leads to a partial order on H_v-structures and posets. Therefore, we can obtain an extremely large number of H_v-structures just putting more elements on any result.

The problem of enumeration and classification of H_v-structures is complicated because we have very great numbers. For example, the number of H_v-groups with three elements, up to isomorphism, is 1.026.462. There are 7.926 abelian; the 1.013.598 are cyclic.

A class of H_v-structures, introduced in [13] and [15], is the following:

Definition 1.1. An H_v-structure is called ***very thin*** iff all hopes are operations except one, which has all results singletons except only one, which is a subset of cardinality more than one. Therefore, in a very thin H_v-structure in a set H there exists a hope $(\cdot)$ and a pair $(a, b) \in H^2$ for which $ab = A$, with card$A > 1$, and all the other products, with respect to any other hopes (so they are operations), are singletons.

Some large classes of H_v-structures are the following [19]:

Definition 1.2. Let $(G, \cdot)$ be groupoid (resp., hypergroupoid) and $f : G \to G$ be any map. We define a hope (∂), called *theta-hope*, we write *∂-hope*, on G as follows:

$$x\partial y = \{f(x) \cdot y, x \cdot f(y)\}, \quad \forall x, y \in G \quad (\text{resp. } \ x\partial y = (f(x) \cdot y) \cup (x \cdot f(y)), \quad \forall x, y \in G$$

If $(\cdot)$ is commutative, then ∂ is commutative. If $(\cdot)$ is *COW*, then ∂ is *COW*.

The motivation for this definition is the map derivative where only the product of functions can be used. The basic property is that if $(G,\cdot)$ is a semigroup then $\forall f$, the (∂) is WASS.

Definition 1.3. (See [12], [15]) Let $(G,\cdot)$ be a groupoid, then for every $P\subset G$, $P\neq\varnothing$, we define the following hopes called *P-hopes*: $\forall x,y\in G$

$$\underline{P}: x\underline{P}y=(xP)y\cup x(Py),\quad \underline{P}_r: x\underline{P}_r y=(xy)P\cup x(yP),\quad \underline{P}_l: x\underline{P}_l y=(Px)y\cup P(xy).$$

The $(G,\underline{P})$, $(G,\underline{P}_r)$ and $(G,\underline{P}_l)$ are called *P-hyperstructures*. The usual case is if $(G,\cdot)$ is semigroup, then $x\underline{P}y=(xP)y\cup x(Py)=xPy$ and $(G,\underline{P})$ is a semihypergroup. In some cases, a depending on the choice of P, the $(G,\underline{P}_r)$ and $(G,\underline{P}_l)$ can be associative or WASS.

A generalization of P-hopes is the following [4]:

Let $(G,\cdot)$ be abelian group, P any subset of G with more than one element. We define the hope $\times_P$ as follows:

$$x\times_P y=\begin{cases} x\cdot P\cdot y=\{x\cdot h\cdot y\mid h\in P\} & ;\text{ if } x\neq e \text{ and } y\neq e\\ \\ x\cdot y & ;\text{ if } x=e \text{ or } y=e\end{cases}$$

We call this hope P_e-*hope*. The hyperstructure $(G,\times_P)$ is an abelian H_v-group.

Let $(H,\cdot)$ be hypergroupoid. We remove $h\in H$, if we take the restriction of $(\cdot)$ in $H-\{h\}$. $\underline{h}\in H$ absorbs $h\in H$ if we replace h by $\underline{h}$. $\underline{h}\in H$ merges with $h\in H$, if we take as the product of any $x\in H$ by $\underline{h}$, the union of the results of x with both h, $\underline{h}$ and consider them in the same class with representative $\underline{h}$.

2 Fundamental Relations

The main tool to study the hyperstructures is the fundamental relation. In 1970 [8] M. Koskas defined in hypergroups the relation β and its transitive closure β^*. This relation connects the hyperstructures with the corresponding classical structures and is defined in H_v-groups as well. T. Vougiouklis [14], [15], [16] and [22] introduced the γ^* and ε^* relations, which are defined, in H_v-rings and H_v-vector spaces, respectively. He also named all these relations β^*, γ^* and ε^*, fundamental relations because they play a very important role in the study of hyperstructures, espicially in their representation theory of them. In 1991, D. Freni [7], proved an open problem that for the classical hypergroups, where the equality is valid, we have $\beta^*=\beta$. However, this problem is open for H_v-groups, therefore, some special classes of them are investigated for which the $\beta^*=\beta$, is valid.

Definition 2.1. The ***fundamental relations*** β^*, γ^*, and ε^* are defined in H_v-groups, H_v-rings, and H_v-vector spaces, respectively, as the smallest equivalences so that the quotient would be group, ring, and vector spaces, respectively.

Remark 2.2. *Let $(G,\cdot)$ be a group and R be any partition in G, then $(G/R,\cdot)$ is an H_v-*group*, so the quotient $(G/R,\cdot)/\beta^*$ is a group, the fundamental one. The classes of the fundamental group $(G/R,\cdot)/\beta^*$ are a union of some of the R-classes.*

The main theorem together with a way to find the fundamental classes is the following:

Theorem 2.3. *Let $(H,\cdot)$ be H_v-*group *and denote by U the set of all finite products of elements of H. Define the relation β in H by $x\beta y$ iff $\{x,y\}\subset u$ where $u\in U$. Then β^* is the transitive closure of β.*

We present a proof for the analogous to the above theorem in the case of an H_v-ring [14], [15], [16] and [6]:

Theorem 2.4. *Let $(R,+,\cdot)$ be an H_v-ring. Denote by U the set of all finite polynomials of elements of R. We define the relation γ in R as follows:*

$$x\ \gamma\ y \quad \textit{iff} \ \ \{x,y\} \subset u, \quad \textit{where} \ \ u \in U.$$

Then, the relation γ^* is the transitive closure of the relation γ.

Proof. Let $\underline{\gamma}$ be the transitive closure of γ, and denote by $\underline{\gamma}(a)$ the class of the element a. First, we prove that the quotient set $R/\underline{\gamma}$ is a ring.

In $R/\underline{\gamma}$ the sum ($\oplus$) and the product ($\otimes$) are defined in the usual manner:

$$\underline{\gamma}(a) \oplus \underline{\gamma}(b) = \{\underline{\gamma}(c) \ : \ c \in \underline{\gamma}(a) + \underline{\gamma}(b)\},$$

$$\underline{\gamma}(a) \otimes \underline{\gamma}(b) = \{\underline{\gamma}(d) : d \in \underline{\gamma}(a) \cdot \underline{\gamma}(b)\}, \quad \forall a,b \in R.$$

Take $a' \in \underline{\gamma}(a)$ and $b' \in \underline{\gamma}(b)$. Then we have $a'\underline{\gamma}\ a$ iff $\exists\ x_1,\dots,x_{m+1}$ with $x_1 = a',\ x_{m+1} = a$ and $u_1,\dots,u_m \in U$ such that $\{x_i,x_{i+1}\} \subset u_i,\ i = 1,\dots,m$ and $b'\underline{\gamma}\ b$ iff $\exists\ y_1,\dots,y_{n+1}$ with $y_1 = b',\ y_{n+1} = b$ and $v_1,\dots,v_n \in U$ such that $\{y_j,y_{j+1}\} \subset v_j,\ j = 1,\dots,n$.

From the above we obtain

$$\{x_i,x_{i+1}\} + y_1 \subset u_i + v_1,\ i = 1,\dots,m-1 \quad \text{and} \quad x_{m+1} + \{y_j,y_{j+1}\} \subset u_m + v_j,\ j = 1,\dots,n.$$

The sums

$$u_i + v_1 = t_i,\ i = 1,\dots,m-1 \quad \text{and} \quad u_m + v_j = t_{m+j-1},\ j = 1,\dots,n,$$

are also polynomials, therefore $t_k \in U$ for all $k \in \{1,\dots,m+n-1\}$.

Now, pick up elements $z_1,\dots,z_{m+n}$ such that

$$z_i \in x_i + y_1,\ i = 1,\dots,n \quad \text{and} \quad z_{m+j} \in x_{m+1} + y_{j+1},\ j = 1,\dots,n,$$

therefore, using the above relations we obtain $\{z_k,z_{k+1}\} \subset t_k,\ k = 1,\dots,m+n-1$.

Thus, every element $z_1 \in x_1 + y_1 = a' + b'$ is $\underline{\gamma}$ equivalent to every element $z_{m+n} \in x_{m+1} + y_{n+1} = a + b$.

Thus $\underline{\gamma}(a) \oplus \underline{\gamma}(b)$ is a singleton so we can write

$$\underline{\gamma}(a) \oplus \underline{\gamma}(b) = \underline{\gamma}(c), \quad \forall c \in \underline{\gamma}(a) + \underline{\gamma}(b).$$

In a similar way, we prove that

$$\underline{\gamma}(a) \otimes \underline{\gamma}(b) = \underline{\gamma}(d), \quad \forall d \in \underline{\gamma}(a) \cdot \underline{\gamma}(b).$$

The WASS and the weak distributivity on R guarantee that the associativity and the distributivity are valid for the quotient R/γ^*. Therefore, R/γ^* is a ring.

Now let σ be an equivalence relation in R such that R/σ is a ring. Denote $\sigma(a)$ the class of a. Then $\sigma(a) \oplus \sigma(b)$ and $\sigma(a) \otimes \sigma(b)$ are singletons, i.e. $\forall a,b \in R$, we have

$$\sigma(a) \oplus \sigma(b) = \sigma(c),\ \forall c \in \sigma(a) + \sigma(b) \quad \text{and} \quad \sigma(a) \otimes \sigma(b) = \sigma(d),\ \forall d \in \sigma(a) \cdot \sigma(b).$$

Thus we can write, $\forall a,b \in R$ and $A \subset \sigma(a)$, $B \subset \sigma(b)$,

$$\sigma(a) \oplus \sigma(b) = \sigma(a+b) = \sigma(A+B) \quad \text{and} \quad \sigma(a) \otimes \sigma(b) = \sigma(ab) = \sigma(A \cdot B).$$

By induction, we extend these relations on finite sums and products. Thus, $\forall u \in U$, we have $\sigma(x) = \sigma(u)$, $\forall x \in u$. Consequently,

$$x \in \gamma(a) \quad \text{implies} \quad x \in \sigma(a),\ \forall x \in R.$$

But σ is transitively closed, so we obtain:

$$x \in \underline{\gamma}(x) \quad \text{implies} \quad x \in \sigma(a).$$

That means that $\underline{\gamma}$ is the smallest equivalence relation in R such that $R/\underline{\gamma}$ is a ring, i.e. $\underline{\gamma} = \gamma^*$. □

An element is called ***single*** if its fundamental class is singleton [15].

Fundamental relations are used for general definitions. Thus we have [14]:

Definition 2.5. An H_v-ring $(R, +, \cdot)$ is called ***H_v-field*** if R/γ^* is a field.

The analogous to Theorem 2.4 on H_v-vector spaces, can be proved:

Let $(V, +)$ be H_v-vector space over the H_v-field F. Denote U the set of all expressions of finite hopes on finite sets of elements of F and V. Define the relation ε, in V, as follows: $x\varepsilon y$ iff $\{x, y\} \subset u$ where $u \in U$. Then ε^* is the transitive closure of ε.

Definition 2.6. Let $(L, +)$ be H_v-vector space over an H_v-field $(F, +, \cdot)$; $\varphi : F \to F/\gamma^*$ the canonical map; $\omega_F = \{x \in F \ : \ \varphi(x) = 0\}$, the core, 0 is the zero of F/γ^*. Let ω_L be the core of $\varphi' : L \to L/\varepsilon^*$ and denote by 0 the zero of L/ε^*, as well. Take the *bracket (commutator) hope*:

$$[\ ,\ ] : L \times L \to P(L) : (x, y) \mapsto [x, y]$$

then L is an ***H_v-Lie algebra*** over F if the following axioms are satisfied:

(L1) The bracket hope is bilinear, i.e.

$[\lambda_1 x_1 + \lambda_2 x_2, y] \cap (\lambda_1[x_1, y] + \lambda_2[x_2, y]) \neq \varnothing$

$[x, \lambda_1 y_1 + \lambda_2 y] \cap (\lambda_1[x, y_1] + \lambda_2[x, y_2]) \neq \varnothing,\ \forall x, x_1, x_2, y, y_1, y_2 \in L$ and $\forall \lambda_1, \lambda_2 \in F$

(L2) $[x, x] \cap \omega_L \neq \varnothing,\ \forall x \in L$

(L3) $([x, [y, z]] + [y, [z, x]] + [z, [x, y]]) \cap \omega_L \neq \varnothing,\ \forall x, y, z \in L$

Definition 2.7. (See [18] and [21]) The H_v-semigroup $(H, \cdot)$ is called ***h/v-group*** if H/β^* is a group.

The H_v-group is a generalization of H_v-group, where a reproductive of classes, is valid: if $\sigma(x)$, $\forall x \in H$, equivalence classes, then $x\sigma(y) = \sigma(xy) = \sigma(x)y$, $\forall x, y \in H$. Similarly, h/v-rings, h/v-fields, h/v-vector spaces etc, are defined.

The ***uniting elements*** method, introduced by Corsini & Vougiouklis in 1989, is the following [2]: Let $\boldsymbol{G}$ be a structure and a not valid property d, described by a set of equations. Take the partition in $\boldsymbol{G}$ for which put in the same class, all pairs of elements that cause the non-validity of d. The quotient by this partition $\boldsymbol{G}/d$ is an H_v-structure. Then, quotient out $\boldsymbol{G}/d$ by β^*, is a stricter structure $(\boldsymbol{G}/d)/\beta^*$ for which the property d is valid.

Theorem 2.8. (See [15]) *Let $(\boldsymbol{R}, +, \cdot)$ be a ring, and $F = \{f_1, \ldots, f_m, f_{m+1}, \ldots, f_{m+n}\}$ be system of equations on $\boldsymbol{R}$ consisting of subsystems $F_m = \{f_1, \ldots, f_m\}$ and $F_n = \{f_{m+1}, \ldots, f_{m+n}\}$. Let σ, σ_m be the equivalence relations defined by the uniting elements using F and F_m respectively, and σ_n the equivalence defined on F_n on the ring $\boldsymbol{R}_m = (\boldsymbol{R}/\sigma_m)/\gamma^*$. Then*

$$(\boldsymbol{R}/\sigma)/\gamma^* \cong (\boldsymbol{R}_m/\sigma_n)/\gamma^*.$$

Theorem 2.9. *Let $(\boldsymbol{H}, \cdot)$ be an H_v-group and $\boldsymbol{H}/\beta^*$ its fundamental group. Suppose that $\boldsymbol{H}/\beta^*$ is not commutative or it is not cyclic, then $(\boldsymbol{H}, \cdot)$ is not COW or cyclic, respectively.*

Proof. Straightforward since if $(\boldsymbol{H}, \cdot)$ is COW or cyclic then its fundamental group $\boldsymbol{H}/\beta^*$ is commutative or cyclic, respectively. □

3 H_v-fields

Definition 3.1. We call ***Raised Very Thin H_v-fields*** the ones obtained from classical rings by enlarging only one result adding only one element, of the underline set, such that the fundamental structure is a field.

Combining the uniting elements procedure with the raise theory we can obtain stricter structures or hyperstructures. So, raising operations or hopes we can obtain more complicated structures as we can see in the following.

Theorem 3.2. *In the ring of integers $(\boldsymbol{Z},+,\cdot)$, we fix a number $m>1$. We raise in the product the special result $0\cdot m$ by setting $0\otimes m=\{0,m\}$ and the rest results remain the same. Then $(\boldsymbol{Z},+,\otimes)$ becomes an* H_v-ring*, with a finite fundamental ring:*

$$(\boldsymbol{Z},+,\otimes)/\gamma^*\cong(\boldsymbol{Z}_m,+,\cdot).$$

If $m=p$, prime, then $(\boldsymbol{Z},+,\otimes)$ is a raised very thin H_v-field*, with the finite fundamental field.*
Raising only the result $a\cdot b$ of two fixed elements $a,b\in\boldsymbol{Z}-\{0,1\}$, by setting $a\otimes b=\{a\cdot b,a\cdot b+m\}$, then we have the same results and $(\boldsymbol{Z},+,\otimes)$ is a raised very thin H_v-field*, where the elements 0 and 1 are scalars.*

Proof. Remark that the expressions of sums and products which contain more than one element are the ones that have at least one time the $0\otimes m$. Adding to $0\otimes m$ the element 1, several times we have the modm equivalence classes. On the other side, by adding or multiplying elements of the same class the results are remaining in one class, the class obtained by using only the representatives. Therefore, the γ^*-classes form a ring isomorphic to $(\boldsymbol{Z}_m,+,\cdot)$.

The rest of the proof is straightforward. Notice only that we can transfer the generalized raised case if we consider the expression $a\otimes b-a\cdot b=\{0,m\}$. □

Theorem 3.3. *In the ring $(\boldsymbol{Z}_n,+,\cdot)$, with $n=ms$ we raise in the product only the result $0\cdot m$ by setting $0\otimes m=\{0,m\}$ and the rest results remain the same. Then*

$$(\boldsymbol{Z}_n,+,\otimes)/\gamma^*\cong(\boldsymbol{Z}_m,+,\cdot).$$

If $m=p$, prime, then $(\boldsymbol{Z}_n,+,\otimes)$ is a raised very thin H_v-field.
Raising only the result $a\cdot b$ of two fixed elements $a,b\in\boldsymbol{Z}_n-\{0,1\}$, by setting $a\otimes b=\{a\cdot b,a\cdot b+m\}$, then we have the same results but $(\boldsymbol{Z}_n,+,\otimes)$ is a raised very thin H_v-field*, where, moreover, the elements 0 and 1 are scalars.*

Proof. Analogous to the above Theorem. □

Now, we focus on raised very thin minimal H_v-fields obtained by a classical field.

Theorem 3.4. *In a field $(\boldsymbol{F},+,\cdot)$, we raise only the product of two elements $a\cdot b$, by $a\otimes b=\{a\cdot b,c\}$, where $c\neq a\cdot b$, and the rest results remain the same. Then we obtain the degenerate, minimal very thin,* H_v-field $(\boldsymbol{F},+,\otimes)/\gamma^*\cong\{0\}$.
Thus, there is no non-degenerate H_v-field *obtained by a field by raising any product.*

Proof. Take any $x\in\boldsymbol{F}-\{0\}$, then from $a\otimes b=\{ab,c\}$ we obtain $(a\otimes b)-ab=\{0,c-ab\}$ and then $(x(c-ab)^{-1})\otimes((a\otimes b)-ab)=\{0,x\}$. thus, $0\gamma x, x\in\boldsymbol{F}-\{0\}$. Which means that every x is in the same fundamental class with 0. Thus, $(\boldsymbol{F},+,\otimes)/\gamma^*\cong\{0\}$. □

Theorem 3.5. *In a field $(\boldsymbol{F},+,\cdot)$, we raise only the sum of two elements $a+b$, by setting $a\oplus b=\{a+b,c\}$, where $c\neq a+b$, and the rest results remain the same. Then we obtain the degenerate, minimal very thin,* H_v-field $(\boldsymbol{F},\oplus,\cdot)/\gamma^*\cong\{0\}$.
Thus, there is no non-degenerate H_v-field *obtained by a field by raising any sum.*

Proof. Take any $x \in \boldsymbol{F} - \{0\}$, then from $a \oplus b = \{a+b, c\}$ we obtain $(a \oplus b) - (a+b) = \{0, c-(a+b)\}$ and then $[x(c-(a+b))^{-1}] \cdot [(a \oplus b) - (a+b)] = \{0, x\}$. Thus, $0 \ \gamma \ x, x \in \boldsymbol{F} - \{0\}$. Which means that every x is in the same fundamental class with the element 0. Thus, $(\boldsymbol{F}, \oplus, \cdot)/\gamma^* \cong \{0\}$. □

The above two theorems state that all H_v-fields obtained from a field by raising any sum or product, are degenerate.

Several results can be obtained by using ∂-hopes [19]: For example, consider the group of integers $(\boldsymbol{Z}, +)$ and $n \neq 0$ be natural number. Take the map f such that $f(0) = n$ and $f(x) = x$, $\forall x \in \boldsymbol{Z} - \{0\}$, then $(\boldsymbol{Z}, \partial)/\beta^* \cong (\boldsymbol{Z}_n, +)$.

Theorem 3.6. *Take the ring of integers $(\boldsymbol{Z}, +, \cdot)$ and fix $n \neq 0$ a natural number. Consider the map f such that $f(0) = n$ and $f(x) = x$, $\forall x \in \boldsymbol{Z} - \{0\}$. Then $(\boldsymbol{Z}, \partial_+, \partial.)$, where ∂_+ and $\partial.$ are the ∂-*hopes *refereed to the sum and the product, respectively, is an H_v*-near-ring*, with*

$$(\boldsymbol{Z}, \partial_+, \partial.)/\gamma^* \cong \boldsymbol{Z}_n.$$

We have the same result if we consider the map f such that $f(n) = 0$ and $f(x) = x$, $\forall x \in \boldsymbol{Z} - \{n\}$.

A special case of the above is for $n = p$, prime, then $(\boldsymbol{Z}, \partial_+, \partial.)$ is an H_v-field.

From the very thin hopes the Attach Construction is obtained [20]:

Definition 3.7. ***(a)*** Let $(H, \cdot)$ be an H_v-semigroup, $v \notin H$. We extend $(\cdot)$ into $\underline{H} = H \cup \{v\}$ by:

$$x \cdot v = v \cdot x = v, \ \forall x \in H \ \text{ and } \ v \cdot v = H.$$

The $(\underline{H}, \cdot)$ is called *attach h/v-group* of $(H, \cdot)$, where $(\underline{H}, \cdot)/\beta^* \cong \boldsymbol{Z}_2$ and v is single. Scalars and units of $(H, \cdot)$ are scalars and units in $(\underline{H}, \cdot)$. If $(H, \cdot)$ is *COW* then $(\underline{H}, \cdot)$ is *COW*.
(b) $(H, \cdot)$ H_v-semigroup, $v \notin H$, $(\underline{H}, \cdot)$ its attached h/v-group. Take $0 \notin \underline{H}$ and define in $\underline{H}_\circ = H \cup \{v, 0\}$ two hopes:

$$\text{hypersum}(+) : 0+0 = x+v = v+x = 0, \ 0+v = v+0 = x+y = v, \ 0+x = x+0 = v+v = H, \ \forall x, y \in H$$

$$\text{hyperproduct } (\cdot) : \text{remains the same as in } \underline{H}, \ \text{moreover, } \ 0 \cdot 0 = v \cdot x = x \cdot 0 = 0, \ \forall x \in \underline{H}.$$

Then $(\underline{H}_\circ, +, \cdot)$ is an h/v-field with $(\underline{H}_\circ, +, \cdot)/\gamma^* \cong \boldsymbol{Z}_3$. $(+)$ is associative, $(\cdot)$ is WASS and weak distributive to $(+)$. 0 is zero absorbing in $(+)$. $(\underline{H}_\circ, +, \cdot)$ is the *attached h/v-field* of $(H, \cdot)$.

Let $(G, \cdot)$ be semigroup and $v \notin G$ be an element appearing in a product ab, where $a, b \in G$, thus the result becomes $a \otimes b = \{ab, v\}$. Then the minimal hope $(\otimes)$ extended in $G' = G \cup \{v\}$ such that $(\otimes)$ contains $(\cdot)$ in the restriction on G, and such that $(G', \otimes)$ is a minimal H_v-semigroup which has a fundamental structure isomorphic to $(G, \cdot)$, is defined as follows:

$$a \otimes b = \{ab, v\}, \quad x \otimes y = xy, \quad \forall (x, y) \in G^2 - \{(a, b)\}$$

$$v \otimes v = abab, \quad x \otimes v = xab \quad \text{and} \quad v \otimes x = abx, \quad \forall x \in G.$$

$(G', \otimes)$ is very thin H_v-semigroup. If $(G, \cdot)$ is commutative then $(G', \otimes)$ is strong commutative.

4 Representations and applications

H_v-structures used in Representation Theory (abbreviate ***rep***) of H_v-groups can be achieved by generalized permutations or by H_v-matrices [6], [15], [17].

$\boldsymbol{H_v}$**-matrix** is a matrix with entries of an H_v-ring. The hyperproduct of two H_v-matrices (a_{ij}) and (b_{ij}), of type $m \times n$ and $n \times r$ respectively, is defined in the usual manner and it is a set of $m \times r$ H_v-matrices. The sum of products of elements of the H_v-ring is the n-ary circle hope on the hyper-sum.

Notation. In a set of matrices or H_v-matrices, we denote by E_{ij} the matrix with 1 in the ij-entry and zero in the rest entries.

The problem of the H_v-matrix reps is the following:

Definition 4.1. Let $(H, \cdot)$ be H_v-group. Find an H_v-ring $(R, +, \cdot)$, a set $M_R = \{(a_{ij}) \mid a_{ij} \in R\}$ and a map $T : H \to M_R : h \mapsto T(h)$, called ***$H_v$-matrix rep***, such that

$$T(h_1h_2) \cap T(h_1)T(h_2) \neq \varnothing, \ \forall h_1, h_2 \in H.$$

If $T(h_1h_2) \subset T(h_1)T(h_2)$, then T is an *inclusion rep.*
If $T(h_1h_2) = T(h_1)T(h_2) = \{T(h) \mid h \in h_1h_2\}$, then T is a *good rep.*
If T is a *good rep* and one to one then it is a *faithful rep.*

The rep problem is simplified in cases such as if the H_v-rings have scalars 0 and 1.

The main theorem of the theory of reps is the following:

Theorem 4.2. *A necessary condition to have an inclusion rep T of an H_v-group $(H, \cdot)$ by $n \times n$, H_v-matrices over the H_v-ring $(R, +, \cdot)$ is the following:*

$\forall \beta^*(x),\ x \in H$ there must exist elements $a_{ij} \in H$, $i, j \in \{1, \dots, n\}$ such that

$$T(\beta^*(a)) \subset \{A = (a'_{ij}) \mid a'_{ij} \in \gamma^*(a_{ij}),\ i, j \in \{1, \dots, n\}\}$$

The inclusion rep $T : H \longrightarrow M_R : a \mapsto T(a) = (a_{ij})$ induces a homomorphic

$$T^* : H/\beta^* \longrightarrow R/\gamma^* : T^*(\beta^*(a)) = [\gamma^*(a_{ij})], \quad \forall \beta^*(a) \in H/\beta^*,$$

where $\gamma^*(a_{ij}) \in R/\gamma^*$ is the ij entry of $T^*(\beta^*(a))$.

An important hope on non-square matrices is defined [5] and [6]:

Definition 4.3. Let $A = (a_{ij}) \in M_{m\times n}$ and $s, t \in N$, $1 \le s \le m$, $1 \le t \le n$. Define a mod-like map, called *helix-projection* of type $\underline{st}$, $\underline{st} : M_{m\times n} \to M_{s\times t} : A \to A\underline{st} = (\underline{a}_{ij})$, where A has entries the sets

$$\underline{a}_{ij} = \{a_{i+\kappa s, j+\lambda t} \mid 1 \le i \le s, \ \ 1 \le j \le t \quad \text{and} \quad \kappa, \lambda \in N, \ \ i + \kappa s \le m, \ \ j + \lambda t \le n\}.$$

$A\underline{st}$ is a set of s × t-matrices $X = (x_{ij})$ such that $x_{ij} \in \underline{a}_{ij}$, $\forall i, j$. Obviously, $A\underline{mn} = A$.
Let $A = (a_{ij}) \in M_{m\times n}$ and $B = (b_{ij}) \in M_{u\times v}$ be matrices.
Denote $s = \min(m, u)$, $t = \min(n, u)$, then we define the ***helix-sum*** by

$$\oplus : M_{m\times n} \times M_{u\times v} \to P(M_{s\times t}) : (A, B) \to A \oplus B = A\underline{st} + B\underline{st} = (\underline{a}_{ij}) + (\underline{b}_{ij}) \subset M_{s\times t},$$

where $(\underline{a}_{ij}) + (\underline{b}_{ij}) = \{(c_{ij}) = (a_{ij} + b_{ij}) \mid a_{ij} \in \underline{a}_{ij} \quad \text{and} \quad b_{ij} \in \underline{b}_{ij}\}$.
Denote $s = \min(n, u)$, then we define the ***helix-product*** by

$$\otimes : M_{m\times n} \times M_{u\times v} \to P(M_{m\times v}) : (A, B) \to A \otimes B = A\underline{ms} \cdot B\underline{sv} = (\underline{a}_{ij}) \cdot (\underline{b}_{ij}) \subset M_{m\times v},$$

where $(\underline{a}_{ij}) \cdot (\underline{b}_{ij}) = \{(c_{ij}) = (\sum a_{it}b_{tj}) \mid a_{ij} \in \underline{a}_{ij} \quad \text{and} \quad b_{ij} \in \underline{b}_{ij}\}$.

The helix-sum is commutative and WASS. The helix-product is WASS.

The definition of a Lie-bracket is immediate, so, the *helix-Lie Algebra* is defined.

Using several classes of H_v-structures one can face several representations [15]:

Let $\boldsymbol{M} = \boldsymbol{M}_{m\times n}$ be a module of $m \times n$ matrices over a ring $\boldsymbol{R}$ and $\boldsymbol{P} = \{P_i \ : \ i \in I\} \subseteq M$. We define, a kind of, a P-hope $\underline{P}$ on $\boldsymbol{M}$ as follows

$$\underline{P} : \boldsymbol{M} \times \boldsymbol{M} \to \boldsymbol{P}(\boldsymbol{M}) : (A, B) \to A\underline{P}B = \{AP_i^t B \ : \ i \in I\} \subseteq \boldsymbol{M}$$

where P^t denotes the transpose of the matrix P.

In last decades the hyperstructures had a variety of applications in other branches of mathematics and in many other sciences. These applications range from biomathematics - conchology, inheritance- and hadronic physics or on leptons to mention but a few. The hyperstructures theory is closely related to fuzzy theory; consequently, hyperstructures can now be widely applicable in industry and production, too. In several books and papers [1], [3], [4], [6] and [22], one can find numerous applications.

The Lie-Santilli theory on isotopies was born in the 1960s to solve Hadronic Mechanics problems. Santilli proposed a lifting of the n-dimensional trivial unit matrix of a normal theory into a nowhere singular, symmetric, real-valued, positive-defined, n-dimensional new matrix [9], [10], [11]. The original theory is reconstructed such as to admit the new matrix as left and right unit. The isofields, needed in this theory correspond to the hyperstructures, were introduced by Santilli & Vougiouklis in 1999 [4], [6], [11].

Definition 4.4. $(F, +, \cdot)$, where $(+)$ is operation and $(\cdot)$ hope, is an ***e-hyperfield*** if the following are valid: $(F, +)$ is an abelian group with unit 0, $(\cdot)$ is WASS, $(\cdot)$ is weak distributive to $(+)$, 0 is absorbing: $0 \cdot x = x \cdot 0 = 0$, $\forall x \in F$, there exist a scalar unit 1, i.e. $1 \cdot x = x \cdot 1 = x$, $\forall x \in F$, and $\forall x \in F$ there is a unique inverse $x^{-1} : 1 \in x \cdot x^{-1} \cap x^{-1} \cdot x$. If the relation: $1 = x \cdot x^{-1} = x^{-1} \cdot x$, is valid, then we have a *strong e-hyperfield.*

The Main e-Construction: Given a group $(G, \cdot)$, e unit, define hopes $(\otimes)$ by:

$$x \otimes y = \{xy, g_1, g_2, \ldots\}, \ \forall x, y \in G - \{e\} \quad \text{and} \quad g_1, g_2, \ldots \in G - \{e\}$$

$(G, \otimes)$ is H_b-group which contains $(G, \cdot)$. $(G, \otimes)$ is e-hypergroup. Moreover, if $\forall x, y$ such that $xy = e$, so $x \otimes y = e$, then $(G, \otimes)$ becomes a strong e-hypergroup.

Example 4.5. In the set of quaternions $\boldsymbol{Q} = \{1, -1, i, -i, j, -j, k, -k\}$, with $i^2 = j^2 = -1$, $ij = -ji = k$, we denote $\underline{i} = \{i, -i\}$, $\underline{j} = \{j, -j\}$, $\underline{k} = \{k, -k\}$ and we define hopes $(*)$ by enlarging few products. For example, $(-1) * k = \underline{k}$, $k * i = \underline{j}$ and $i * j = \underline{k}$. Then $(\boldsymbol{Q}, *)$ is strong e-hypergroup.

5 On 2×2 Very Thin H_v-matrix representations

From now to the end we focus on the small non-degenerate H_v-fields on $(\boldsymbol{Z}_n, +, \cdot)$, which in isotheory, satisfy the following conditions:

1. very thin minimal,
2. COW (non-commutative),
3. they have the elements 0 and 1, scalars,
4. if an element has an inverse element, this is unique.

Therefore, we cannot raise the result if it is 1 and we cannot put 1 in enlargement.

We present some known results and examples on the topic [20], [21] and [23], along with some new ones.

Theorem 5.1. *The multiplicative H_v-fields on $(\boldsymbol{Z}_4, +, \cdot)$, with non-degenerate fundamental field, satisfying the above 4 conditions, are the following isomorphic ones:*
The only product which is set is $2 \otimes 3 = \{0, 2\}$ or $3 \otimes 2 = \{0, 2\}$.
Fundamental classes: $[0] = \{0, 2\}$, $[1] = \{1, 3\}$ and we have $(\boldsymbol{Z}_4, +, \otimes)/\gamma^* \cong (\boldsymbol{Z}_2, +, \cdot)$.

Example 5.2. Take the 2×2 upper triangular H_v-matrices on the above H_v-field $(\boldsymbol{Z}_4,+,\otimes)$ of the case that only $2\otimes 3=\{0,2\}$ is a hyperproduct:

$$I=E_{11}+E_{22},\quad a=E_{11}+E_{12}+E_{22},\quad b=E_{11}+2E_{12}+E_{22},\quad c=E_{11}+3E_{12}+E_{22},$$

$$d=E_{11}+3E_{22},\quad e=E_{11}+E_{12}+3E_{22},\quad f=E_{11}+2E_{12}+3E_{22},\quad g=E_{11}+3E_{12}+3E_{22},$$

then, for $\boldsymbol{X}=\{I,a,b,c,d,e,f,g\}$, we obtain the following multiplicative table:

$\otimes$	$\boldsymbol{I}$	$\boldsymbol{a}$	$\boldsymbol{b}$	$\boldsymbol{c}$	$\boldsymbol{d}$	$\boldsymbol{e}$	$\boldsymbol{f}$	$\boldsymbol{g}$
$\boldsymbol{I}$	I	a	b	c	d	e	f	g
$\boldsymbol{a}$	a	b	c	I	g	d	e	f
$\boldsymbol{b}$	b	c	I	a	d,f	e,g	d,f	e,g
$\boldsymbol{c}$	c	I	a	b	e	f	g	d
$\boldsymbol{d}$	d	e	f	g	I	a	b	c
$\boldsymbol{e}$	e	f	g	d	c	I	a	b
$\boldsymbol{f}$	f	g	d	e	I,b	a,c	I,b	a,c
$\boldsymbol{g}$	g	d	e	f	a	b	c	b

The $(\boldsymbol{X},\otimes)$ is COW H_v-group where the fundamental classes are $\underline{I}=\{I,b\}$, $\underline{a}=\{a,c\}$, $\underline{d}=\{d,f\}$, $\underline{e}=\{e,g\}$ and the fundamental group is isomorphic to $(\boldsymbol{Z}_2\times\boldsymbol{Z}_2,+)$. There is only one unit and every element has a unique double inverse. Only f has one more right inverse element d, since $f\otimes d=\{I,b\}$. $(\boldsymbol{X},\otimes)$ is not cyclic.

Example 5.3. Consider the 2×2 upper triangular H_v-matrices on the above H_v-field $(\boldsymbol{Z}_4,+,\otimes)$ of the case that only $2\otimes 3=\{0,2\}$ is a hyperproduct:

$$a=E_{11}+E_{22},\quad a_1=E_{11}+E_{12}+E_{22},\quad a_2=E_{11}+2E_{12}+E_{22},\quad a_3=E_{11}+3E_{12}+E_{22},$$

$$b=E_{11}+3E_{22},\quad b_1=E_{11}+E_{12}+3E_{22},\quad b_2=E_{11}+2E_{12}+3E_{22},\quad b3=E_{11}+3E_{12}+3E_{22},$$

$$c=3E_{11}+E_{22},\quad c_1=3E_{11}+E_{12}+E_{22},\quad c_2=3E_{11}+2E_{12}+E_{22},\quad c3=3E_{11}+3E_{12}+E_{22},$$

$$d=3E_{11}+3E_{22},\quad d_1=3E_{11}+E_{12}+3E_{22},\quad d_2=3E_{11}+2E_{12}+3E_{22},\quad d_3=3E_{11}+3E_{12}+3E_{22},$$

then, for $\boldsymbol{X}=\{a,a_1,a_2,a_3,b,b_1,b_2,b_3,c,c_1,c_2,c_3,d,d_1,d_2,d_3\}$, we obtain the following multiplicative table:

$\otimes$	$\boldsymbol{a}$	$\boldsymbol{a_1}$	$\boldsymbol{a_2}$	$\boldsymbol{a_3}$	$\boldsymbol{b}$	$\boldsymbol{b_1}$	$\boldsymbol{b_2}$	$\boldsymbol{b_3}$	$\boldsymbol{c}$	$\boldsymbol{c_1}$	$\boldsymbol{c_2}$	$\boldsymbol{c_3}$	$\boldsymbol{d}$	$\boldsymbol{d_1}$	$\boldsymbol{d_2}$	$\boldsymbol{d_3}$
$\boldsymbol{a}$	a	a_1	a_2	a_3	b	b_1	b_2	b_3	c	c_1	c_2	c_3	d	d_1	d_2	d_3
$\boldsymbol{a_1}$	a_1	a_2	a_3	a	b_3	b	b_1	b_2	c_1	c_2	c_3	c	d_3	d	d_1	d_2
$\boldsymbol{a_2}$	a_2	a_3	a	a_1	b, b_2	b_1, b_3	b, b_2	b_1, b_3	c_2	c_3	c	c_1	d, d_2	d_1, d_3	d, d_2	d_1, d_3
$\boldsymbol{a_3}$	a_3	a	a_1	a_2	b_1	b_2	b_3	b	c_3	c	c_1	c_2	d_1	d_2	d_3	d
$\boldsymbol{b}$	b	b_1	b_2	b_3	a	a_1	a_2	a_3	d	d_1	d_2	d_3	c	c_1	c_2	c_3
$\boldsymbol{b_1}$	b_1	b_2	b_3	b	a_3	a	a_1	a_2	d_1	d_2	d_3	d	c_3	c	c_1	c_2
$\boldsymbol{b_2}$	b_2	b_3	b	b_1	a, a_2	a_1, a_3	a, a_2	a_1, a_3	d_2	d_3	d	d_1	c, c_2	c_1, c_3	c, c_2	c_1, c_3
$\boldsymbol{b_3}$	b_3	b	b_1	b_2	a_1	a_2	a_3	a	d_3	d	d_1	d_2	c_1	c_2	c_3	c
$\boldsymbol{c}$	c	c_3	c_2	c_1	d	d_3	d_2	d_1	a	a_3	a_2	a_1	b	b_3	b_2	b_1
$\boldsymbol{c_1}$	c_1	c	c_3	c_2	d_3	d_2	d_1	d	a_1	a	a_3	a_2	b_3	b_2	b_1	b
$\boldsymbol{c_2}$	c_2	c_1	c	c_3	d, d_2	d_1, d_3	d, d_2	d_1, d_3	a_2	a_1	a	a_3	b, b_2	b_1, b_3	b, b_2	b_1, b_3
$\boldsymbol{c_3}$	c_3	c_2	c_1	c	d_1	d	d_3	d_2	a_3	a_2	a_1	a	b_1	b	b_3	b_2
$\boldsymbol{d}$	d	d_3	d_2	d_1	c	c_3	c_2	c_1	b	b_3	b_2	b_1	a	a_3	a_2	a_1
$\boldsymbol{d_1}$	d_1	d	d_3	d_2	c_3	c_2	c_1	c	b_1	b	b_3	b_2	a_3	a_2	a_1	a
$\boldsymbol{d_2}$	d_2	d_1	d	d_3	c, c_2	c_1, c_3	c, c_2	c_1, c_3	b_2	b_1	b	b_3	a, a_2	a_1, a_3	a, a_2	a_1, a_3
$\boldsymbol{d_3}$	d_3	d_2	d_1	d	c_1	c	c_3	c_2	b_3	b_2	b_1	b	a_1	a	a_3	a_2

The $(\boldsymbol{X}, \otimes)$ is a COW H_v-group where the fundamental classes are $\underline{a} = \{a, a_2\}$, $\underline{a}_1 = \{a_1, a_3\}$, $\underline{b} = \{b, b_2\}$, $\underline{b}_1 = \{b_1, b_3\}$, $\underline{c} = \{c, c_2\}$, $\underline{c}_1 = \{c_1, c_3\}$, $\underline{d} = \{d, d_2\}$, $\underline{d}_1 = \{d_1, d_3\}$, with multiplicative table the following:

$\underline{\otimes}$	$\underline{\boldsymbol{a}}$	$\underline{\boldsymbol{a_1}}$	$\underline{\boldsymbol{b}}$	$\underline{\boldsymbol{b_1}}$	$\underline{\boldsymbol{c}}$	$\underline{\boldsymbol{c_1}}$	$\underline{\boldsymbol{d}}$	$\underline{\boldsymbol{d_1}}$
$\underline{\boldsymbol{a}}$	$\underline{a}$	$\underline{a}_1$	$\underline{b}$	$\underline{b}_1$	$\underline{c}$	$\underline{c}_1$	$\underline{d}$	$\underline{d}_1$
$\underline{\boldsymbol{a_1}}$	$\underline{a}_1$	$\underline{a}$	$\underline{b}_1$	$\underline{b}$	$\underline{c}_1$	$\underline{c}$	$\underline{d}_1$	$\underline{d}$
$\underline{\boldsymbol{b}}$	$\underline{b}$	$\underline{b}_1$	$\underline{a}$	$\underline{a}_1$	$\underline{d}$	$\underline{d}_1$	$\underline{c}$	$\underline{c}_1$
$\underline{\boldsymbol{b_1}}$	$\underline{b}_1$	$\underline{b}$	$\underline{a}_1$	$\underline{a}$	$\underline{d}_1$	$\underline{d}$	$\underline{c}_1$	$\underline{c}$
$\underline{\boldsymbol{c}}$	$\underline{c}$	$\underline{c}_1$	$\underline{d}$	$\underline{d}_1$	$\underline{a}$	$\underline{a}_1$	$\underline{b}$	$\underline{b}_1$
$\underline{\boldsymbol{c_1}}$	$\underline{c}_1$	$\underline{c}$	$\underline{d}_1$	$\underline{d}$	$\underline{a}_1$	$\underline{a}$	$\underline{b}_1$	$\underline{b}$
$\underline{\boldsymbol{d}}$	$\underline{d}$	$\underline{d}_1$	$\underline{c}$	$\underline{c}_1$	$\underline{b}$	$\underline{b}_1$	$\underline{a}$	$\underline{a}_1$
$\underline{\boldsymbol{d_1}}$	$\underline{d}_1$	$\underline{d}$	$\underline{c}_1$	$\underline{c}$	$\underline{b}_1$	$\underline{b}$	$\underline{a}_1$	$\underline{a}$

Moreover, in $(\boldsymbol{X}, \otimes)$ there is only one unit and every element has unique double inverse. The element b_2 is left inverse to b and b_2 because $a \in b_2 b$ and $a \in b_2 b_2$. The element d_2 is left inverse to d and d_2 because $a \in d_2 d$, $a \in d_2 d_2$. $(\boldsymbol{X}, \otimes)$ is not cyclic, since, from Theorem 2.9, the $(\underline{\boldsymbol{X}}, \underline{\otimes})$ is not cyclic.

Example 5.4. Consider the 2×2 upper triangular H_v-matrices on the above H_v-field $(\boldsymbol{Z}_4, +, \otimes)$ of the case that only $3 \otimes 2 = \{0, 2\}$ is a hyperproduct:

$$a = E_{11} + E_{22}, \quad a_1 = E_{11} + E_{12} + E_{22}, \quad a_2 = E_{11} + 2E_{12} + E_{22}, \quad a_3 = E_{11} + 3E_{12} + E_{22},$$

$$b = E_{11} + 3E_{22}, \quad b_1 = E_{11} + E_{12} + 3_{E22}, \quad b_2 = E_{11} + 2E_{12} + 3E_{22}, \quad b_3 = E_{11} + 3E_{12} + 3E_{22},$$

$$c = 3E_{11} + E_{22}, \quad c_1 = 3E_{11} + E_{12} + E_{22}, \quad c_2 = 3E_{11} + 2E_{12} + E_{22}, \quad c_3 = 3E_{11} + 3E_{12} + E_{22},$$

$$d = 3E_{11} + 3E_{22}, \quad d_1 = 3E_{11} + E_{12} + 3E_{22}, \quad d_2 = 3E_{11} + 2E_{12} + 3E_{22}, \quad d_3 = 3E_{11} + 3E_{12} + 3E_{22},$$

then, for $\boldsymbol{X} = \{a, a_1, a_2, a_3, b, b_1, b_2, b_3, c, c_1, c_2, c_3, d, d_1, d_2, d_3\}$, we obtain the following table:

The $(\boldsymbol{X}, \otimes)$ is a COW H_v-group with fundamental classes: $\underline{a} = \{a, a_2\}$, $\underline{a}_1 = \{a_1, a_3\}$, $\underline{b} = \{b, b_2\}$, $\underline{b}_1 = \{b_1, b_3\}$, $\underline{c} = \{c, c_2\}$, $\underline{c}_1 = \{c_1, c_3\}$, $\underline{d} = \{d, d_2\}$, $\underline{d}_1 = \{d_1, d_3\}$, with table as the above example.

$\otimes$	$\boldsymbol{a}$	$\boldsymbol{a_1}$	$\boldsymbol{a_2}$	$\boldsymbol{a_3}$	$\boldsymbol{b}$	$\boldsymbol{b_1}$	$\boldsymbol{b_2}$	$\boldsymbol{b_3}$	$\boldsymbol{c}$	$\boldsymbol{c_1}$	$\boldsymbol{c_2}$	$\boldsymbol{c_3}$	$\boldsymbol{d}$	$\boldsymbol{d_1}$	$\boldsymbol{d_2}$	$\boldsymbol{d_3}$
$\boldsymbol{a}$	a	a_1	a_2	a_3	b	b_1	b_2	b_3	c	c_1	c_2	c_3	d	d_1	d_2	d_3
$\boldsymbol{a_1}$	a_1	a_2	a_3	a	b_3	b	b_1	b_2	c_1	c_2	c_3	c	d_3	d	d_1	d_2
$\boldsymbol{a_2}$	a_2	a_3	a	a_1	b_2	b_3	b	b_1	c_2	c_3	c	c_1	d_2	d_3	d	d_1
$\boldsymbol{a_3}$	a_3	a	a_1	a_2	b_1	b_2	b_3	b	c_3	c	c_1	c_2	d_1	d_2	d_3	d
$\boldsymbol{b}$	b	b_1	b_2	b_3	a	a_1	a_2	a_3	d	d_1	d_2	d_3	c	c_1	c_2	c_3
$\boldsymbol{b_1}$	b_1	b_2	b_3	b	a_3	a	a_1	a_2	d_1	d_2	d_3	d	c_3	c	c_1	c_2
$\boldsymbol{b_2}$	b_2	b_3	b	b_1	a_2	a_3	a	a_1	d_2	d_3	d	d_1	c_2	c_3	c	c_1
$\boldsymbol{b_3}$	b_3	b	b_1	b_2	a_1	a_2	a_3	a	d_3	d	d_1	d_2	c_1	c_2	c_3	c
$\boldsymbol{c}$	c	c_3	c, c_2	c_1	d	d_3	d, d_2	d_1	a	a_3	a, a_2	a_1	b	b_3	b, b_2	b_1
$\boldsymbol{c_1}$	c_1	c	c_1, c_3	c_2	d_3	d_2	d_1, d_3	d	a_1	a	a_1, a_3	a_2	b_3	b_2	b_1, b_3	b
$\boldsymbol{c_2}$	c_2	c_1	c, c_2	c_3	d_2	d_1	d, d_2	d_3	a_2	a_1	a, a_2	a_3	b_2	b_1	b, b_2	b_3
$\boldsymbol{c_3}$	c_3	c_2	c_1, c_3	c	d_1	d	d_1, d_3	d_2	a_3	a_2	a_1, a_3	a	b_1	b	b_1, b_3	b_2
$\boldsymbol{d}$	d	d_3	d, d_2	d_1	c	c_3	c, c_2	c_1	b	b_3	b, b_2	b_1	a	a_3	a, a_2	a_1
$\boldsymbol{d_1}$	d_1	d	d_1, d_3	d_2	c_3	c_2	c_1, c_3	c	b_1	b	b_1, b_3	b_2	a_3	a_2	a_1, a_3	a
$\boldsymbol{d_2}$	d_2	d_1	d, d_2	d_3	c_2	c_1	c, c_2	c_3	b_2	b_1	b, b_2	b_3	a_2	a_1	a, a_2	a_3
$\boldsymbol{d_3}$	d_3	d_2	d_1, d_3	d	c_1	c	c_1, c_3	c_2	b_3	b_2	b_1, b_3	b	a_1	a	a_1, a_3	a_2

Moreover, in $(\boldsymbol{X}, \otimes)$ there is only one unit a, and every element has unique double inverse. The element c_2 is right inverse to c and c_2 because $a \in cc_2$, $a \in c_2c_2$. The element d_2 is right inverse to d and d_2 because $a \in dd_2$, $a \in d_2d_2$. $(\boldsymbol{X}, \otimes)$ is not cyclic, since, from Theorem 2.9, the $(\underline{\boldsymbol{X}}, \underline{\otimes})$ is not cyclic.

Theorem 5.5. *All multiplicative H_v-fields on $(\boldsymbol{Z}_6, +, \cdot)$, with non-degenerate fundamental field, satisfying the above 4 conditions, with one hyperproduct, are the following isomorphic cases:*

(I) $2 \otimes 3 = \{0, 3\}$, $2 \otimes 4 = \{2, 5\}$, $3 \otimes 4 = \{0, 3\}$, $3 \otimes 5 = \{0, 3\}$, $4 \otimes 5 = \{2, 5\}$

Fundamental classes: $[0] = \{0, 3\}$, $[1] = \{1, 4\}$, $[2] = \{2, 5\}$ *and* $(\boldsymbol{Z}_6, +, \otimes)/\gamma^* \cong (\boldsymbol{Z}_3, +, \cdot)$.

(II) $2 \otimes 3 = \{0, 2\}$ *or* $2 \otimes 3 = \{0, 4\}$, $2 \otimes 4 = \{0, 2\}$ *or* $\{2, 4\}$, $2 \otimes 5 = \{0, 4\}$ *or* $2 \otimes 5 = \{2, 4\}$, $3 \otimes 4 = \{0, 2\}$ *or* $\{0, 4\}$, $3 \otimes 5 = \{3, 5\}$, $4 \otimes 5 = \{0, 2\}$ *or* $\{2, 4\}$.

In all cases, fundamental classes are $[0] = \{0, 2, 4\}$, $[1] = \{1, 3, 5\}$ *and* $(\boldsymbol{Z}_6, +, \otimes)/\gamma^* \cong (\boldsymbol{Z}_2, +, \cdot)$.

Example. In the H_v-field $(\boldsymbol{Z}_6, +, \otimes)$ where only the hyperproduct is $2 \otimes 4 = \{2, 5\}$, take the H_v-matrices of type $\underline{i} = E_{11} + iE_{12} + 4E_{22}$, where $i = 0, 1, \ldots, 5$, then the multiplicative table of the hyperproduct of those H_v-matrices is

$\otimes$	$\underline{0}$	$\underline{1}$	$\underline{2}$	$\underline{3}$	$\underline{4}$	$\underline{5}$
$\underline{0}$	$\underline{0}$	$\underline{1}$	$\underline{2}$	$\underline{3}$	$\underline{4}$	$\underline{5}$
$\underline{1}$	$\underline{4}$	$\underline{5}$	$\underline{0}$	$\underline{1}$	$\underline{2}$	$\underline{3}$
$\underline{2}$	$\underline{2}, \underline{5}$	$\underline{0}, \underline{3}$	$\underline{1}, \underline{4}$	$\underline{2}, \underline{5}$	$\underline{0}, \underline{3}$	$\underline{1}, \underline{4}$
$\underline{3}$	$\underline{0}$	$\underline{1}$	$\underline{2}$	$\underline{3}$	$\underline{4}$	$\underline{5}$
$\underline{4}$	$\underline{4}$	$\underline{5}$	$\underline{0}$	$\underline{1}$	$\underline{2}$	$\underline{3}$
$\underline{5}$	$\underline{2}$	$\underline{3}$	$\underline{4}$	$\underline{5}$	$\underline{0}$	$\underline{1}$

Classes: $[\underline{0}] = \{\underline{0}, \underline{3}\}$, $[\underline{1}] = \{\underline{1}, \underline{4}\}$, $[\underline{2}] = \{\underline{2}, \underline{5}\}$ and fundamental group isomorphic to $(\boldsymbol{Z}_3, +)$. $(\boldsymbol{Z}_6, \otimes)$ is h/v-group which is cyclic where $\underline{2}$ is generator of period 4 and $\underline{4}$ is generator of period 5.

Example 5.6. Consider the 2×2 upper triangular H_v-matrices on the above H_v-field $(\boldsymbol{Z}_6, +, \otimes)$ of the case that only $4 \otimes 5 = \{2, 5\}$ is a hyperproduct. We set

$$a = E_{11} + E_{22}, \quad a_1 = E_{11} + E_{12} + E_{22}, \quad a_2 = E_{11} + 2E_{12} + E_{22},$$

$$a_3 = E_{11} + 3E_{12} + E_{22}, \quad a_4 = E_{11} + 4E_{12} + E_{22}, \quad a_5 = E_{11} + 5E_{12} + E_{22},$$

$$b = E_{11} + 5E_{22}, \quad b_1 = E_{11} + E_{12} + 5E_{22}, \quad b_2 = E_{11} + 2E_{12} + 5E_{22},$$

$$b_3 = E_{11} + 3E_{12} + 5E_{22}, \quad b_4 = E_{11} + 4E_{12} + 5E_{22}, \quad b_5 = E_{11} + 5E_{12} + 5E_{22},$$

$$c = 5E_{11} + E_{22}, \quad c_1 = 5E_{11} + E_{12} + E_{22}, \quad c_2 = 5E_{11} + 2E_{12} + E_{22},$$

$$c_3 = 5E_{11} + 3E_{12} + E_{22}, \quad c_4 = 5E_{11} + 4E_{12} + E_{22}, \quad c_5 = 5E_{11} + 5E_{12} + E_{22},$$

$$d = 5E_{11} + 5E_{22}, \quad d_1 = 5E_{11} + E_{12} + 5E_{22}, \quad d_2 = 5E_{11} + 2E_{12} + 5E_{22},$$

$$d_3 = 5E_{11} + 3E_{12} + 5E_{22}, \quad d_4 = 5E_{11} + 4E_{12} + 5E_{22}, \quad d_5 = 5E_{11} + 5E_{12} + 5E_{22},$$

then, for $\boldsymbol{X} = \{a, a_1, a_2, a_3, a_4, a_5, b, b_1, b_2, b_3, b_4, b_5, c, c_1, c_2, c_3, c_4, c_5, d, d_1, d_2, d_3, d_4, d_5\}$, we obtain the table:

$\otimes$	a	a_1	a_2	a_3	a_4	a_5	b	b_1	b_2	b_3	b_4	b_5	c	c_1	c_2	c_3	c_4	c_5	d	d_1	d_2	d_3	d_4	d_5
a	a	a_1	a_2	a_3	a_4	a_5	b	b_1	b_2	b_3	b_4	b_5	c	c_1	c_2	c_3	c_4	c_5	d	d_1	d_2	d_3	d_4	d_5
a_1	a_1	a_2	a_3	a_4	a_5	a	b_5	b	b_1	b_2	b_3	b_4	c_1	c_2	c_3	c_4	c_5	c	d_5	d	d_1	d_2	d_3	d_4
a_2	a_2	a_3	a_4	a_5	a	a_1	b_4	b_5	b	b_1	b_2	b_3	c_2	c_3	c_4	c_5	c	c_1	d_4	d_5	d	d_1	d_2	d_3
a_3	a_3	a_4	a_5	a	a_1	a_2	b_3	b_4	b_5	b	b_1	b_2	c_3	c_4	c_5	c	c_1	c_2	d_3	d_4	d_5	d	d_1	d_2
a_4	a_4	a_5	a	a_1	a_2	a_3	b_2, b_5	b, b_3	b_1, b_4	b_2, b_5	b, b_3	b_1, b_4	c_4	c_5	c	c_1	c_2	c_3	d_2, d_5	d, d_3	d_1, d_4	d_2, d_5	d, d_3	d_1, d_4
a_5	a_5	a	a_1	a_2	a_3	a_4	b_1	b_2	b_3	b_4	b_5	b	c_5	c	c_1	c_2	c_3	c_4	d_1	d_2	d_3	d_4	d_5	d
b	b	b_1	b_2	b_3	b_4	b_5	a	a_1	a_2	a_3	a_4	a_5	d	d_1	d_2	d_3	d_4	d_5	c	c_1	c_2	c_3	c_4	c_5
b_1	b_1	b_2	b_3	b_4	b_5	b	a_5	a	a_1	a_2	a_3	a_4	d_1	d_2	d_3	d_4	d_5	d	c_5	c	c_1	c_2	c_3	c_4
b_2	b_2	b_3	b_4	b_5	b	b_1	a_4	a_5	a	a_1	a_2	a_3	d_2	d_3	d_4	d_5	d	d_1	c_4	c_5	c	c_1	c_2	c_3
b_3	b_3	b_4	b_5	b	b_1	b_2	a_3	a_4	a_5	a	a_1	a_2	d_3	d_4	d_5	d	d_1	d_2	c_3	c_4	c_5	c	c_1	c_2
b_4	b_4	b_5	b	b_1	b_2	b_3	a_2, a_5	a, a_3	a_1, a_4	a_2, a_5	a, a_3	a_1, a_4	d_4	d_5	d	d_1	d_2	d_3	c_2, c_5	c, c_3	c_1, c_4	c_2, c_5	c, c_3	c_1, c_4
b_5	b_5	b	b_1	b_2	b_3	b_4	a_1	a_2	a_3	a_4	a_5	a	d_5	d	d_1	d_2	d_3	d_4	c_1	c_2	c_3	c_4	c_5	c
c	c	c_5	c_4	c_3	c_2	c_1	d	d_5	d_4	d_3	d_2	d_1	a	a_5	a_4	a_3	a_2	a_1	b	b_5	b_4	b_3	b_2	b_1
c_1	c_1	c	c_5	c_4	c_3	c_2	d_5	d_4	d_3	d_2	d_1	d	a_1	a	a_5	a_4	a_3	a_2	b_5	b_4	b_3	b_2	b_1	b
c_2	c_2	c_1	c	c_5	c_4	c_3	d_4	d_3	d_2	d_1	d	d_5	a_2	a_1	a	a_5	a_4	a_3	b_4	b_3	b_2	b_1	b	b_5
c_3	c_3	c_2	c_1	c	c_5	c_4	d_3	d_2	d_1	d	d_5	d_4	a_3	a_2	a_1	a	a_5	a_4	b_3	b_2	b_1	b	b_5	b_4
c_4	c_4	c_3	c_2	c_1	c	c_5	d_2, d_5	d_1, d_4	d, d_3	d_2, d_5	d_1, d_4	d, d_3	a_4	a_3	a_2	a_1	a	a_5	b_2, b_5	b_1, b_4	b, b_3	b_2, b_5	b_1, b_4	b, b_3
c_5	c_5	c_4	c_3	c_2	c_1	c	d_1	d	d_5	d_4	d_3	d_2	a_5	a_4	a_3	a_2	a_1	a	b_1	b	b_5	b_4	b_3	b_2
d	d	d_5	d_4	d_3	d_2	d_1	c	c_5	c_4	c_3	c_2	c_1	b	b_5	b_4	b_3	b_2	b_1	a	a_5	a_4	a_3	a_2	a_1
d_1	d_1	d	d_5	d_4	d_3	d_2	c_5	c_4	c_3	c_2	c_1	c	b_1	b	b_5	b_4	b_3	b_2	a_5	a_4	a_3	a_2	a_1	a
d_2	d_2	d_1	d	d_5	d_4	d_3	c_4	c_3	c_2	c_1	c	c_5	b_2	b_1	b	b_5	b_4	b_3	a_4	a_3	a_2	a_1	a	a_5
d_3	d_3	d_2	d_1	d	d_5	d_4	c_3	c_2	c_1	c	c_5	c_4	b_3	b_2	b_1	b	b_5	b_4	a_3	a_2	a_1	a	a_5	a_4
d_4	d_4	d_3	d_2	d_1	d	d_5	c_2, c_5	c_1, c_4	c, c_3	c_2, c_5	c_1, c_4	c, c_3	b_4	b_3	b_2	b_1	b	b_5	a_2, a_5	a_1, a_4	a, a_3	a_2, a_5	a_1, a_4	a, a_3
d_5	d_5	d_4	d_3	d_2	d_1	d	c_1	c	c_5	c_4	c_3	c_2	b_5	b_4	b_3	b_2	b_1	b	a_1	a	a_5	a_4	a_3	a_2

The $(\boldsymbol{X}, \otimes)$ is a COW H_v-group with fundamental classes:

$$\underline{a} = \{a, a_3\}, \quad \underline{a}_1 = \{a_1, a_4\} \quad , \underline{a}_2 = \{a_2, a_5\}, \quad \underline{b} = \{b, b_3\}, \quad \underline{b}_1 = \{b_1, b_4\} \quad , \underline{b}_2 = \{b_2, b_5\},$$

$$\underline{c} = \{c, c_3\}, \quad \underline{c}_1 = \{c_1, c_4\}, \quad \underline{c}_2 = \{c_2, c_5\}, \quad \underline{d} = \{d, d_3\}, \quad \underline{d}_1 = \{d_1, d_4\}, \quad \underline{d}_2 = \{d_2, d_5\},$$

and the fundamental group $(\underline{\boldsymbol{X}}, \underline{\otimes})$ is defined with the table:

Theorem 5.7. *All multiplicative H_v-fields defined on $(\boldsymbol{Z}_9, +, \cdot)$, which have a non-degenerate fundamental field and satisfy the above 4 conditions, are the following isomorphic cases: We have the only one hyperproduct,*

$$2 \otimes 3 = \{0, 6\} \quad or \ \{3, 6\}, \quad 2 \otimes 4 = \{2, 8\} \quad or \ \{5, 8\}, \quad 2 \otimes 6 = \{0, 3\} \quad or \ \{3, 6\}, \quad 2 \otimes 7 = \{2, 5\} \quad or \ \{5, 8\},$$

$$2 \otimes 8 = \{1, 7\} \quad or \ \{4, 7\}, \quad 3 \otimes 4 = \{0, 3\} \quad or \ \{3, 6\}, \quad 3 \otimes 5 = \{0, 6\} \quad or \ \{3, 6\}, \quad 3 \otimes 6 = \{0, 3\} \ or \ \{0, 6\},$$

$\underline{\otimes}$	$\underline{\boldsymbol{a}}$	$\underline{\boldsymbol{a_1}}$	$\underline{\boldsymbol{a_2}}$	$\underline{\boldsymbol{b}}$	$\underline{\boldsymbol{b_1}}$	$\underline{\boldsymbol{b_2}}$	$\underline{\boldsymbol{c}}$	$\underline{\boldsymbol{c_1}}$	$\underline{\boldsymbol{c_2}}$	$\underline{\boldsymbol{d}}$	$\underline{\boldsymbol{d_1}}$	$\underline{\boldsymbol{d_2}}$
$\underline{\boldsymbol{a}}$	$\underline{a}$	$\underline{a}_1$	$\underline{a}_2$	$\underline{b}$	$\underline{b}_1$	$\underline{b}_2$	$\underline{c}$	$\underline{c}_1$	$\underline{c}_2$	$\underline{d}$	$\underline{d}_1$	$\underline{d}_2$
$\underline{\boldsymbol{a}}_1$	$\underline{a}_1$	$\underline{a}_2$	$\underline{a}$	$\underline{b}_2$	$\underline{b}$	$\underline{b}_1$	$\underline{c}_1$	$\underline{c}_2$	$\underline{c}$	$\underline{d}_2$	$\underline{d}$	$\underline{d}_1$
$\underline{\boldsymbol{a}}_2$	$\underline{a}_2$	$\underline{a}$	$\underline{a}_1$	$\underline{b}_1$	$\underline{b}_2$	$\underline{b}$	$\underline{c}_2$	$\underline{c}$	$\underline{c}_1$	$\underline{d}_1$	$\underline{d}_2$	$\underline{d}$
$\underline{\boldsymbol{b}}$	$\underline{b}$	$\underline{b}_1$	$\underline{b}_2$	$\underline{a}$	$\underline{a}_1$	$\underline{a}_2$	$\underline{d}$	$\underline{d}_1$	$\underline{d}_2$	$\underline{c}$	$\underline{c}_1$	$\underline{c}_2$
$\underline{\boldsymbol{b}}_1$	$\underline{b}_1$	$\underline{b}_2$	$\underline{b}$	$\underline{a}_2$	$\underline{a}$	$\underline{a}_1$	$\underline{d}_1$	$\underline{d}_2$	$\underline{d}$	$\underline{c}_2$	$\underline{c}$	$\underline{c}_1$
$\underline{\boldsymbol{b}}_2$	$\underline{b}_2$	$\underline{b}$	$\underline{b}_1$	$\underline{a}_1$	$\underline{a}_2$	$\underline{a}$	$\underline{d}_2$	$\underline{d}$	$\underline{d}_1$	$\underline{c}_1$	$\underline{c}_2$	$\underline{c}$
$\underline{\boldsymbol{c}}$	$\underline{c}$	$\underline{c}_2$	$\underline{c}_1$	$\underline{d}$	$\underline{d}_2$	$\underline{d}_1$	$\underline{a}$	$\underline{a}_2$	$\underline{a}_1$	$\underline{b}$	$\underline{b}_2$	$\underline{b}_1$
$\underline{\boldsymbol{c}}_1$	$\underline{c}_1$	$\underline{c}$	$\underline{c}_2$	$\underline{d}_2$	$\underline{d}_1$	$\underline{d}$	$\underline{a}_1$	$\underline{a}$	$\underline{a}_2$	$\underline{b}_2$	$\underline{b}_1$	$\underline{b}$
$\underline{\boldsymbol{c}}_2$	$\underline{c}_2$	$\underline{c}_1$	$\underline{c}$	$\underline{d}_1$	$\underline{d}$	$\underline{d}_2$	$\underline{a}_2$	$\underline{a}_1$	$\underline{a}$	$\underline{b}_1$	$\underline{b}$	$\underline{b}_2$
$\underline{\boldsymbol{d}}$	$\underline{d}$	$\underline{d}_2$	$\underline{d}_1$	$\underline{c}$	$\underline{c}_2$	$\underline{c}_1$	$\underline{b}$	$\underline{b}_2$	$\underline{b}_1$	$\underline{a}$	$\underline{a}_2$	$\underline{a}_1$
$\underline{\boldsymbol{d}}_1$	$\underline{d}_1$	$\underline{d}$	$\underline{d}_2$	$\underline{c}_2$	$\underline{c}_1$	$\underline{c}$	$\underline{b}_1$	$\underline{b}$	$\underline{b}_2$	$\underline{a}_2$	$\underline{a}_1$	$\underline{a}$
$\underline{\boldsymbol{d}}_2$	$\underline{d}_2$	$\underline{d}_1$	$\underline{d}$	$\underline{c}_1$	$\underline{c}$	$\underline{c}_2$	$\underline{b}_2$	$\underline{b}_1$	$\underline{b}$	$\underline{a}_1$	$\underline{a}$	$\underline{a}_2$

$3 \otimes 7 = \{0,3\}$ *or* $\{3,6\}$, $3 \otimes 8 = \{0,6\}$ *or* $\{3,6\}$, $4 \otimes 5 = \{2,5\}$ *or* $\{2,8\}$, $4 \otimes 6 = \{0,6\}$ *or* $\{3,6\}$,
$4 \otimes 8 = \{2,5\}$ *or* $\{5,8\}$, $5 \otimes 6 = \{0,3\}$ *or* $\{3,6\}$, $5 \otimes 7 = \{2,8\}$ *or* $\{5,8\}$, $5 \otimes 8 = \{1,4\}$ *or* $\{4,7\}$,
$6 \otimes 7 = \{0,6\}$ *or* $\{3,6\}$, $6 \otimes 8 = \{0,3\}$ *or* $\{3,6\}$, $7 \otimes 8 = \{2,5\}$ *or* $\{2,8\}$

In all the above cases the fundamental classes are $[0] = \{0,3,6\}$, $[1] = \{1,4,7\}$, $[2] = \{2,5,8\}$, *and we have* $(\boldsymbol{Z}_9, +, \otimes)/\gamma^* \cong (\boldsymbol{Z}_3, +, \cdot)$.

Example 5.8. 8 Consider the 2×2 upper triangular H_v-matrices on the above H_v-field $(\boldsymbol{Z}_9, +, \otimes)$ of the case that only $2 \otimes 8 = \{4,7\}$ is a hyperproduct. We set, for $i = 1, 2, \ldots, 8$,

$$a = E_{11} + E_{22}, \quad a_i = E_{11} + iE_{12} + E_{22},$$

$$b = E_{11} + 8E_{22}, \quad b_i = E_{11} + iE_{12} + 8E_{22},$$

then, for $\boldsymbol{X} = \{a, a_1, \ldots, a_8, b, b_1, \ldots, b_8\}$, we obtain the following table:

$\otimes$	$\boldsymbol{a}$	$\boldsymbol{a_1}$	$\boldsymbol{a_2}$	$\boldsymbol{a_3}$	$\boldsymbol{a_4}$	$\boldsymbol{a_5}$	$\boldsymbol{a_6}$	$\boldsymbol{a_7}$	$\boldsymbol{a_8}$	$\boldsymbol{b}$	$\boldsymbol{b_1}$	$\boldsymbol{b_2}$	$\boldsymbol{b_3}$	$\boldsymbol{b_4}$	$\boldsymbol{b_5}$	$\boldsymbol{b_6}$	$\boldsymbol{b_7}$	$\boldsymbol{b_8}$
$\boldsymbol{a}$	a	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	b	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8
$\boldsymbol{a_1}$	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a	b_8	b	b_1	b_2	b_3	b_4	b_5	b_6	b_7
$\boldsymbol{a_2}$	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a	a_1	b_4, b_7	b_5, b_8	b, b_6	b_1, b_7	b_2, b_8	b, b_3	b_1, b_4	b_2, b_5	b_3, b_6
$\boldsymbol{a_3}$	a_3	a_4	a_5	a_6	a_7	a_8	a	a_1	a_2	b_6	b_7	b_8	b	b_1	b_2	b_3	b_4	b_5
$\boldsymbol{a_4}$	a_4	a_5	a_6	a_7	a_8	a	a_1	a_2	a_3	b_5	b_6	b_7	b_8	b	b_1	b_2	b_3	b_4
$\boldsymbol{a_5}$	a_5	a_6	a_7	a_8	a	a_1	a_2	a_3	a_4	b_4	b_5	b_6	b_7	b_8	b	b_1	b_2	b_3
$\boldsymbol{a_6}$	a_6	a_7	a_8	a	a_1	a_2	a_3	a_4	a_5	b_3	b_4	b_5	b_6	b_7	b_8	b	b_1	b_2
$\boldsymbol{a_7}$	a_7	a_8	a	a_1	a_2	a_3	a_4	a_5	a_6	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b	b_1
$\boldsymbol{a_8}$	a_8	a	a_1	a_2	a_3	a_4	a_5	a_6	a_7	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b
$\boldsymbol{b}$	b	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	a	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8
$\boldsymbol{b_1}$	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b	a_8	a	a_1	a_2	a_3	a_4	a_5	a_6	a_7
$\boldsymbol{b_2}$	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b	b_1	a_4, a_7	a_5, a_8	a, a_6	a_1, a_7	a_2, a_8	a, a_3	a_1, a_4	a_2, a_5	a_3, a_6
$\boldsymbol{b_3}$	b_3	b_4	b_5	b_6	b_7	b_8	b	b_1	b_2	a_6	a_7	a_8	a	a_1	a_2	a_3	a_4	a_5
$\boldsymbol{b_4}$	b_4	b_5	b_6	b_7	b_8	b	b_1	b_2	b_3	a_5	a_6	a_7	a_8	a	a_1	a_2	a_3	a_4
$\boldsymbol{b_5}$	b_5	b_6	b_7	b_8	b	b_1	b_2	b_3	b_4	a_4	a_5	a_6	a_7	a_8	a	a_1	a_2	a_3
$\boldsymbol{b_6}$	b_6	b_7	b_8	b	b_1	b_2	b_3	b_4	b_5	a_3	a_4	a_5	a_6	a_7	a_8	a	a_1	a_2
$\boldsymbol{b_7}$	b_7	b_8	b	b_1	b_2	b_3	b_4	b_5	b_6	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a	a_1
$\boldsymbol{b_8}$	b_8	b	b_1	b_2	b_3	b_4	b_5	b_6	b_7	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a

The $(\boldsymbol{X}, \otimes)$ is a COW H_v-group with fundamental classes: $\underline{a} = \{a, a_3, a_6\}$, $\underline{a}_1 = \{a_1, a_4, a_7\}$, $\underline{a}_2 = \{a_2, a_5, a_8\}$, $\underline{b} = \{b, b_3, b_6\}$, $\underline{b}_1 = \{b_1, b_4, b_7\}$, $\underline{b}_2 = \{b_2, b_5, a_b\}$, and the fundamental group $(\underline{\boldsymbol{X}}, \underline{\otimes})$ is defined with the table:

$\underline{\otimes}$	$\underline{\boldsymbol{a}}$	$\underline{\boldsymbol{a_1}}$	$\underline{\boldsymbol{a_2}}$	$\underline{\boldsymbol{b}}$	$\underline{\boldsymbol{b_1}}$	$\underline{\boldsymbol{b_2}}$
$\underline{\boldsymbol{a}}$	$\underline{a}$	$\underline{a}_1$	$\underline{a}_2$	$\underline{b}$	$\underline{b}_1$	$\underline{b}_2$
$\underline{\boldsymbol{a_1}}$	$\underline{a}_1$	$\underline{a}_2$	$\underline{a}$	$\underline{b}_2$	$\underline{b}_1$	$\underline{b}$
$\underline{\boldsymbol{a_2}}$	$\underline{a}_2$	$\underline{a}$	$\underline{a}_1$	$\underline{b}_1$	$\underline{b}_2$	$\underline{b}$
$\underline{\boldsymbol{b}}$	$\underline{b}$	$\underline{b}_1$	$\underline{b}_2$	$\underline{a}$	$\underline{a}_1$	$\underline{a}_2$
$\underline{\boldsymbol{b_1}}$	$\underline{b}_1$	$\underline{b}_2$	$\underline{b}$	$\underline{a}_2$	$\underline{a}$	$\underline{a}_1$
$\underline{\boldsymbol{b_2}}$	$\underline{b}_2$	$\underline{b}$	$\underline{b}_1$	$\underline{a}_1$	$\underline{a}_2$	$\underline{a}$

Example 5.9. Consider the 2×2 upper triangular H_v-matrices on the above H_v-field $(\boldsymbol{Z}_9, +, \otimes)$ of the case that only $2 \otimes 8 = \{4, 7\}$ is a hyperproduct. We set $i = 1, 2, \ldots, 8$,

$$a = E_{11} + E_{22}, \quad a_i = E_{11} + iE_{12} + E_{22},$$

$$b = E_{11} + 4E_{22}, \quad b_i = E_{11} + iE_{12} + 4E_{22},$$

$$c = E_{11} + 7E_{22}, \quad c_i = E_{11} + iE_{12} + 7E_{22},$$

then, for $\boldsymbol{X} = \{a, a_1, \ldots, a_8, b, b_1, \ldots, b_8, c, c_1, \ldots, c_8\}$, we obtain the following table:

$\otimes$	$\boldsymbol{a}$	$\boldsymbol{a_1}$	$\boldsymbol{a_2}$	$\boldsymbol{a_3}$	$\boldsymbol{a_4}$	$\boldsymbol{a_5}$	$\boldsymbol{a_6}$	$\boldsymbol{a_7}$	$\boldsymbol{a_8}$	$\boldsymbol{b}$	$\boldsymbol{b_1}$	$\boldsymbol{b_2}$	$\boldsymbol{b_3}$	$\boldsymbol{b_4}$	$\boldsymbol{b_5}$	$\boldsymbol{b_6}$	$\boldsymbol{b_7}$	$\boldsymbol{b_8}$	$\boldsymbol{c}$	$\boldsymbol{c_1}$	$\boldsymbol{c_2}$	$\boldsymbol{c_3}$	$\boldsymbol{c_4}$	$\boldsymbol{c_5}$	$\boldsymbol{c_6}$	$\boldsymbol{c_7}$	$\boldsymbol{c_8}$
$\boldsymbol{a}$	a	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	b	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	c	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8
$\boldsymbol{a_1}$	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a	b_4	b_5	b_6	b_7	b_8	b	b_1	b_2	b_3	c_7	c_8	c	c_1	c_2	c_3	c_4	c_5	c_6
$\boldsymbol{a_2}$	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a	a_1	b_2, b_8	b, b_3	b_1, b_4	b_2, b_5	b_3, b_6	b_4, b_7	b, b_6	b_1, b_7	b_4, b_7	c_5	c_6	c_7	c_8	c	c_1	c_2	c_3	c_4
$\boldsymbol{a_3}$	a_3	a_4	a_5	a_6	a_7	a_8	a	a_1	a_2	b_3	b_4	b_5	b_6	b_7	b_8	b	b_1	b_2	c_3	c_4	c_5	c_6	c_7	c_8	c	c_1	c_2
$\boldsymbol{a_4}$	a_4	a_5	a_6	a_7	a_8	a	a_1	a_2	a_3	b_7	b_8	b	b_1	b_2	b_3	b_4	b_5	b_6	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c
$\boldsymbol{a_5}$	a_5	a_6	a_7	a_8	a	a_1	a_2	a_3	a_4	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b	b_1	c_8	c	c_1	c_2	c_3	c_4	c_5	c_6	c_7
$\boldsymbol{a_6}$	a_6	a_7	a_8	a	a_1	a_2	a_3	a_4	a_5	b_6	b_7	b_8	b	b_1	b_2	b_3	b_4	b_5	c_6	c_7	c_8	c	c_1	c_2	c_3	c_4	c_5
$\boldsymbol{a_7}$	a_7	a_8	a	a_1	a_2	a_3	a_4	a_5	a_6	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b	c_4	c_5	c_6	c_7	c_8	c	c_1	c_2	c_3
$\boldsymbol{a_8}$	a_8	a	a_1	a_2	a_3	a_4	a_5	a_6	a_7	b_5	b_6	b_7	b_8	b	b_1	b_2	b_3	b_4	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c	c_1
$\boldsymbol{b}$	b	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	c	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	a	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8
$\boldsymbol{b_1}$	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b	c_4	c_5	c_6	c_7	c_8	c	c_1	c_2	c_3	a_7	a_8	a	a_1	a_2	a_3	a_4	a_5	a_6
$\boldsymbol{b_2}$	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b	b_1	c_2, c_8	c, c_3	c_1, c_4	c_2, c_5	c_3, c_6	c_4, c_7	c_5, c_8	c, c_6	c_1, c_7	a_5	a_6	a_7	a_8	a	a_1	a_2	a_3	a_4
$\boldsymbol{b_3}$	b_3	b_4	b_5	b_6	b_7	b_8	b	b_1	b_2	c_3	c_4	c_5	c_6	c_7	c_8	c	c_1	c_2	a_3	a_4	a_5	a_6	a_7	a_8	a	a_1	a_2
$\boldsymbol{b_4}$	b_4	b_5	b_6	b_7	b_8	b	b_1	b_2	b_3	c_7	c_8	c	c_1	c_2	c_3	c_4	c_5	c_6	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a
$\boldsymbol{b_5}$	b_5	b_6	b_7	b_8	b	b_1	b_2	b_3	b_4	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c	c_1	a_8	a	a_1	a_2	a_3	a_4	a_5	a_6	a_7
$\boldsymbol{b_6}$	b_6	b_7	b_8	b	b_1	b_2	b_3	b_4	b_5	c_6	c_7	c_8	c	c_1	c_2	c_3	c_4	c_5	a_6	a_7	a_8	a	a_1	a_2	a_3	a_4	a_5
$\boldsymbol{b_7}$	b_7	b_8	b	b_1	b_2	b_3	b_4	b_5	b_6	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c	a_4	a_5	a_6	a_7	a_8	a	a_1	a_2	a_3
$\boldsymbol{b_8}$	b_8	b	b_1	b_2	b_3	b_4	b_5	b_6	b_7	c_5	c_6	c_7	c_8	c	c_1	c_2	c_3	c_4	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a	a_1
$\boldsymbol{c}$	c	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	a	a_1	a_2	a_3	a_4	a_5	a_6	a_7	c_8	b	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8
$\boldsymbol{c_1}$	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c	a_4	a_5	a_6	a_7	a_8	a	a_1	a_2	a_3	b_7	b_8	b	b_1	b_2	b_3	b_4	b_5	b_6
$\boldsymbol{c_2}$	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c	c_1	a_2, a_8	a, a_3	a_1, a_4	a_2, a_5	a_3, a_6	a_4, a_7	a_5, a_8	a, a_6	a_1, a_7	b_5	b_6	b_7	b_8	b	b_1	b_2	b_3	b_4
$\boldsymbol{c_3}$	c_3	c_4	c_5	c_6	c_7	c_8	c	c_1	c_2	a_3	a_4	a_5	a_6	a_7	a_8	a	a_1	a_2	b_3	b_4	b_5	b_6	b_7	b_8	b	b_1	b_2
$\boldsymbol{c_4}$	c_4	c_5	c_6	c_7	c_8	c	c_1	c_2	c_3	a_7	a_8	a	a_1	a_2	a_3	a_4	a_5	a_6	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b
$\boldsymbol{c_5}$	c_5	c_6	c_7	c_8	c	c_1	c_2	c_3	c_4	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a	a_1	b_8	b	b_1	b_2	b_3	b_4	b_5	b_6	b_7
$\boldsymbol{c_6}$	c_6	c_7	c_8	c	c_1	c_2	c_3	c_4	c_5	a_6	a_7	a_8	a	a_1	a_2	a_3	a_4	a_5	b_6	b_7	b_8	b	b_1	b_2	b_3	b_4	b_5
$\boldsymbol{c_7}$	c_7	c_8	c	c_1	c_2	c_3	c_4	c_5	c_6	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a	b_4	b_5	b_6	b_7	b_8	b	b_1	b_2	b_3
$\boldsymbol{c_8}$	c_8	c	c_1	c_2	c_3	c_4	c_5	c_6	c_7	a_5	a_6	a_7	a_8	a	a_1	a_2	a_3	a_4	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b	b_1

The $(\boldsymbol{X}, \otimes)$ is a COW H_v-group with fundamental classes: $\underline{a} = \{a, a_3, a_6\}$, $\underline{a}_1 = \{a_1, a_4, a_7\}$, $\underline{a}_2 = \{a_2, a_5, a_8\}$, $\underline{b} = \{b, b_3, b_6\}$, $\underline{b}_1 = \{b_1, b_4, b_7\}$, $\underline{b}_2 = \{b_2, b_5, a_b\}$, $\underline{c} = \{c, c_3, c_6\}$, $\underline{c}_1 = \{c_1, c_4, c_7\}$, $\underline{c}_2 = \{c_2, c_5, c_8\}$, and the fundamental group $(\underline{\boldsymbol{X}}, \underline{\otimes})$ is defined with the table:

$\underline{\otimes}$	$\underline{\boldsymbol{a}}$	$\underline{\boldsymbol{a}}_{\boldsymbol{1}}$	$\underline{\boldsymbol{a}}_{\boldsymbol{2}}$	$\underline{\boldsymbol{b}}$	$\underline{\boldsymbol{b}}_{\boldsymbol{1}}$	$\underline{\boldsymbol{b}}_{\boldsymbol{2}}$	$\underline{\boldsymbol{c}}$	$\underline{\boldsymbol{c}}_{\boldsymbol{1}}$	$\underline{\boldsymbol{c}}_{\boldsymbol{2}}$
$\underline{\boldsymbol{a}}$	$\underline{a}$	$\underline{a}_1$	$\underline{a}_2$	$\underline{b}$	$\underline{b}_1$	$\underline{b}_2$	$\underline{\boldsymbol{c}}$	$\underline{\boldsymbol{c}}_{\boldsymbol{1}}$	$\underline{\boldsymbol{c}}_{\boldsymbol{2}}$
$\underline{\boldsymbol{a}}_{\boldsymbol{1}}$	$\underline{a}_1$	$\underline{a}_2$	$\underline{a}$	$\underline{b}_1$	$\underline{b}_2$	$\underline{b}$	$\underline{c}_1$	$\underline{c}_2$	$\underline{c}$
$\underline{\boldsymbol{a}}_{\boldsymbol{2}}$	$\underline{a}_2$	$\underline{a}$	$\underline{a}_1$	$\underline{b}_2$	$\underline{b}$	$\underline{b}_1$	$\underline{c}_2$	$\underline{c}$	$\underline{c}_1$
$\underline{\boldsymbol{b}}$	$\underline{b}$	$\underline{b}_1$	$\underline{b}_2$	$\underline{c}$	$\underline{c}_1$	$\underline{c}_2$	$\underline{a}$	$\underline{a}_1$	$\underline{a}_2$
$\underline{\boldsymbol{b}}_{\boldsymbol{1}}$	$\underline{b}_1$	$\underline{b}_2$	$\underline{b}$	$\underline{c}_1$	$\underline{c}_2$	$\underline{c}$	$\underline{a}_1$	$\underline{a}_2$	$\underline{a}$
$\underline{\boldsymbol{b}}_{\boldsymbol{2}}$	$\underline{b}_2$	$\underline{b}$	$\underline{b}_1$	$\underline{c}_2$	$\underline{c}$	$\underline{c}_1$	$\underline{a}_2$	$\underline{a}$	$\underline{a}_1$
$\underline{\boldsymbol{c}}$	$\underline{c}$	$\underline{c}_1$	$\underline{c}_2$	$\underline{a}$	$\underline{a}_1$	$\underline{a}_2$	$\underline{b}$	$\underline{b}_1$	$\underline{b}_2$
$\underline{\boldsymbol{c}}_{\boldsymbol{1}}$	$\underline{c}_1$	$\underline{c}_2$	$\underline{c}$	$\underline{a}_1$	$\underline{a}_2$	$\underline{a}$	$\underline{b}_1$	$\underline{b}_2$	$\underline{b}$
$\underline{\boldsymbol{c}}_{\boldsymbol{2}}$	$\underline{c}_2$	$\underline{c}$	$\underline{c}_1$	$\underline{a}_2$	$\underline{a}$	$\underline{a}_1$	$\underline{b}_2$	$\underline{b}$	$\underline{b}_1$

Theorem 5.10. *All multiplicative H_v-fields on $(\boldsymbol{Z}_{10}, +, \cdot)$, with a non-degenerate fundamental field, and satisfy the above 4 conditions, are the following isomorphic cases:*
(I) *We have the only one hyperproduct,*

$$2 \otimes 4 = \{3, 8\}, \quad 2 \otimes 5 = \{0, 5\}, \quad 2 \otimes 6 = \{2, 7\}, \quad 2 \otimes 7 = \{4, 9\}, \quad 2 \otimes 9 = \{3, 8\},$$

$$3 \otimes 4 = \{2, 7\}, \quad 3 \otimes 5 = \{0, 5\}, \quad 3 \otimes 6 = \{3, 8\}, \quad 3 \otimes 8 = \{4, 9\}, \quad 3 \otimes 9 = \{2, 7\},$$

$$4 \otimes 5 = \{0, 5\}, \quad 4 \otimes 6 = \{4, 9\}, \quad 4 \otimes 7 = \{3, 8\}, \quad 4 \otimes 8 = \{2, 7\},$$

$$5 \otimes 6 = \{0, 5\}, \quad 5 \otimes 7 = \{0, 5\}, \quad 5 \otimes 8 = \{0, 5\}, \quad 5 \otimes 9 = \{0, 5\},$$

$$6 \otimes 7 = \{2, 7\}, \quad 6 \otimes 8 = \{3, 8\}, \quad 6 \otimes 9 = \{4, 9\}, \quad 7 \otimes 9 = \{3, 8\}, \quad 8 \otimes 9 = \{2, 7\}.$$

In all these cases the fundamental classes are
$[0] = \{0, 5\}$, $[1] = \{1, 6\}$, $[2] = \{2, 7\}$, $[3] = \{3, 8\}$, $[4] = \{4, 9\}$ *and* $(\boldsymbol{Z}_{10}, +, \otimes)/\gamma^* \cong (\boldsymbol{Z}_5, +, \cdot)$.
(II) *The cases with classes* $[0] = \{0, 2, 4, 6, 8\}$ *and* $[1] = \{1, 3, 5, 7, 9\}$, *and with fundamental field* $(\boldsymbol{Z}_{10}, +, \otimes)/\gamma^* \cong (\boldsymbol{Z}_2, +, \cdot)$, *are described as follows: In the multiplicative table only the results above the diagonal, we raise each of the products by putting one element of the same class of the results. We do not raise setting* 1, *and we cannot raise only the* $3 \otimes 7 = 1$. *The number of those* H_v-fields *is* 103.

Conflict of Interest: The author declares no conflict of interest.

References

[1] P. Corsini and V. Leoreanu, Application of Hyperstructure Theory, *Springer, Boston*, (2003).

[2] P. Corsini and T. Vougiouklis, From groupoids to groups through hypergroups, *Rend. Mat.*, 9 (1989), 173-181.

[3] B. Davvaz and V. Leoreanu-Fotea, Hyperring Theory and Applications, *International Academic Press, USA*, (2007).

[4] B. Davvaz, R. M. Santilli and T. Vougiouklis, Multi-valued Hypermathematics for characterization of matter and antimatter systems, *J. Comp. Meth. Sci. Eng. (JCMSE)*, 13 (2013), 37-50.

[5] B. Davvaz, S. Vougioukli and T. Vougiouklis, On the multiplicative H_v-rings derived from helix hyperoperations, *Util. Math.*, 84 (2011), 53-63.

[6] B. Davvaz and T. Vougiouklis, A Walk Through Weak Hyperstructures, H_v-Structures, *World Scientific, Singapore*, (2019).

[7] D. Freni, Una nota sul cuore di un ipergruppo e sulla chiusura transitiva β^* *di* β, *Rivista Mat. Pura Appl.*, 8 (1991), 153-156.

[8] M. Koskas, Groupoides, demi-groupoides et hypergroups, *J. Math. Pures Appl.*, 49(9) (1970), 155-192.

[9] R. M. Santilli, Embedding of Lie-algebras into Lie-admissible algebras, *Nuovo Cimento*, 51 (1967), 570-576.

[10] R. M. Santilli, Studies on A. Einstein, B. Podolsky and N. Rosen argument that quantum mechanics is not a complete theory, I: Basic methods, *Ratio Mathematica*, 38 (2020), 5-69.

[11] R. M. Santilli and T. Vougiouklis, Isotopies, Genotopies, Hyperstructures and their Applications, *New Frontiers in Mathematics, Hadronic Press, Palm Harbor, United States*, (1996).

[12] T. Vougiouklis, Generalization of P-hypergroups, *Rend. Circolo Mat. Palermo, Ser.II*, 36 (1987), 114-121.

[13] T. Vougiouklis, The very thin hypergroups and the S-construction, *Combinatorics88, Incidence Geom. Comb. Str.*, 2 (1991), 471-477.

[14] T. Vougiouklis, The fundamental relation in hyperrings, The general hyperfield, 4^{th}AHA, Xanthi (1990), *World Scientific, Singapore*, (1991).

[15] T. Vougiouklis, Hyperstructures and their Representations, *Monographs in Math., Hadronic, Palm Harbor, United States*, (1994).

[16] T. Vougiouklis, Some remarks on hyperstructures, *Contemporary Math., Amer. Math. Society*, 184 (1995), 427-431.

[17] T. Vougiouklis, On H_v-rings and H_v-representations, *Discrete Math.*, 208/209, (1999), 615-620.

[18] T. Vougiouklis, The h/v-structures, *Discrete Math.*, 6 (2003), 235-243.

[19] T. Vougiouklis, ∂-operations and H_v-fields, *Acta Math. Sinica, Engl. Ser.*, 24 (2008), 1067-1078.

[20] T. Vougiouklis, From H_v-rings to H_v-fields, *Int. J. Algebr. Hyperstructures. Appl.*, 1(1) (2014), 1-13.

[21] T. Vougiouklis, H_v-fields, h/v-fields, *Ratio Mathematica*, 33 (2017), 181-201.

[22] T. Vougiouklis, Fundamental Relations in H_v-structures, The Judging from the Results proof, *J. algebr. hyperstrucres log. algebr.*, 1(1) (2020), 21-36.

[23] T. Vougiouklis, Minimal H_v-fields, *Ratio Mathematica*, 38 (2020), 313-328.

Thomas Vougiouklis
School of Science and Education
Democritus University of Thrace
Greece
E-mail: tvougiou@eled.duth.gr, thovou@gmail.com