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Markov Characteristics for IFSP and IIFSP

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Abstract. As the research object of modern nonlinear science, a fractal theory has been an important research content for scholars since it comes into the world. Moreover, iterated function system (IFS) is a significant research object of fractal theory. On the other hand, the Markov process plays an important role in the stochastic process. In this paper, the iterated function system with probability(IFSP) and the infinite function system with probability(IIFSP) are investigated by using interlink, period, recurrence and some related concepts. Then, some important properties are obtained, such as: 1. The sequence of stochastic variable $\{\zeta_n, (n \geq 0)\}$ is a homogenous Markov chain. 2. The sequence of stochastic variable $\{\zeta_n, (n \geq 0)\}$ is an irreducible ergodic chain. 3. The distribution of transition probability $p_{ij}^{(n)}$ based on $n \to \infty$ is a stationary probability distribution. 4. The state space can be decomposed of the union of the finite(or countable) mutually disjoint subsets, which are composed of non-recurrence states and recurrence states respectively.

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1 Introduction

As the research object of modern nonlinear science, a fractal theory has been an important research content for scholars since it comes into the world. One of its most important features is that fractal can truly describe natural phenomena. It was founded by an American mathematician Mandelbrot in the 1970s. With the vigorous promotion of Mandelbrot, fractal makes people's understanding of object shape change from regular to irregular gradually, and provides a new mathematical tool for the research of nonlinear characteristics and irregular phenomena. Fractal is being applied and explored in many fields with a new concept and theory, and its research has also greatly expanded the human cognitive domain [8, 10].

There are a lot of both this and that phenomenon, which have no clear boundary in the real world. This makes it necessary to go through a continuous repetition and accumulation change process from complete coincidence to complete non coincidence. In other words, the fractal set in nature cannot be described by the characteristic function with only two values of 0 or 1 in classical set theory. In 1965, American cybernetic expert Zadeh extended the value range of characteristic function in classical set theory from $\{0, 1\}$ to the closed interval $[0, 1]$, which is the core idea of fuzzy set. In order to apply fractal set to this fuzzy phenomenon, Xie Heping et al gave the concept of fuzzy fractal in 1990.

Iterated function system (IFS) is a significant research object of fractal theory founded by Hutchinson [6]. As the framework of fractal theory, the affine transformation is a critical mathematical tool. According

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to the self similar formation of the part and whole about the objective object, the overall shape is iterated on the basis of some affine transformations with a given probability until emerging a pretty fractal figure. With the help of the powerful iterative computing ability of a computer, IFS applies the essence of fractal theory such as self similarity, a multiplicity of levels and unity of rules at different levels to the field of computer graphics, and can produce many graphics with infinite detail and exquisite texture.

Markov process plays an important role in the stochastic process. The mathematician Markov proposed an interesting chain in 1907 called Markov chain nowadays. So far, it has formed a branch of mathematics with rich content, complete theory and wide application. The Markov process is a significant method for researching the discrete state space based on the theory of stochastic process [18].

The random fractal is another important field of the fractal theory. Because it is closely related to nature, it has become hot research in recent years. Moreover, the random iterative function system is also an important research object of the random fractal, and many scholars have done a lot of research work in this field. Such as Joanna used two methods to construct a gradually stable IFSP [11]. Weihrauch et al studied the IFSP from the perspective of measure [3, 9, 21, 22]. Andrzej et al researched the fractal dimension through IFSP [1, 4, 5, 16]. There is also some literature have researched the related problems between IFSP and Markov process [7, 13, 15, 19]. In particular, John et al used the Markov process to study the linear properties of IFS as early as 1990 [12]. However, some specific Markov Characteristics for IFSP and IIFSP almost no one has studied yet. Therefore, Our main purpose is to find and prove specific Markov Characteristics for IFSP and IIFSP in this paper.

In this article, we first review some important concepts and properties of the stochastic process and fractal theory in section 2. Then, we will introduce the homogeneous property for IFSP in section 3. In section 4, we will introduce the ergolic property for IFSP. In section 5, we will introduce the distribution property for IFSP. In section 6, we will introduce the decomposition of state space for IFSP. Then, we will extend the Markov Characteristics to IIFSP in section 7. The last section is conclusion and future work.

2 Preliminaries

Before studying the Markov Characteristics for IFSP and IIFSP, we introduce some important concepts that will be useful later in the subsections. First, we begin with the concept of the fractal theory.

Definition 2.1. [14] Let $y_n \in Y$ be a point sequence, if there exists a positive integer number $N(\varepsilon)$, such that $d(y_m, y_n) < \varepsilon$ for all $m, n > N(\varepsilon)$ and $\forall \varepsilon > 0$, then y_n is said to be a Cauchy sequence. Further, (Y, d) is known as a complete metric space, if each Cauchy sequence in Y converges to a point y in Y.

According to the complete metric space, we give the definitions of iterated function system(IFS) and hyperbolic iterated function system (HIFS).

Definition 2.2. [2, 14, 17] Suppose (Y, d) be a complete metric space, if there exists a family of continuous functions $f_k(k \in \{1, 2, \cdots, N\})$: Y → Y. Then

$$\{Y; f_k, k \in \{1, 2, \cdots, N\}\}$$

is called an iterated function system(IFS). Further,

$$\{Y; H; f_k, k \in \{1, 2, \cdots, N\}\}$$

is called a hyperbolic iterated function system (HIFS), if $f_k(k \in \{1, 2, \cdots, N\})$ is the contraction mapping (for $\forall x, y \in X$, there exists $0 \leq \alpha < 1$, such that $d(f_k(x), f_k(y)) \leq \alpha d(x, y))$ based on (Y, d).

With the help of IFS, HIFS and probability vector, we will introduce the definition of IFSP and HIFSP which are the most important concepts in this paper.

Definition 2.3. [2, 14, 17] Let $\{Y; f_k, k \in \{1, 2, \cdots, N\}\}$ be an IFS, and $P = \{p_1, p_2, \cdots, p_N\}$ be a probability vector, where $\sum_{i=1}^{N} p_i = 1$ and $p_i \geq 0$ for all $i \in \{1, 2, \cdots, N\}$. We have $P(\zeta_n = i) = p_i$ for the given independent random variables sequence $\zeta_1, \zeta_2, \cdots, \zeta_n, \cdots$, where $i \in \{1, 2, \cdots, N\}; n \in \{1, 2, 3, \cdots\}$. Then we call it is an iterated function system with probability(IFSP), denoted by

$$\{Y; P; f_k, k \in \{1, 2, \cdots, N\}\}.$$

Further,

$$\{Y; H; P; f_k, k \in \{1, 2, \cdots, N\}\}$$

is known as a hyperbolic iterated function system (HIFSP), if $f_k (k \in \{1, 2, \cdots, N\})$ is the contraction mapping based on (Y, d).

Theorem 2.4. *[2, 14, 17] Let $\{Y; H; f_k, k \in \{1, 2, \cdots, N\}\}$ be a HIFS based on (Y, d). Then $A \subset Y$ is a unique non-empty compact subset, such that*

$$A = f(A) = \bigcup_{k=1}^{N} f_k(A), \tag{1}$$

and $f^n(A_0) \longrightarrow A$, where A_0 is any element of the all nonempty compact subsets of Y. A is the attractor (or invariant set) of IFS.

The next theorem will give the relationship of probability between the random iterative sequence and the attractor of IFS through hausdorff distance h. $(h(A, B) = max\{d(A, B), d(B, A)\}, d(A, B) = max_{a \in A}\{min_{b \in B} d(a, b)\})$.

Theorem 2.5. *[2, 14, 17] Let $\{Y; H; P; f_k, k \in \{1, 2, \cdots, N\}\}$ be a HIFSP. For any $y_0 \in Y$, let $y_{n+1} = f_{\zeta_n}(y_n), n \in \{0, 1, 2, \cdots\}$., where $P(\zeta_n = i) = p_i, i \in \{1, 2, \cdots, N\}$. Then there exists $n_0 = n_0(\varepsilon)$ and $k_0 = k_0(\varepsilon)$, for any $\varepsilon > 0$, we have*

$$P\{h(\{y_n, y_{n+1}, \cdots, y_{n+k}\}, A) < \varepsilon\} > 1 - \varepsilon, \tag{2}$$

if $n \geq n_0$ and $k \geq k_0$, where A is the attractor (or invariant set) of IFS.

The above theorem tells us such a fact that if we remove n_0 items in front of the random iterative sequence $\{y_n\}$. So the possibility of the hausdorff distance between sufficiently long sequence $\{y_n, y_{n+1}, \cdots, y_{n+k}\}$ and the attractor is less than ε will exceed $1 - \varepsilon$. We have introduced the related concepts and properties of IFS in the previous. And we will introduce some significant concepts about Markov process in the sequel, which play an important role in this paper.

Definition 2.6. [18, 20, 23] Let $(\Omega, \mathcal{F}, P)$ be a probability space, and $\{X(n), (n \geq 0)\}$ be a random sequence. Then we have

$$\begin{aligned} P\{X(t_{m+1}) = i_{m+1} | X(t_1) = i_1, X(t_2) = i_2, \cdots, X(t_m) = i_m\} \\ = P\{X(t_{m+1}) = i_{m+1} | X(t_m) = i_m\}, \end{aligned} \tag{3}$$

if for any $m \geq 1$, and nonnegative integer $t_1 < t_2 < \cdots < t_m < t_{m+1}$, where $i_1, i_2, \cdots, i_{m+1} \in E$, E is the state space of $\{X(n), (n \geq 0)\}$. If the conditional probabilities at both ends of the equation are meaningful, then $\{X(n), (n \geq 0)\}$ is a Markov chain.

The above equation is often called the markov attribute(or the attribute of no aftereffect) in the Markov process. We find the random variables at each time have a certain dependence(i.e.,non independence) in the above Definition. More specifically, the past only affects the present, not the future.

Definition 2.7. [18, 20, 23] The equation

$$p^{\mu}_{ij}(m) = P\{X(m+\mu) = j|X(m) = i\}, i, j \in E, \mu \geq 1, \tag{4}$$

is said to be the $\mu - step$ transition probability of transferring to state j after μ steps, if the system is in state i at m.

It is said to be one-step transition probability apparently if $\mu = 1$, and short for transition probability. As we know, $p^{\mu}_{ij}(m)$ has the following important properties since it is a probability.

$$\begin{cases} p^{(\mu)}_{ij}(m) \geq 0, \quad j \in E, \\ \sum_{j\in I} p^{(\mu)}_{ij}(m) = \sum_{j\in I} P\{X(m+\mu) = j|X(m) = i\} = 1. \end{cases} \tag{5}$$

Then the matrix

$$P^{(\mu)}(m) = (p^{(\mu)}_{ij}(m))_{i,j\in E}, \quad m \in T = \{0, 1, 2, \cdots\},$$

is said to be the k steps transition matrix of $\{X(m)\}$.

It is not difficult to see $\{p^{(\mu)}_{ij}(m), j \in E\}$ is a probability distribution for any given $i \in E$ and $m \geq 0, \mu \geq 1$. We will introduce other important concepts called absolute probability and initial probability in the next Definition.

Definition 2.8. [18, 20, 23] $p_j(\mu) = P\{X(\mu) = j, j \in E\}$ is known as absolute probability, if μ is a nonnegative integer. Particularly, $p_j = p_j(0) = P\{X(0) = j, j \in E\}$ is known as initial probability.

Similarly, $p_j(\mu)$ and p_j also has the same properties as below:

$$\begin{cases} p_j(\mu) \geq 0, \quad j \in E, \\ \sum_{j\in I} p_j(\mu) = 1, \end{cases} \tag{6}$$

$$\begin{cases} p_j \geq 0, \quad j \in E, \\ \sum_{j\in I} p_j = 1, \end{cases} \tag{7}$$

Therefore, $\{p_j(\mu), (\mu \geq 0)\}$ and $\{p_j\}$ are both probability distributions. Particularly, $\{p_j\}$ is also called initial distribution. And $\{p_j(\mu), (\mu \geq 0)\}$ is the one dimension distribution in Markov chain known as absolute distribution commonly.

Theorem 2.9. *[18, 20, 23] Let $\{X(n), n \geq 0\}$ be a Markov chain. Then the following formula holds for any nonnegative integer μ, ν, m.*

$$p^{(\mu+\nu)}_{ij}(m) = \sum_{s\in E} p^{(\mu)}_{is}(m) p^{(\nu)}_{sj}(m+\mu), i, j \in E. \tag{8}$$

The above equation is known as Chapman–Kolmogorov equation, abbreviated as C-K equation. The C-K equation is an important result of transition probability.

3 The homogeneous property for IFSP

Homogeneity is a very important mathematical property, which describes the change characteristics of transition probability. In this section, we will discuss the homogeneity for IFSP through transition probability in Markov process.

Definition 3.1. [18, 20, 23] The Markov chain $\{X(n), n \geq 0\}$ is called homogeneous, if its one-step transition probability $\{p_{ij}(m), i, j \in E\}$ is independent of m, where E is the state space.

We will consider stochastic dynamical system which is determined by IFSP in Definition 2.3. In the stochastic dynamical system:

$$y_{n+1} = f_{\zeta_n}(y_n), n \in \{0, 1, 2, \cdots\}.$$

As we know, the steps of this iterative process are: first, we take the origin point $y_0 \in Y$, then we take the value of probability p_{j_0}

$$y_1 = f_{j_0}(y_0),$$

further, we take the value of probability p_{j_1}

$$y_2 = f_{j_1}(y_1).$$

In the same way, we obtain a random iterative sequence $\{y_n, (n \geq 0)\}$ by iterating one by one. What interests us is to select a random variable sequence $\{\zeta_n, (n \geq 0)\}$ that is determined by the random iterative sequence $\{y_n, (n \geq 0)\}$. $(\zeta_n = g(y_n) = j_n, j_n \in \{1, 2, \cdots, N\})$

Proposition 3.2. *Let the random variable sequence $\{\zeta_n, (n \geq 0)\}$ be determined by the random iterative sequence $\{y_n, (n \geq 0)\}$ on the IFSP, then $\{\zeta_n, (n \geq 0)\}$ is a finite homogeneous Markov chain.*

Proof. Due to the characteristic of random variable sequence $\{\zeta_n, (n \geq 0)\}$ in IFSP, it's not hard for us to find out y_{n+1} is determined by $y_0, y_1, \cdots, y_n$, then the following conditional probability equation holds:

$$P\{y_{n+1} = f_{j_n}(y_n)|y_1 = f_{j_0}(y_0), y_2 = f_{j_1}(y_1), \cdots, y_n = f_{j_{n-1}}(y_{n-1})\}$$

$$= P\{y_{n+1} = f_{j_n}(y_n)|y_n = f_{j_{n-1}}(y_{n-1})\}. \tag{9}$$

Where $j_0, j_1, \cdots, j_n \in \{1, 2, \cdots, N\}$, since $\zeta_n = j_n; n \in \{0, 1, 2, \cdots\}$. Thus $\{1, 2, \cdots, N\}$ is the state space of random variable sequence $\{\zeta_n, (n \geq 0)\}$, ζ_n is determined by x_n. So we obtain:

$$P\{\zeta_n = j_n|\zeta_0 = j_0, \zeta_1 = j_1, \cdots, \zeta_{n-1} = j_{n-1}\}$$

$$P\{\zeta_n = j_n|\zeta_{n-1} = j_{n-1}\}, \tag{10}$$

where $j_0, j_1, \cdots, j_n \in \{1, 2, \cdots, N\}$, Obviously, the probabilities at both ends of the equation make sense. So $\{\zeta_n, (n \geq 0)\}$ is a Markov chain. We will proof it is finite and homogeneous in the next.

In the iterative process of IFSP, we let

$$p_{ij}(m) = P\{\zeta_{m+1} = j|\zeta_m = i\}, i, j \in \{1, 2, \cdots, N\}$$

is the transition probability from state i to state j after m-th iteration of the stochastic system. Obviously, $p_{ij}(m)$ is the one-step transition probability of $\{\zeta_n, (n \geq 0)\}$. The iterative process is independent of m, and N is a finite positive integer in the state space. Thus $\{\zeta_n, (n \geq 0)\}$ is a finite homogeneous Markov chain. This completes the proof. □

Example 3.3. (General random walk) There is a particle in the line segment $[1, 3]$. It can only stay at the three points $1, 2, 3$, one movement per second. The move rule is: the particle is at any one of the points $1, 2, 3$ before moving, it either stays where it is or moves to any of the remaining three points in the next second, the probability are both $\frac{1}{3}$.

Firstly, according to the above example, we can construct a model of IFSP. i.e.,let

$$\{Y; P; f_k, k \in \{1, 2, 3\}\}$$

be an IFSP, where $\{x_n, n \geq 0\}$ is a random iterative sequence,

$$P = \{p_1, p_2, p_3\} = \{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\}$$

is the probability vector,and

$$f_{\zeta_n}(y) = \zeta_n, \quad \zeta_n \in \{1, 2, 3\}, \quad y \in \{1, 2, 3\}$$

is the iterated function.

Secondly, we can let $X(n) = \zeta_n = i$ be the particle is at point i at $t = n$ $(i = 1, 2, 3, n = 0, 1, 2, \cdots)$ through the above analysis. Then $\{y_n, (n \geq 0)\}$ is a random sequence, and the state space is $E = \{1, 2, 3\}$. Thus, ζ_n is a finite homogeneous Markov chain based on IFSP, owing to

$$P\{X(m+1) = j|A, X(m) = i\} = P\{X(m+1) = j|X(m) = i\}$$
$$= p_{ij}(m) = \frac{1}{3},$$

where A is known as any one event, which is determined by $X(0), \cdots, X(m-1)$.

Finally, we will give the transition probability matrix (one step) of $\{X(m)\}$ in the following.

$$P = (p_{ij}) = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

It not difficult to obtain the two step (or n step) transition probability matrix of $\{X(m)\}$ in the following.

$$P^{(2)} = (p_{ij}^{(2)}) = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$
$$= (p_{ij}^{(n)}) = P^{(n)}.$$

Example 3.4. (The random walk that cannot cross the wall) There is a particle in the line segment $[1, 5]$. It can only stay at the five points $1, 2, 3, 4, 5$, one movement per second. The move rule is: the particle is at any one of the points $2, 3, 4$ before moving, it either stays where it is, or move one space to the left, or move one space to the right in the next second, the probability are both $\frac{1}{3}$. If the particle is at 5 before moving, then it will move to point 4 with a probability of 1 in the next second. If the particle is at 1 before moving, then it will move to point 2 with a probability of 1 in the next second. Because 1 and 5 are “insurmountable walls” of the particle.

Firstly, similar to Example 3.3, we can construct a model of IFSP. i.e., let

$$\{Y; P; f_k, k \in \{1, 2, 3\}\}$$

be an IFSP, where $\{x_n, n \geq 0\}$ is a random iterative sequence.

$$f_1(y) = y, f_2(y) = y - 1, f_3(y) = y + 1$$

are the iterated functions,

$$P = \{p_1, p_2, p_3\} = \{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\}$$

is the probability vector, if the particle is at any one of the points $2, 3, 4$ before moving.

$$P = \{p_1, p_2, p_3\} = \{0, 0, 1\}$$

is the probability vector, if the particle is at the point 1 before moving.

$$P = \{p_1, p_2, p_3\} = \{0, 1, 0\}$$

is the probability vector, if the particle is at the point 5 before moving. It's quite easy to know, $\zeta_n \epsilon \{1, 2, 3\}$ which is determined by $X(n) = i$, so it is a finite homogeneous Markov chain based on IFSP due to Proposition 3.2.

Secondly, we can let $X(n) = i$ be the particle is at point i at time $t = n$ $(i = 1, 2, 3, 4, 5, n = 0, 1, 2, \cdots)$. Then $\{y_n, (n \geq 0)\}$ is a random sequence, and the state space is $E = \{1, 2, 3, 4, 5\}$. Thus, $X(n) = i$ is also a finite homogeneous Markov chain, owing to

$$P\{X(m+1) = j|A, X(m) = i\} = P\{X(m+1) = j|X(m) = i\}$$

$$= p_{ij}(m) = \begin{cases} 1, & if|j-i| = 1, i = 1, 5, \\ \frac{1}{3}, & if|j-i| \leq 1, i = 2, 3, 4, \\ 0, & otherwise, \end{cases}$$

where A is known as any one event, which is determined by $X(0), \cdots, X(m-1)$.

Finally, we will give the transition probability matrix (one step) of $\{X(m)\}$ in the following.

$$= P = (p_{ij}) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

It not difficult to obtain the transition probability matrix (two step) of $\{X(m)\}$ in the following.

$$P^{(2)} = (p_{ij}^{(2)}) = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ \frac{1}{9} & \frac{5}{9} & \frac{2}{9} & \frac{1}{9} & 0 \\ \frac{1}{9} & \frac{2}{9} & \frac{1}{3} & \frac{2}{9} & \frac{1}{9} \\ 0 & \frac{1}{9} & \frac{2}{9} & \frac{5}{9} & \frac{1}{9} \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}.$$

4 The ergodic property for IFSP

The development of the physical system can be regarded as a random process from the viewpoint of a quantitative relationships. The physical system always reaches equilibrium after a period of time, if there is no significant change in the reasons affecting the development of the system. It is of great significance to expose the internal law of this phenomenon with mathematical theory. This law is called ergodicity in a random process. More specifically, The ergodic property is to study the limit case of transition probability $p_{ij}^{(m)}$, for $m \to \infty$.

"Recurrent" is an important concept in the Markov chain. We can use it to further reveal many characteristics of the state. For the state j, we can pull-in random variables

$$F_j = min\{m : X(m) = j, (m \geq 1)\},$$

It indicates the time when the system enters the state j for the first time. If the set on the right of the above formula is empty (i.e. for any $m \geq 1, Y(m) \doteq j$), we stipulate $min\phi = +\infty$, and let

$$g_{ij}^{(m)} = P\{F_j = m|X(0) = i\}, m \geq 1,$$

be the probability that the system first reaches state j after m steps from state i. Now, we let

$$g_{ij} = \sum_{m=1}^{\infty} g_{ij}^{(m)} = \sum_{m=1}^{\infty} P\{F_j = m|(X0) = i\}$$

$$= P\{F_j < +\infty|X(0) = i\},$$

be the probability that the system will arrive sooner or later reaches state j from state i. In particular, g_{jj} means the probability that the system starting from state j and returning to state j sooner or later if $i = j$.

Definition 4.1. [18, 20, 23] The state j is said to be recurrence, if $g_{jj} = 1$; The state j is known as non-recurrence (or transience), if $g_{jj} < 1$.

Definition 4.2. [18, 20, 23] The greatest common divisor T of the positive integer set $\{m : m \geq 1, p_{jj}^{(m)} > 0\}$ is said to be the period of state j for state j, if the set is non empty. The state j is called periodic, if $T > 1$. The state j is known as aperiodic, if $T = 1$. The period of the state j cannot be defined, if the positive integer set $\{m : m \geq 1, p_{jj}^{(m)} > 0\}$ is an empty set.

Remark 4.3. Given recurrence state j, owing to

$$g_{jj} = P\{F_j < +\infty\} = 1,$$

this shows that starting from state i must return to itself. We can further subdivide the recurrence state, because of

$$g_{jj} = \sum_{m=1}^{\infty} g_{jj}^{(m)} = 1,$$

thus $g_{jj}^{(m)}$ is a probability distribution. We can describe this phenomenon with mathematical expectation, i.e

$$\nu_j = \sum_{m=1}^{\infty} m g_{jj}^{(m)} = \sum_{m=1}^{\infty} mP\{F_j = m|X(0) = j\}$$

$$= E\{F_j|X(0) = j\}.$$

It not hard to get $\nu_j \geq 1$, which signifies the mean of times (or steps) that the system starting from state j and also returning to state j. The state j is said to be positive recurrence, if $\nu_j < +\infty$. And the state j is known as null recurrence, if $\nu_j = +\infty$. Then the aperiodic and positive recurrence state is called ergodic state.

Lemma 4.4. *[18, 20, 23] Let j be a recurrence state and its period is T, then*

$$\lim_{m\to\infty} p_{jj}^{(mT)} = \frac{T}{\nu_j}.$$

The right end of the equation is equal to zero when $\nu_j = +\infty$. Most notably, the necessary and sufficient condition of the positive recurrence state j is

$$\overline{\lim_{m\to\infty}} p_{jj}^{(m)} > 0.$$

Definition 4.5. [18, 20, 23] For the state i and j, if there exists $m \geq 1$ satisfy $p_{ij}^{(m)} > 0$, i.e., Starting from the state i, after certain m steps, it can reach the state j. Then it is called the state i can reach state j, denoted by $i \to j$. Then, the state i and j are said to be interlinked, if $j \to i$ hold simultaneously. A chain is called irreducible, if any two states are interlinked in this chain.

Based on the above definitions and lemma, we will investigate the ergodic property for Markov chain $\{\zeta_n, (n \geq 0)\}$ in next.

Theorem 4.6. *Let $\{\zeta_n, (n \geq 0)\}$ be the random variable sequence which is determined by the random iterative sequence $\{y_n, (n \geq 0)\}$ in the IFSP. Then $\{\zeta_n, (n \geq 0)\}$ is an irreducible ergodic chain.*

Proof. Let $\{1, 2, \cdots, N\}$ be the state space of random variable sequence $\{\zeta_n, (n \geq 0)\}$, j is an any state in the state space$\{1, 2, \cdots, N\}$. According to the iterative process in IFSP, it is obvious that the positive integer set $\{n : n \geq 1, p_{jj}^{(n)} > 0\}$ is an empty set, and its greatest common divisor is $T = 1$. Therefore, the state j is called aperiodic owing to Definition 4.2.

Next, we will show the state j of the random variable sequence $\{\zeta_n, (n \geq 0)\}$ is a positive recurrence. It is not difficult to verify that the upper limit of transition probability of the state j returns to itself after n-step iteration is always greater than zero. i.e.

$$\overline{\lim_{n\to\infty}} p_{jj}^{(n)} > 0.$$

This completes the proof due to Lemma 4.4. Thus the state i of the random variable sequence $\{\zeta_n, (n \geq 0)\}$ is ergodic.

Finally, we obtain the state i and j are interlinked, for $\forall i, j \in \{1, 2, \cdots, N\}$, by the arbitrariness of iteration in IFSP. i.e.,

$$i \longleftrightarrow j.$$

It shows this chain is irreducible. Therefore, $\{\zeta_n, (n \geq 0)\}$ is an irreducible ergodic chain. □

5 The distribution property for IFSP

The law of probability distribution is used to describe the random value of probability variables. The stationary distribution is an important type of probability distribution, which has a certain kind of invariable property. It is often used to describe some characteristics of Markov process. In this section, we will consider the relationship between IFSP and Markov chain through the property of stationary distribution. The definition of stationary distribution in Markov chain and some properties will be given in the sequel.

Definition 5.1. [18, 20, 23] A probability distribution $\{u_j, j \in E\}$ is called stationary in the homogeneous Markov chain, if it satisfies

$$u_j = \sum_{i\in I} u_i p_{ij}, j \in E.$$

Remark 5.2. For the stationary distribution $\{u_j\}$, if $n \geq 1$ is integer number. It not difficult to verify the following equation hold

$$u_j = \sum_{i \in I} u_i p_{ij}^{(n)}, j \in E$$

Therefore, the initial distribution of the homogeneous Markov chain is stationary.

In the light of the ergodic in the previous section, the Markov chain $\{y_n, (n \geq 0)\}$ is known as ergodic, if there exists a constant π_j which be independent of i such that the following equation, for all state i and j of the homogeneous Markov chain $\{y_n, (n \geq 0)\}$.

$$\lim_{n \to \infty} p_{ij}^{(n)} = \pi_j.$$

It means the probability of transferring to the state j is approach to a constant π_j no matter what state the system starts, if the "transition step" n is large enough. We will give an important property related to the constant π_j in the next lemma.

Lemma 5.3. *[18, 20, 23] Let $\{y_n, (n \geq 0)\}$ be a finite homogeneous Markov chain (without losing generality, we can set the state space $E = \{1, 2, \cdots, N\}$). The Markov chain is ergodic, if there exists positive integer t for all state i and j satisfy*

$$p_{ij}^{(n)} > 0.$$

Therefore $\lim_{n \to \infty} p_{ij}^{(n)} = \pi_j$, where the constant π_j is independent of i. Moreover, $\pi_j (j \in \{1, 2, \cdots, N\})$ is the unique solution in the following formulas

$$\pi_j = \sum_{i=1}^{N} \pi_i p_{ij}, \quad j \in \{1, 2, \cdots, N\},$$

if it satisfies the conditions

$$\pi_j > 0, \quad j \in \{1, 2, \cdots, N\}, \quad \sum_{j=1}^{N} \pi_j = 1.$$

In the next theorem, we will discuss the distribution property of random iterative sequences based on IFSP.

Theorem 5.4. *Let $\{\zeta_n, (n \geq 0)\}$ be the random variable sequence which is determined by the random iterative sequence $\{y_n, (n \geq 0)\}$ on the IFSP. The state $i, j \in \{1, 2, \cdots, N\}$ are any two states of the state space. Then the limit distribution of transition probability $p_{ij}^{(n)}$ is a stationary probability distribution, if the state i transfers to the state j after n-step iteration.*

Proof. $\{\zeta_n, (n \geq 0)\}$ is an irreducible ergodic (aperiodic and positive recurrence) chain owing to Theorem 4.6. Then we will obtain the following equation through Lemma 5.3.

$$\lim_{n \to \infty} p_{ij}^{(n)} = \frac{1}{\psi_j}, \quad j \in \{1, 2, \cdots, N\}.$$

where $\frac{1}{\psi_j}$ is similar to π_j in Lemma 5.3. On the other hand, according to Definition 2.7 we get

$$\sum_{j=1}^{N} p_{ij}^{(n)} = 1.$$

Through the C-K equation

$$p_{ij}^{(\mu+\nu)} = \sum_{k=1}^{N} p_{ik}^{(\mu)} p_{kj}^{(\nu)}.$$

Without loss of generality, let $\mu = m$, and $\nu = n$. Then if $m \to \infty$, we gain

$$\frac{1}{\psi_j} = \sum_{k=1}^{N} (\frac{1}{\psi_k}) p_{kj}^{(n)}, \quad j \in \{1, 2, \cdots, N\}, \quad n \geq 1. \tag{11}$$

Thus $\frac{1}{\psi_j}$ is a stationary distribution. We will proof it is also a probability distribution. Also let $n \to \infty$, we have

$$\frac{1}{\psi_j} = \sum_{k=1}^{N} (\frac{1}{\psi_k})(\lim_{n\to\infty} p_{kj}^{(n)})$$

$$= \sum_{k=1}^{N} (\frac{1}{\psi_k})(\frac{1}{\psi_j}), \quad j \in \{1, 2, \cdots, N\}. \tag{12}$$

From the above equation, we immediately get

$$\sum_{k=1}^{N} \frac{1}{\psi_k} = 1.$$

Therefore, the limit of $p_{ij}^{(n)}$, i.e. $\{\frac{1}{\psi_j}, j \in \{1, 2, \cdots, N\}\}$ is a probability distribution. This completes the proof. □

6 The decomposition of state space for IFSP

The state space is an important concept in Markov process. The system state and the minimum number of variables in the system can be determined by the ordered set of variables known as the state. Therefore, the set of all possible states in the system constitutes a state space. It can considered to be the space with state variables as the coordinate axis. In this section, we will make a new state space that has a little different from the above, then extend the related properties of the Markov process to the IFSP.

Definition 6.1. $\{Y; P; f_1, f_2, \cdots, f_N\}$ is an IFSP, then

$$E = \{y : f_{\zeta_n}(y_n) = y, n \in \{0, 1, 2, \cdots\}\}$$

is the state space based on IFSP, where $\{y_n, (n \geq 0)\}$ is an random iterative sequence related to IFSP.

Definition 6.2. The subset D of the state space E based on IFSP is called a closed set, if the state inside D cannot reach the state outside D. i.e., $p_{ij} = 0$, for any $i \in D$, and $j \in E - D$.

As can be seen from the above definition, once the particle enters a closed set, it will always move in it and cannot reach the outside, denoted by

$$p_{ij}^{n} = 0, n \geq 1.$$

It not difficult to find E is the maximum closed set, and the minimum closed set is compose by all absorption state j, i.e. $p_{jj} = 1$.

Proposition 6.3. *A is the minimum closed set, if A is the attractor of a HIFS.*

According to Theorem 2.4, if A is the attractor of a HIFS, it can be deduced that once the random iterative sequences $y(n)$ enter to the attractor A, it can not get out. i.e.,

$$P_{ij} = 0, i \in A, j \in E - A.$$

Proposition 6.4. *Let $\{Y; H; P; f_k, k \in \{1, 2, \cdots, N\}\}$ be a HIFSP, if A is the minimum closed set based on it, then for any $y_0 \in Y$, let $y_{n+1} = f_{\zeta_n}(y_n), n \in \{0, 1, 2, \cdots\}.$, where $P(\zeta_n = i) = p_i, i \in \{1, 2, \cdots, N\}$, there exists $n^* = n^*(\varepsilon)$ and $k^* = k^*(\varepsilon)$, for any $\varepsilon > 0$, such that if $n \geq n^*$ and $k \geq k^*$, we have*

$$P\{h(\{y_n, y_{n+1}, \cdots, y_{n+k}\}, A) < \varepsilon\} > 1 - \varepsilon.$$

Proof. A is the minimum closed set based on a HIFSP in the above proposition, it not difficult to verify A is an attractor based on the HIFS in nature. Therefore, according to Theorem 2.5, the relationship between the minimum closed set A and the random iterative sequence $Y(n)$ can be obtained as follows.

$$P\{h(\{y_n, y_{n+1}, \cdots, y_{n+k}\}, A) < \varepsilon\} > 1 - \varepsilon.$$

□

Definition 6.5. [18] The closed set D is said to be irreducible, if D does not contain non empty true closed sets. The Markov chain $\{Y(n), (n \geq 0)\}$ is an irreducible chain, if its state space E is an irreducible set. i.e.there are no non empty sets except E. Otherwise, it is a reducible chain.

Proposition 6.6. *Let the random variable sequence $\{\zeta_n, (n \geq 0)\}$ be determined by the random iterative sequence $\{y_n, (n \geq 0)\}$ based on the HIFS, then $\{\zeta_n, (n \geq 0)\}$ is an reducible chain. Otherwise, $\{\zeta_n, (n \geq 0)\}$ is an irreducible chain.*

In virtue of Definition 6.1, $E = \{y : f_{\zeta_n}(y_n) = y, n \in \{0, 1, 2, \cdots\}\}$ is the state space based on IFS. If the IFS is hyperbolic, then $A \subset E$ is the attractor of IFS. Therefore E has non empty true closed sets. so $\{\zeta_n, (n \geq 0)\}$ is an reducible chain. On the other hand, if the IFS is not hyperbolic, $\{\zeta_n, (n \geq 0)\}$ is an irreducible chain.

Proposition 6.7. *Let the random variable sequence $\{\zeta_n, (n \geq 0)\}$ be determined by the random iterative sequence $\{y_n, (n \geq 0)\}$ based on the HIFSP. If $\{\zeta_n, (n \geq 0)\}$ is an reducible chain, then there exists $n^* = n^*(\varepsilon)$ and $k^* = k^*(\varepsilon)$, for any $\varepsilon > 0$, such that if $n \geq n^*$ and $k \geq k^*$, we have*

$$P\{h(\{y_n, y_{n+1}, \cdots, y_{n+k}\}, A) < \varepsilon\} > 1 - \varepsilon,$$

where A is the attractor of the IFS. Otherwise, $\{\zeta_n, (n \geq 0)\}$ is an irreducible chain.

Proof. The key to solving proposition is to ascertain the relationship between the attractor of the IFS and the random iterative sequence $Y(n)$. Similar to the above, we get the following formula.

$$P\{h(\{y_n, y_{n+1}, \cdots, y_{n+k}\}, A) < \varepsilon\} > 1 - \varepsilon.$$

Therefore, the conclusion of the proposition is tenable. □

Lemma 6.8. *[18] The homogeneous Markov chain is called irreducible, if and only if any two states in its state space are interlinked.*

Proposition 6.9. *Let the random variable sequence $\{\zeta_n, (n \geq 0)\}$ be determined by the random iterative sequence $\{y_n, (n \geq 0)\}$ based on the IFSP. $\{\zeta_n, (n \geq 0)\}$ is an reducible chain, if the IFSP is hyperbolic.*

Proof. If IFSP is hyperbolic, we can deduce that the IFSP must have a attractor A. Then there exists an n^*, such that if $n > n^*$, we have $x_n \in A$. That is to say, the random iterative sequence $\{y_n\}$ can not get out of A, if $n > n^*$.

Now we suppose E is the state space of the IFSP, if $j \in A, i \in E - A$, it not difficult to get $i \to j$ is true, and $j \to i$ is not true. i.e. the states A and $E - A$ are not interlinked. Therefore, $\{\zeta_n, (n \geq 0)\}$ is an reducible chain due to Lemma 6.8. □

Remark 6.10. The random variable sequence $\{\zeta_n, (n \geq 0)\}$ are determined by the random iterative sequence $\{y_n, (n \geq 0)\}$ based on the IFSP. It is not difficult to verify that $\{\zeta_n, (n \geq 0)\}$ is an irreducible chain, if the IFSP is not hyperbolic.

Lemma 6.11. *[18] The equivalence class $E(i)$ is irreducible, if it is a closed set.*

Theorem 6.12. *Let E be the state space of an IFSP. Then it can be decompose of mutually disjoint subsets which are the union of the finite(or countable) of states $G, D_1, D_2, \cdots$. i.e.,*

$$E = G \bigcup D_1 \bigcup D_2 \bigcup \cdots, \tag{13}$$

where G is the set that compose of the all non-recurrence states, and every $D_n (n = 1, 2, \cdots)$ is the closed set that compose of the recurrence states.

Proof. Let F be the set that compose of all the recurrence states based on the IFSP, and $G = E - F$ be the set that compose of all the non-recurrence states based on the IFSP. Take $i_1 \in F$ arbitrarily, denoted by $D_1 = E(i_1)$.

Now, let any $j \in E(i_1), k \in E$, if $j \leftrightarrow i$, then j and i are interlinked, thus $k \in E(i_1)$, $D_1 = E(i_1)$ is a closed set. Therefore, D_1 is an irreducible closed set owing to Lemma 6.11.

Finally, take any $i_2 \in F - D_1$, denoted by $D_2 = E(i_2)$. D_2 can be verified is an irreducible closed set as above. Go on like this, we get $D_1, D_2, D_3, \cdots$, and all of the recurrence state closed sets $\{D_n\}$ are mutually disjoint.Moreover,

$$F = D1 \bigcup D_2 \bigcup D_3 \bigcup \cdots.$$

This completes the proof. □

Example 6.13. Let $Y \times Z = [0, 1] \times [0, 1]$,

$$f_1 \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix},$$

$$f_2 \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix},$$

$$f_3 \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix}.$$

and $P_1 = P_2 = P_3 = 1/3$, we can construct a HIFSP, denoted by $\{Y; P; f_k, k \in \{1, 2, 3\}\}$. The attractor A of $\{Y; P; f_k, k \in \{1, 2, 3\}\}$ is called Sierpinski right triangle in fractal theory.

Through the above analysis, Sierpinski right triangle is the minimum closed set of $\{Y; P; f_k, k \in \{1, 2, 3\}\}$, and it is irreducible. Let E be the state space of $\{Y; P; f_k, k \in \{1, 2, 3\}\}$, then it can be decomposed of

$$E = A \bigcup E - A,$$

where A is the minimum closed set that compose of the all recurrence states, and $E - A$ is the set that compose of the non-recurrence states.

7 The Markov characteristics for IIFSP

We have investigated the Markov characteristics for IFSP in the above sections, and obtained many interesting results. However, these results are based on finite state space. That is to say, the iterative functions in the iterated function system (IFS) must be finite. We will study further what Markov characteristics will emerge if the iterative functions in the iterated function system are changed to denumerably infinite in the sequel, which also leads to denumerably infinite state space. The definition of IIFSP based on IFSP will be given first.

Definition 7.1. Let $\{Y; f_k, k \in \{1, 2, \cdots, N, \cdots\}\}$ be an IIFS, $P = \{p_1, p_2, \cdots, p_N, \cdots\}$ is a probability vector, where $p_i \geq 0$ for all $i \in 1, 2, \cdots, N, \cdots$, and$\sum_{i=1}^{\infty} p_i = 1$. We have $P(\zeta_n = i) = p_i$ for the independent random variables sequence $\zeta_1, \zeta_2, \cdots, \zeta_n, \cdots$, where $i \in \{1, 2, \cdots, N, \cdots\}, n \in \{1, 2, 3, \cdots\}$. Then it is called infinite iterated function system with probability(IIFSP),denoted by $\{Y; P; f_k, k \in \{1, 2, \cdots, N, \cdots\}\}$.

Corollary 7.2. *Let the random variable sequence $\{\zeta_n, (n \geq 0)\}$ be determined by the random iterative sequence $\{y_n, (n \geq 0)\}$ based on the IIFSP, then $\{\zeta_n, (n \geq 0)\}$ is an infinite homogeneous Markov chain.*

Corollary 7.3. *Let $\{\zeta_n, (n \geq 0)\}$ be the random variable sequence which is determined by the random iterative sequence $\{y_n, (n \geq 0)\}$ in the IIFSP. Then $\{\zeta_n, (n \geq 0)\}$ is an irreducible ergodic chain.*

Corollary 7.2 and Corollary 7.3 are the important extension of Proposition 3.2 and Theorem 4.6. The proof thought and process are also similar to the previous two theorems. Therefore, we omit the proof of Corollary 7.2 and Corollary 7.3 here.

Theorem 7.4. *Let $\{\zeta_n, (n \geq 0)\}$ be the random variable sequence which is determined by the random iterative sequence $\{y_n, (n \geq 0)\}$ in the IIFSP. $i, j \in Z^+$ are any two states in the state (positive integer) space of $\{\zeta_n, (n \geq 0)\}$, where $Z^+ = \{1, 2, \cdots, N, \cdots\}$. Then the limit distribution of transition probability $p_{ij}^{(n)}$ is said to be a stationary probability distribution, if the state i transfers to j after n-step iteration.*

Theorem 7.4 are an important extension of Theorem 5.4, but the proof thought and process is different to Theorem 5.4. We will give the detailed proof process in the following.

Proof. Similar to Theorem 5.4, $\{\zeta_n, (n \geq 0)\}$ is an irreducible ergodic (aperiodic and positive recurrence) chain owing to Theorem 4.6. Then we will obtain the following equation through Lemma 5.3.

$$\lim_{n \to \infty} p_{ij}^{(n)} = \frac{1}{\psi_j}, j \in Z^+.$$

On the other hand,

$$\sum_{j=1}^{N} p_{ij}^{(n)} \leq \sum_{j=1}^{\infty} p_{ij}^{(n)} = \sum_{j \in Z^+} p_{ij}^{(n)} = 1.$$

Let $n \to \infty$, and $N \to \infty$, then we get

$$\sum_{j=1}^{\infty} \frac{1}{\psi_j} \leq 1.$$

According to the C-K equation again, we have

$$\sum_{k=1}^{N} p_{ik}^{(\mu)} p_{kj}^{(\nu)} \leq \sum_{k=1}^{\infty} p_{ik}^{(\mu)} p_{kj}^{(\nu)} = p_{ij}^{(\mu+\nu)}.$$

Without loss of generality, suppose $\mu = m$, and $\nu = n$, and let $m \to \infty$, and $N \to \infty$, then we get

$$\sum_{k=1}^{\infty} (\frac{1}{\psi_k}) p_{kj}^{(n)} \le \frac{1}{\psi_j}, \quad j, n \in Z^+. \tag{14}$$

We will prove the equal sign of the above formula is also true by means of counter evidence for $j, n \in Z^+$ in the sequel. Suppose for some j or n, the equal sign of the above formula does not hold, then we get

$$\sum_{k=1}^{\infty} \frac{1}{\psi_k} = \sum_{k=1}^{\infty} (\frac{1}{\psi_k}) [\sum_{j=1}^{\infty} p_{kj}^{(n)}]$$

$$= \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} (\frac{1}{\psi_k}) p_{kj}^{(n)} < \sum_{j=1}^{\infty} \frac{1}{\psi_j} \le 1.$$

This is a contradiction. Thus, for all $j, n \in Z^+$, the equal sign hold, i.e.,

$$\frac{1}{\psi_j} = \sum_{k=1}^{\infty} (\frac{1}{\psi_k}) p_{kj}^{(n)}, \quad j, n \in Z^+. \tag{15}$$

Thus $\frac{1}{\psi_j}$ is a stationary distribution. We will proof it is also a probability distribution. The following inequality hold due to the above equation.

$$\sum_{k=1}^{N} (\frac{1}{\psi_k}) p_{kj}^{(n)} \le \frac{1}{\psi_j} \le \sum_{k=1}^{N} (\frac{1}{\psi_k}) p_{kj}^{(n)} + \sum_{k=N+1}^{\infty} (\frac{1}{\psi_k}).$$

Then let $k \to \infty$,$N \to \infty$ again, we get

$$\frac{1}{\psi_j} = \sum_{k=1}^{\infty} (\frac{1}{\psi_k}) \lim_{n \to \infty} (p_{kj}^{(n)})$$

$$= \sum_{k=1}^{\infty} (\frac{1}{\psi_k}) \frac{1}{\psi_j}, \quad j \in Z^+.$$

Through the above formula, we can get the following immediately.

$$\sum_{k=1}^{\infty} \frac{1}{\psi_k} = 1.$$

Therefore, the limit of $p_{ij}^{(n)}$, i.e. $\{\frac{1}{\psi_j}, j \in Z^+\}$ is a probability distribution. This completes the proof. □

Corollary 7.5. *Let E be the state space of an IIFSP. Then it can be decompose of mutually disjoint subsets which are the union of the finite (or countable) of states $G, D_1, D_2, \cdots$. i.e.,*

$$E = G \bigcup D_1 \bigcup D_2 \bigcup \cdots, \tag{16}$$

where G is the set that compose of the all non-recurrence states. And every $D_n (n = 1, 2, \cdots)$ is the closed set that compose of the recurrence states.

The proof process is similar to Theorem 6.12, we omit the proof here.

8 Conclusions

In this article, we research the Markov Characteristics for IFSP and IIFSP through interlink, period, recurrence and some related concepts and properties on the basis of predecessors's work. Then, there are four important results are obtained as follows:

1. The sequence of stochastic variable $\{\zeta_n, (n \geq 0)\}$ is a homogenous Markov chain.
2. The sequence of stochastic variable $\{\zeta_n, (n \geq 0)\}$ is an irreducible ergodic chain.
3. The distribution of transition probability $p_{ij}^{(n)}$ based on $n \to \infty$ is a stationary probability distribution.
4. The state space can be decomposed of the union of the finite(or countable) mutually disjoint subsets, which are composed of non-recurrence states and recurrence states respectively.

In the future, we can further study IFSP by some important theories in stochastic processes and fuzzy fractal, such as martingale theory, Poisson process, renewal process et al. These studies will not only enrich the fractal theory, but also enhance the relationship between random fractal and real life.

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