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Fuzzy Mathematics and Nonstandard Analysis Application to the Theory of Relativity

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Abstract. In this paper, we extend some results of nonstandard analysis to include concepts from fuzzy mathematics. We then apply our results to issues from special and general relativity and the theory of light-clocks. The extension includes concepts of fuzzy numbers, continuity, and differentiability. Our goal is to provide a new research area in fuzzy mathematics.

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1 Introduction

The purpose of this paper is to open the door for a new research area in fuzzy mathematics. This new area is based on nonstandard analysis. Mathematicians and physicists solved problems by considering infinitesimally small pieces of a shape, or movement along a path by infinitesimal amounts. Infinitesmals were ultimately rejected as mathematically unsound. However, in 1960 Abraham Robinson developed nonstandard analysis by rigorously extending the reals $\mathbb{R}$ to a field $\mathbb{R}^*$ which includes infintesimal numbers and infinite numbers. The goal was to create a system of analysis that was more intuitively appealing than standard analysis, but without losing any rigor of standard analysis, [1]. The standard notation for the field of hyperreals is ${}^*\mathbb{R}$, but for our purposes the notation $\mathbb{R}^*$ is easier to work with.

Our approach to introducing these concepts to fuzzy mathematics essentially involves replacing the interval $[0,1]$ with an extension of it to $\mathbb{R}^*$. There are two possible extensions. One is replacing $[0,1]$ with its natural extension $[0,1]^*$ or with $]^-0,1^+[$. A few scholars have replaced $[0,1]$ with $]^-0,1^+[$ in the definition of certain fuzzy sets, but have never used it in their research. These extensions have been discussed in [8]. In [3], Herrmann applies nonstandard analysis to explain issues from special and general relativity and the theory of light-clocks. In this paper, we extend some of the results in [3] to nonstandard fuzzy analysis. We do this in terms of nonstandard fuzzy functions and nonstandard fuzzy numbers. Related works can be seen in [4, 6, 7].

We let $\mathbb{N}$ denote the positive integers, and $\mathbb{R}$ the set of real numbers. We let $\wedge$ denote minimum or infimum and $\vee$ denote maximum or supremum. If X is a set, $\mathcal{P}(X)$ denotes the power set of X. If X and Y are sets, $X\backslash Y$ denotes set difference. If Y is a subset of X, we sometimes write Y^c for $X\backslash Y$.

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In Section 2, we present some basic concepts, definitions, and results from nonstandard analysis. The material presented here is a summary of known basic results needed in our presentation.

In Section 3, we extend these concepts to fuzzy mathematics. We first review the basic definitions and results of fuzzy numbers. We then extend the notion of a fuzzy number to that of a nonstandard fuzzy number.

In Section 4, we review some results concerning continuity and differentiability of functions that are pertinent to nonstandard analysis and which can be used in our application to the theory of relativity established in [3].

In Section 5, we show how the concepts of fuzzy numbers, continuity and differentiability can be applied to nonstandard analysis and an application to the theory of relativity. Much of the discussion is from [3].

2 Nonstandard Analysis

Much of the following is from [1].

Definition 2.1. *(Free Ultrafilter) A* ***filter*** *$\mathcal{U}$ on a set J is a subset of $\mathcal{P}(J)$ satisfying properties $(1)-(3)$. A filter $\mathcal{U}$ is called an* ***ultrafilter*** *if it satisfies (4) and an ultrafilter is called* ***free*** *if it satisfies (5).*

(1) *Proper: $\emptyset \notin \mathcal{U}$,*

(2) *Finite intersection property: If $A, B \in \mathcal{U}$, then $A \cap B \in \mathcal{U}$.*

(3) *Superset property: If $A \in \mathcal{U}$ and $A \subseteq B \subseteq J$, then $B \in \mathcal{U}$,*

(4) *Maximality: For all $A \subseteq J$, either $A \in \mathcal{U}$ or $J \backslash A \in \mathcal{U}$,*

(5) *Freeness: $\mathcal{U}$ contains no finite subsets.*

It is important to note that by $(1),(2)$ and (4), if $A \subseteq J$, then either $A \in \mathcal{U}$ or $J \backslash A \in \mathcal{U}$, but not both.

Lemma 2.2. *Let U be an ultrafilter on $\mathbb{N}$ and let $\{A_1, ..., A_n\}$ be a finite collection of disjoint subsets of $\mathbb{N}$ such that $\cup_{j=1}^{n} A_i = \mathbb{N}$. Then $A_i \in \mathcal{U}$ for exactly one $i \in \{1, ..., n\}$.*

Proof. Suppose that $\mathcal{U}$ contains no A_i. Then by (4), $\mathcal{U}$ contains A_i^c for each i. Thus by (2), contains $\cap_{i=1}^{n} A_i^c = (\cup_{i=1}^{n} A_i)^c = \mathbb{N}^c = \emptyset$, contrary to (1). Thus $\mathcal{U}$ contains some A_i. Suppose that $\mathcal{U}$ contains A_i and $A_j, i \neq j$. Then by (2), $\mathcal{U}$ contains $A_i \cap A_j = \emptyset$, contrary to (1). □

The proof of the next result uses Zorn' Lemma.

Lemma 2.3. *(Ultrafilter) Let A be a set and $F_0 \subseteq \mathcal{P}(A)$ be a filter on A. Then F_0 can be extended to an ultrafilter $\mathcal{F}$ on A.*

Proposition 2.4. *Free Ultrafilters exist.*

Definition 2.5. *Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$. Let $\mathbb{R}^{\mathbb{N}}$ denote the set of all real-valued sequences. Define the relation $=_{\mathcal{U}}$ on $\mathbb{R}^{\mathbb{N}}$ by $\forall (a_n), (b_n) \in \mathbb{R}^{\mathbb{N}}$, $(a_n) =_{\mathcal{U}} (b_n)$ if and only if $\{n \in \mathbb{N} | \ a_n = b_n\} \in \mathcal{U}$.*

Proposition 2.6. *$=_{\mathcal{U}}$ is an equivalence relation on $\mathbb{R}^{\mathbb{N}}$.*

Proof. Let $[(a_n)]_{\mathcal{U}}$ denote the equivalence class of $=_{\mathcal{U}}$ determined by (a_n).

Let $\mathbb{R}^* = \{[(a_n)]_U | \ a_n \in \mathbb{R}, \ n = 1, 2, ...\}$. Define addition $+$ and multiplication $\bullet$ on $\mathbb{R}^*$ as follows: $\forall [(a_n)]_{\mathcal{U}}, [(b_n)]_{\mathcal{U}} \in R^*$.

$$\begin{aligned} [(a_n)]_{\mathcal{U}} +_U [(b_n)]_{\mathcal{U}} &= [(a_n + b_n)]_{\mathcal{U}}, \\ [(a_n)]_{\mathcal{U}} \bullet_U [(b_n)]_{\mathcal{U}} &= [(a_n \bullet b_n)]_{\mathcal{U}}. \end{aligned}$$

It follows that $\mathbb{R}^*$ is a field under these operations. The result actually holds from the Transfer Principle. □

Theorem 2.7. *$(\mathbb{R}^*, +_{\mathcal{U}}, \bullet_{\mathcal{U}})$ is a field.*

Definition 2.8. *Define $\leq_{\mathcal{U}}$ on $\mathbb{R}^*$ as follows: $\forall[(a_n)]_{\mathcal{U}}, [(b_n)]_{\mathcal{U}} \in \mathbb{R}^*$, $[(a_n)]_{\mathcal{U}} \leq_U [(b_n)]_{\mathcal{U}}$ if and only if $\{n \in \mathbb{N}|\ a_n \leq b_n\} \in \mathcal{U}$.*

Let $[(a_n)]_{\mathcal{U}}, [(b_n)]_{\mathcal{U}}, [(b_n]_{\mathcal{U}} \in \mathbb{R}^*$. Suppose that $[(a_n)]_{\mathcal{U}} \leq [(b_n)]_{\mathcal{U}}$ and $[(a_n)]_{\mathcal{U}} \leq [(b_n)]_{\mathcal{U}}$. Then $\{j \in \mathbb{N}|\ a_j \leq b_j\} \in \mathcal{U}$ and $\{j \in \mathbb{N}|\ a_j \leq b_j\} \in \mathcal{U}$ By the finite intersection property, it follows that $\{j \in \mathbb{N}|\ a_j \leq c_j\} \in \mathcal{U}$ and so $[(a_n)]_{\mathcal{U}} \leq [(c_n)]_{\mathcal{U}}$.

Let $[(a_n)]_{\mathcal{U}}, [(b_n)]_{\mathcal{U}} \in \mathbb{R}^*$. Let $X = \{j \in \mathbb{N}|\ a_j \leq b_j\}$. Then either $X \in \mathcal{U}$ or $\mathbb{N}\backslash X \in \mathcal{U}$. If $X \in \mathcal{U}$, then $[(a_n)]_{\mathcal{U}} \leq_{\mathcal{U}} [(b_n)]_{\mathcal{U}}$. If $X \notin U$, then $\mathbb{N}\backslash X \in \mathcal{U}$, but $\mathbb{N}\backslash X = \{j \in \mathbb{N}|\ a_n > b_n\}n$ and so $[(a_n)]_{\mathcal{U}} >_U [(b_n)]_{\mathcal{U}}$. Thus $\leq_U$ is a total ordering on $\mathbb{R}^*$.

Definition 2.9. *A hyperreal number $[(a_n)]_{\mathcal{U}}$ in $\mathbb{R}^*$ is said to be **infinitesimal** if $[(a_n)]_{\mathcal{U}} \leq_{\mathcal{U}} [(j)]_{\mathcal{U}}$ for every $j \in \mathbb{N}$ and **infinite** if $[(j)]_{\mathcal{U}} \leq_{\mathcal{U}} [(a_n)]_{\mathcal{U}}$ for every $j \in \mathbb{N}$.*

Consider $[(1, 2, 3, ...)]_{\mathcal{U}}$. Let $j \in \mathbb{N}$. Since $\mathcal{U}$ is free, it contains all cofinite subsets. Thus $\mathcal{U}$ contains $\{m \in \mathbb{N}|\ m \geq j\}$. Hence $[(1, 2, 3, ...)]_{\mathcal{U}} \geq_{\mathcal{U}} [(j, j, j, ...)]_{\mathcal{U}}$ for all $j \in \mathbb{N}$. Thus $\mathbb{R}^*$ contains infinite elements. Similarly, it can be shown that $[(1, \frac{1}{2}, \frac{1}{3}, ...)]_{\mathcal{U}} \leq_{\mathcal{U}} [(\frac{1}{j}, \frac{1}{j}, \frac{1}{j}, ...)]_{\mathcal{U}}$ for fixed $j \in \mathbb{N}$. Hence $\mathbb{R}^*$ contains infinitesimal elements.

Define the function $f : \mathbb{R} \to \mathbb{R}^*$ by for all $a \in \mathbb{R}, f(a) = [(a, a, a, ...)]_{\mathcal{U}}$. It is easily shown that f is a one-to-one function of $\mathbb{R}$ into $\mathbb{R}^*$ that preserves addition and multiplication. It also follows easily for all $a, b \in \mathbb{R}$ that $a \leq b$ if and only if $f(a) \leq_{\mathcal{U}} f(b)$.

We review some postulates given in [2] that a nonstandard universe should possess. Actually, this universe has been rigorously constructed.

Let $\mathbb{R}^*$ denote a nonstandard universe with the following properties:

$(NS1)$ $(\mathbb{R}, +, \cdot, 0, 1, <)$ is an ordered subfield of $(\mathbb{R}^*, +, \cdot, 0, 1, <)$.

$(NS2)$ $\mathbb{R}^*$ has a positive infinitesimal element, that is $\varepsilon \in \mathbb{R}^*$ such that $\varepsilon > 0$, but $\varepsilon < r$ for all positive real numbers r.

$(NS3)$ For all $n \in \mathbb{N}$ and every function $f : \mathbb{R}^n \to \mathbb{R}$, there is a natural extension $f : (\mathbb{R}^*)^n \to \mathbb{R}^*$. The natural extensions of the field operations $+, \cdot : \mathbb{R}^2 \to \mathbb{R}$ coincide with the field operations in $\mathbb{R}^*$. Similarly, for every $A \subseteq R^n$, there is a subset $A^* \subseteq (\mathbb{R}^*)^n$ such that $A^* \cap \mathbb{R}^n = A$.

$(NS4)$ $\mathbb{R}^*$, equipped with the above assignments of extensions of functions and subsets, behaves logically like $\mathbb{R}$.

Definition 2.10. *$\mathbb{R}^*$ is called the ordered field of hyperreals.*

Now ε has an additive inverse $-\varepsilon$. It is easily seen that $-\varepsilon$ is a negative infinitesimal. Since $\varepsilon \neq 0$, it has a multiplicative inverse ε^{-1}. For any positive real number $r, \epsilon^{-1} > r$ since $\varepsilon < r$. Thus ε^{-1} is a positive infinite element and $-\varepsilon^{-1}$ is a negative infinite element.

Definition 2.11. (1) *Let $\mathbb{R}_{fin} = \{x \in \mathbb{R}^*|\ |x| \leq n$ for some $n \in \mathbb{N}\}$. $\mathbb{R}_{fin}$ is called the set of **finite hyperreals**.*

(2) *Let $\mathbb{R}_{\inf} = \mathbb{R}^*\backslash\mathbb{R}_{fin}$. $\mathbb{R}_{\inf}$ is called the set of **infinite hyperreals**.*

(3) *Let $\mu(0) = \{x \in \mathbb{R}^*|\ |x| \leq \frac{1}{n}$, for all $n \in \mathbb{N}\}$. $\mu(0)$ is called the set of **infinitesimal hyperreals**.*

We see that $\mu(0) \subseteq \mathbb{R}_{fin}, \mathbb{R} \subseteq \mathbb{R}_{fin}$, and $\mu(0) \cap \mathbb{R} = \{0\}$. If $\delta \in \mu(0)\backslash\{0\}$, then $\delta^{-1} \notin \mathbb{R}_{fin}$.

Proposition 2.12. (1) *$\mathbb{R}_{fin}$ is a subring of $\mathbb{R}^*$.*

(2) *$\mu(0)$ is an ideal of $\mathbb{R}_{fin}$.*

Definition 2.13. *Define the relation $\approx$ on $\mathbb{R}^*$ by for all $x, y \in \mathbb{R}^*, x \approx y$ if and only if $x - y \in \mu(0)$. If $x \approx y$, we say that x and y are **infinitely close**.*

It follows immediately that $\approx$ is an equivalence relation on $\mathbb{R}^*$. It also follows that $\approx$ is a congruence relation on $\mathbb{R}_{fin}$. This follows since $\mu(0)$ is an ideal of $\mathbb{R}_{fin}$.

Theorem 2.14. *(Existence of Standard Parts) Let $r \in \mathbb{R}_{fin}$. Then there exists a unique $s \in \mathbb{R}$ such that $r \approx s$. We call s the **standard part** of r and write $st(r) = s$.*

Corollary 2.15. $\mathbb{R}_{fin} = \mathbb{R} + \mu(0)$ *and* $\mathbb{R} \cap \mu(0) = \{0\}$.

Corollary 2.16. *Define $st : \mathbb{R}_{fin} \to \mathbb{R}$ by for all $r \in \mathbb{R}, st(r) = s$, where s is the standard part of r. Then st is a homomorphism of $\mathbb{R}_{fin}$ onto $\mathbb{R}$ such that $Ker(st) = \mu(0)$.*

Corollary 2.17. *The quotient ring $\mathbb{R}_{fin}/\mu(0)$ is isomorphic to $\mathbb{R}$, $\mu(0)$ is a maximal ideal of $\mathbb{R}_{fin}$, and is in fact the unique maximal ideal of $\mathbb{R}_{fin}$.*

Proof. Let $a \in \mathbb{R}_{fin} \backslash \mu(0)$. Then $a^{-1} \in \mathbb{R}^*$. However, $a^{-1} \notin \mathbb{R}^* \backslash \mathbb{R}_{fin}$ since $a \notin \mu(0)$. Thus $a^{-1} \in \mathbb{R}_{fin}$. That is, every element in $\mathbb{R}_{fin}$, but not in $\mu(0)$ has an inverse.

Let F_0 be the filter consisting of all cofinite subsets of $\mathbb{N}$. Let U be a free ultrafilter. Let $A \in F_0$. Then A or A^c is in U. However, A^c is not in U since A^c is finite. Thus $A \in U$. Hence $F_0 \subseteq U$.

Let (x_i) and (y_i) be sequences of real numbers. Define the relation $\simeq$ by $(x_i) \simeq (y_i)$ if and only if $\{i \in \mathbb{N} | \; x_i = y_i\} \in U$. Then $\simeq$ is an equivalence relation. Let $[(x_i)]_U$ denote the equivalence class of (x_i) with respect to $\simeq$.

Hence $[(x_i)]_U = [(y_i)]_U$ if and only if $\{i \in \mathbb{N} | \; x_i = y_i\} \in U$. $\square$

If we replace the notation $\leq_U$ by $\leq$, we have the following.

Definition 2.18. *Let $[(x_1, x_2, ...)]_U \leq [(y_1, y_2, ...)]_U$ if and only if $\{i \in \mathbb{N} | x_i \leq y_i\} \in U$.*

Define $\geq, <, >$ on $\mathbb{R}^*$ similarly.

Definition 2.11(3) becomes $\mu(0) = \{[(x_i)]_U | [(x_i)]_U < [(r, r, , ...)]_U$ for all $r \in \mathbb{R}, r > 0\}$.

Definition 2.13 is equivalent to $[(x_i)]_U \approx [(y_i)]_U$ if and only if $[(x_i)]_U - [(y_i)]_U \in \mu(0)$, i.e., $[(x_i - y_i)]_U < [(r, r, ...)]_U$ for all $r \in \mathbb{R}, r > 0$.

Definition 2.19. [[10], p.10] *Let $A \subseteq \mathbb{R}$. The **natural extension** of A to $\mathbb{R}^*$ is the set A^* defined to be the set of all $[(r_n)]_U$ such that $\{n \in \mathbb{N} | \; r_n \in A\} \in U$.*

Definition 2.20. [[10], p.10] *Let $f : X \to \mathbb{R}$, where X is a subset of $\mathbb{R}$. The natural extension of f to $\mathbb{R}^*$ is the function $f^* : X^* \to \mathbb{R}^*$ defined as follows:*

$$f^*([(r_n)]_U) = [(f(r_n))]_U.$$

Consequently, the natural extension of $[0, 1]$ to $\mathbb{R}^*$ is $[0, 1]^* = \{x \in \mathbb{R}^* | \; 0 \leq x \leq 1\}$.

Proposition 2.21. *Let $[a, b]$ be a closed interval in $\mathbb{R}$. Then $[a, b]^* = \{x \in \mathbb{R}^* | \; a \leq x \leq b\}$.*

Proof. We have that

$$\begin{aligned} [(r_n)]_U \in [a, b]^* &\Leftrightarrow \{n \in \mathbb{N} | \; r_n \in [a, b]\} \in U \\ &\Leftrightarrow \{n \in \mathbb{N} | \; a \leq r_n \leq b\} \in U \\ &\Leftrightarrow a = [(a, a, ...)]_U \leq [(r_n)]_U \leq [(b, b, ...)]_U = b. \end{aligned}$$

Consequently, the natural extension of $[0, 1]$ to $\mathbb{R}^*$ is $[0, 1]^* = \{x \in \mathbb{R}^* | 0 \leq x \leq 1\}$. $\square$

It is shown in [8] that $[0, 1]^* \neq]^-0, 1^+[= [0, 1] + \mu(0)$.

Let $a = [(a, a, a, ...,)]_U$ and $m = [(1, 1/2, 1/3, ...)]_U$. Then $a + m > a$. Define $A(a) = a$ for all $a \in \mathbb{R}$. Let A^* denote the natural extension of A to $\mathbb{R}^*$. Then $A^*([(x_1, x_2, ...)]_U) = [(A(x_1), A(x_2), ...]_U = [(x_1, x_2, ...)]_U$. Thus $A^*(a + m) > A(a)$.

3 Fuzzy Numbers

We review some basics of fuzzy numbers.

Definition 3.1. [[6], p.97] *Let A be a fuzzy subset of $\mathbb{R}$. Then A is a fuzzy number if the following conditions hold.*
(1) *There exists $x \in \mathbb{R}$ such that $A(x) = 1$.*
(3) *A^α is a closed bounded interval for all $\alpha \in (0, 1]$.*
(3) *The support of A is bounded.*

Theorem 3.2. [[6]. p.98] *Let A be a fuzzy subset of $\mathbb{R}$. Then A is a fuzzy number if and only if there is a closed interval $[c, d]$ and functions $l : (-\infty, c) \to [0, 1], r : (d, \infty) \to [0, 1]$, and $a, b \in \mathbb{R}, a \le c \le d \le b$ such that*

$$A(x) = \begin{cases} 1 & ;\ \text{if } x \in [c, d] \\ l(x) & ;\ \text{if } x \in (-\infty, c) \\ r(x) & ;\ \text{if } x \in (d, \infty), \end{cases}$$

where l is monotonic increasing, continuous from the right and such that $l(x) = 0$ for $x \in (-\infty, a); r$ is monotonic decreasing, continuous from the left and such that $r(x) = 0$ for $x \in (b, \infty)$.

Theorem 3.3. [[6], p.41] *Let A be a fuzzy subset of $\mathbb{R}$. Then $A = \cup_{\alpha\in[0,1]\alpha} A$, where ${}_\alpha A(x) = \alpha A^a(x)$ and $(\cup_{\alpha\in[0,1]\alpha} A)(x) = \vee\{_\alpha(A)(x)|\ x \in [0, 1]\}$ for all $x \in \mathbb{R}$.*

We next proceed to the second method for developing fuzzy arithmetic, which is the extension principle. Employing this principle, standard arithmetic operations on real numbers are extended to fuzzy numbers.

Let $*$ denote any of the four basic arithmetic operations and let A, B denote fuzzy numbers. Then define $A * B$ by for all $z \in \mathbb{R}$.

$$(A * B)(z) = \vee\{A(x) \wedge B(y)|\ z = x * y, x, y \in \mathbb{R}\}.$$

Theorem 3.4. [6] *Let $* \in \{+, -, \bullet, /\}$ and let A, B denote continuous fuzzy numbers. Then the fuzzy subset $A * B$ is a continuous fuzzy number.*

Let A be a fuzzy subset of $\mathbb{R}$. Let A^* be the natural extension of A to $\mathbb{R}^*$. Let $B = 1 - A$, i.e., for all $x \in \mathbb{R}, B(x) = 1 - A(x)$. Let B^* be the natural extension of B to $\mathbb{R}^*$. Let $[(x_1, x_2, ..., x_n, ...)]_U$. Then

$$\begin{aligned} B^*([(x_1, x_2, ..., x_n, ...)]_U) &= [(B(x_1), B(x_2), ..., B(x_n), ...]_U \\ &= [(1 - A(x_1), 1 - A(x_2), ..., 1 - A(x_n), ...]_U \\ &= [(1, 1, ..., 1, ...)]_U - [A(x_1), A(x_2), ..., A(x_n), ...]_U \\ &= 1 - A^*([(x_1, x_2, ..., x_n, ...)]_U. \end{aligned}$$

In the following, let A and B be continuous fuzzy numbers. Let A^*, B^*, and $(A + B)^*$ denote the natural extensions of A, B, and $A + B$ to $\mathbb{R}^*$.

Definition 3.5. Define $A^* + B^*$ as follows:

$$\begin{aligned} (A^* + B^*)(a + m) &= (A + B)(a) + m,\ \text{if } a \in \mathbb{R}, m \in \mu(0), \\ (A^* + B^*)(x) &= 0 \text{ if } x \in \mathbb{R}\backslash\mathbb{R}_{fin}. \end{aligned}$$

Let $a \in \mathbb{R}$. Then $(A^* + B^*)(a) = (A + B)(a) = (A + B)^*(a)$. Now $(A + B)^*(a + m) \approx (A + B)(a) \approx (A^* + B^*)(a + m)$ since $(A^* + B^*)(a + m) = (A^* + B^*)(a) + m$.

Definition 3.6. A^* is a nonstandard fuzzy number if the following properties hold:

(1) There exist $x \in \mathbb{R}^*$ such that $A^*(x) \approx 1$.

(2) $\forall \alpha \in [0,1]^*$, there exists $c_\alpha, d_\alpha \in [0,1]^*$ such that $c_\alpha \leq d_\alpha$ and $A^{*\alpha} = \{x \in \mathbb{R}^* | \ c_\alpha \lessapprox x \lessapprox d_\alpha\}$.

(3) There exists $c, d \in \mathbb{R}_{fin}$ such that $c \leq d$ and $\mathrm{NSupp}(A^*) \subseteq \{x \in R^* | \ c \lessapprox x \lessapprox d\}$.

Theorem 3.7. *Suppose A is continuous. Then A is a fuzzy number if and only if A^* is a nonstandard fuzzy number.*

Proof. Suppose A is a fuzzy number.

(1) Then there exists $x \in \mathbb{R}$ such that $A(x) = 1$. Hence $A^*(x) = 1$.

(2) Let $\alpha \in [0,1]^*$ and $y \in \mathbb{R}^*$. Suppose $A^*(y) \gtrapprox \alpha$. Since A^* is microcontinuous, $A^*(y) \approx A(st(y))$ and so $A(st(y)) \gtrapprox \alpha$. Thus $A(st(y)) \geq st(\alpha)$. Hence there exist $c_{st(\alpha)}, d_{st(\alpha)} \in [0,1]$ such that $c_{st(\alpha)} \leq d_{st(\alpha)}$ and $c_{st(\alpha)} \leq st(y) \leq d_{st(\alpha)}$. Thus $c_{st(\alpha)} \lessapprox y \lessapprox d_{st(\alpha)}$.

(3) Suppose $A^*(y) \notin \mu(0)$, where $y \in \mathbb{R}^*$. Now $A^*(y) \approx A(st(y))$. Thus $A(st(y)) > 0$. Hence there exists $c, d \in R$ such that $c \leq st(y) \leq d$. Thus $c \lessapprox y \lessapprox d$.

Conversely, suppose A^* is a nonstandard fuzzy number.

(1) Then there exists $y \in \mathbb{R}^*$ such that $A^*(y) \approx 1$. Hence $A(st(y)) = A^*(st(y)) = 1$.

(2) Let $\alpha \in [0,1]$ and $x \in \mathbb{R}$. Suppose $A(x) \geq \alpha$. Then $A^*(x) = A(x) \geq \alpha$. Thus $A^*(x) \gtrapprox \alpha$. Hence there exists $c_\alpha, d_\alpha \in [0,1]$ with $c \leq d$ such that $c_\alpha \lessapprox x \lessapprox d_\alpha$. Since $x \in R, c_\alpha \leq x \leq d_\alpha$.

(3) Suppose $A(x) > 0$, where $x \in \mathbb{R}$. Then $A^*(x) > 0$ and so $x \notin \mu(0)$. Thus there exists $c, d \in [0,1]^*$ such that $c \lessapprox x \lessapprox d$. Since $x \in \mathbb{R}, st(c) \leq x \leq st(d)$. Thus $\mathrm{Supp}(A) \subseteq [st(c), st(d)]$. □

Proposition 3.8. *Let C and D be nonstandard fuzzy subsets of $\mathbb{R}^*$. If C is a nonstandard fuzzy number and $C(y) \approx D(y)$ for all $y \in \mathbb{R}^*$, then D is a nonstandard fuzzy number.*

Proof. There exists $y \in \mathbb{R}$ such that $A(y) \approx 1$. Thus $B(y) \approx 1$.

Let $y \in R^*$. Let $\alpha \in [0,1]^*$. Then $B(y) \gtrapprox \alpha$ if and only if $A(y) \gtrapprox \alpha$. Thus B^α is bounded.

Now $A(y) \notin \mu(0)$ if and only if $B(y) \notin \mu(0)$ since $A(y) \approx B(y)$. Hence $\mathrm{NSupp}(B) = \mathrm{NSupp}(A)$. Thus $\mathrm{NSupp}(B)$ is bounded. □

Corollary 3.9. *Let A and B be fuzzy subsets of $\mathbb{R}$. Then $A + B$ is a fuzzy number if and only if $A^* + B^*$ is a nonstandard fuzzy number.*

Proof. $A + B$ is a fuzzy number if and only if $(A + B)^*$ is a fuzzy number. Now $(A + B)^*(y) \approx (A^* + B^*)(y)$ for all $y \in \mathbb{R}^*$. □

Proposition 3.10. *Let $a \in \mathbb{R}$ and $m \in \mu(0)$. Let $m = m' + m''$, where $m', m'' \in \mu(0)$. Then $(A^* + B^*)(a + m) = \vee\{(A^* \circ st)(b + m') \wedge (B^* \circ st(c + m'')|a = b + c\}$.*

Proof. For all such m', m'' (held fixed),

$$\begin{aligned} \vee\{(A^* \circ st)(b + m') \wedge (B^* \circ st(c + m'')|a &= b + c\} \\ &= \vee\{A^*(b) \wedge B^*(c)|a = b + c\} \\ &= \vee\{A(b) \wedge B(c)|a = b + c\} \\ &= (A + B)(a). \end{aligned}$$

Let A be a fuzzy subset of $\mathbb{R}$. Assume there exist real numbers a, b with $a \leq b$ such that $A(y) = 0$ for all $y \notin [a, b]$. □

Proposition 3.11. *If A^* is the natural extension of A to $\mathbb{R}^*$, then $A^*(y) = 0$ for all $y \in \mathbb{R}^* \backslash [a, b]^*$.*

Proof. Suppose $[(y_n)]_U \in \mathbb{R}^* \backslash [a, b]^*$. Then $\{n \in \mathbb{N} | \ a \leq y_n \leq b\} \notin U$ else $[(y_n)]_U \in [a, b]^*$. Hence $\{n \in \mathbb{N} | y_n \notin [a, b]\} \in U$ since either $\{n \in \mathbb{N} | \ a \leq y_n \leq b\} \in U$ or $\{n \in \mathbb{N} | \ a \leq y_n \leq b\}^c \in U$, but not both. Thus $\{n \in \mathbb{N} | \ A(y_n) = 0\} \in U$. Hence $A^*([(y_n)]_U) = [A(y_n)]_U = [(0, 0, ...)]_U$. It follows that A^* maps every element of $\mathbb{R}^* \backslash \mathbb{R}_{fin}$ to 0. □

4 Continuity and Differentiation

Definition 4.1. (See [10]) *(Nonstandard Definition of Continuity) Let* $f:\mathbb{R}\to\mathbb{R}$ *and* $a\in\mathbb{R}$. *Then* f *is continuous at* a *if and only if* $\forall\delta\approx 0, f^*(a+\delta)-f(a)\approx 0$, *where* f^* *is the natural extension of* f *to* $\mathbb{R}^*$.

Let $A:\mathbb{R}\to\mathbb{R}$ (or $[0,1]$) and let A^* be the natural extension of A to $\mathbb{R}^*\to\mathbb{R}^*$(or $[0,1]^*$). Let $a\in\mathbb{R}$. Then in $\mathbb{R}^*, a=[(a,a,...,a,..)]_U$ and $A^*(a)=[(A(a),A(a),...,A(a),...)]_U=A(a)$. That is, $A^*|_{\mathbf{R}}=A$.

Definition 4.2. [[10], p.11] *Let* $X\subseteq\mathbb{R}^*$. *Then a function* $f:X\to\mathbb{R}^*$ *is said to be* ***microcontinuous*** *at* $x_0\in X$ *if* $x\approx x_0$ *implies* $f(x)\approx f(x_0)$ *for all* $x\in X$.

Let $f:\mathbb{R}\to\mathbb{R}$ and $a\in\mathbb{R}$. If f is continuous at a, then f^* is microcontinuous at $a+m$ for all $m\in\mu(0)$. Note that if f is continuous at a, then $f^*(a+m)\approx f(a)$ for all $m\in\mu(0)$ and so $f(a+m)\approx f(a+m')$ for all $m,m'\in\mu(0)$.

Theorem 4.3. [[10], p.11] *A function* $f:X\to\mathbb{R}$ *is continuous at* $c\in\mathbb{R}$ *if and only if* f^* *is microcontinuous at* c.

Theorem 4.4. [[12], p.21] *The nonstandard definition of continuity given above is equivalent to the classic definition of continuity:* f *is continuous at* a *if and only if* $lim_{x\to a}f(x)=f(a)$.

Let A be a continuous fuzzy number. Then $\forall a\in\mathbb{R}$ and $\forall m\in\mu(0), A^*(a+m)\approx A(a)$. Since $A^*(a)=A(a)$ and $A^*:\mathbb{R}\backslash\mathbb{R}_{fin}\to\{0\}$,we have a “picture” of A^*.

Let A^* be the natural extension of A to $\mathbb{R}^*$. We could define A^* to be a nonstandard fuzzy number if A is a fuzzy number. Note $A^*|_{\mathbb{R}}=A$.

We need to review some results dealing with the differentiation and integration of fuzzy functions in order to make their connection to the work of [H] concerning relativity.

We denote the space of all fuzzy-valued functions on $[a,b]$ by $\mathcal{F}[a,b]$, or simply $\mathcal{F}$.

Definition 4.5. *Define the fuzzy subset* $\widetilde{F}$ *of* $\mathbb{R}$ *by* $\forall y\in\mathbb{R}$, $\widetilde{F}(y)=\int_a^b\widetilde{f}(x)(y)dx=\vee\{\wedge\{\widetilde{f}(x)(g(x))|\ a\leq x\leq b\}|g\in\mathcal{I}(a,b),y=\int_a^b g(t)dt\}$, *where* $\mathcal{I}(a,b)$*denotes the set of all integrable functions whose domain is* $[a,b]$.

Definition 4.6. *Let* $\widetilde{f}$ *be a fuzzy-valued function with level sets* $[f^-(a,x),f^+(\alpha,x)]$ *such that* $f^-(a,\_)$ *and* $f^+(\alpha,\_)$ *are integrable functions on the interval* $[a,b]$. *Let* $\widetilde{I}\in\widetilde{P}(\mathbb{R})$ *be defined by* $\forall y\in\mathbb{R}$,

$$\widetilde{I}(y)=\begin{cases}\vee\{\alpha\in(0,1)|\int_a^b f^-(\alpha,x)dx\leq y\leq\int_a^b\widetilde{f}(\alpha,x)dx\}, & \\ 0 & ;\ otherwise\end{cases}$$

Theorem 4.7. [[9], p.40] *Let* $\widetilde{I}$ *be defined as in Definition 4.6 Then* $\widetilde{I}=\widetilde{F}$.

Definition 4.8. [[10], p.12] *Let* $f:A\to\mathbb{R}$. *We say that* f *is differentiable at* $x_0\in A$ *if there exists* $L\in\mathbb{R}^*$ *such that for every nonzero infinitesimal* ε, *we have*

$$\frac{f^*(x_0+\varepsilon)-f^*(x_0)}{\varepsilon}\approx L.$$

If so, we define the derivative of f *at* x_0 *to be the standard part of* L, $f'(x_0)=st(L)$.

Theorem 4.9. [9] *Let $\widetilde{f}$ be a function of $\mathbb{D}$ into $\widetilde{P}(\mathbb{R})$ such that $\widetilde{f}(x)$ is a fuzzy number for all x in $\mathbb{D}$. Suppose that $\forall x \in \mathbb{D}, \forall \alpha \in [0,1], \widetilde{f}(x)_\alpha$ is a closed bounded interval. Then there exist unique functions f^-, f^+ of $[0,1] \times \mathbb{D}$ into $\mathbb{R}$ such that*

(1) $\forall x \in \mathbb{D},\ f^-(\_, x)\ (f^+(\_, x))$ *is a nondecreasing (nonincreasing) function of* α.

(2) $\forall (\alpha, x) \in [0,1] \times \mathbb{D},\ f^-(\alpha, x) \leq f^+(\alpha, x)$.

(3) $\forall (\alpha, x) \in [0,1] \times \mathbb{D},\ f(x)_\alpha = [f^-(\alpha, x), f^+(\alpha, x)]$.

(4) $\forall x \in \mathbb{D},\ f^-(1, x) = f^+(1, x)$.

Theorem 4.10. [9] *Let g and h be functions of $[0,1] \times \mathbb{D}$ into $\mathbb{R}$ such that $\forall x \in \mathbb{D},\ g(\_, x)\ (h(\_, x))$ is a nondecreasing (nonincreasing) function of α and $\forall (\alpha, x) \in [0,1] \times \mathbb{D}, g(\alpha, x) \leq h(\alpha, x)$. Let $\widetilde{f}$ be the function $\mathbb{D} \times \mathbb{R}$ into $[0,1]$ defined as follows: $\forall (x, y) \in \mathbb{D} \times \mathbb{R}$,*

$$\widetilde{f}(x,y) = \begin{cases} \vee\{\beta \in [0,1] | y \in g(\beta, x), h(\beta, x)]\} & ;\ \text{if } y \in [g(0,x), h(0,x)]\} \\ 0 & ;\ \text{otherwise.} \end{cases}$$

If $\forall x \in \mathbb{D},\ g(\_, x)$ and $h(\_, x)$ are continuous from the left, then $\widetilde{f}(x)_\alpha = [g(\alpha, x), h(\alpha, x)] \forall \alpha \in [0,1]$.

Theorem 4.11. [9] *Let g and h be functions of $[0,1] \times \mathbb{D}$ into $\mathbb{R}$ such that $\forall x \in \mathbb{D}, g(\_, x)\ (h(\_, x))$ is a nondecreasing (nonincreasing) function of α and $\forall (\alpha, x) \in [0,1] \times \mathbb{D}, g(\alpha, x) \leq h(\alpha, x)$. Suppose there exists a function $\widetilde{f}$ of $[0,1] \times \mathbb{D}$ into $[0,1]$ such that $\forall \alpha \in [0,1], \widetilde{f}(x)_\alpha = [g(\alpha, x), h(\alpha, x)]$. Then $\forall x \in \mathbb{D}, g(\_, x)$ and $h(\_, x)$ are continuous functions of α from the left. Furthermore,*

$$\widetilde{f}(x)(y) = \begin{cases} \vee\{\beta \in [0,1] | y \in g(\beta, x), h(\beta, x)]\} & ;\ \text{if } y \in [g(0,x), h(0,x)]\} \\ 0 & ;\ \text{otherwise.} \end{cases}$$

Let $x \in \mathbb{D}$ and hold x fixed. Let $f(x)^*$ be the natural extension of $f(x)$ to $\mathbb{R}^*$. ($f(x)$ is a fuzzy number so $f(x) : \mathbb{R} \to [0,1]$.We assume $f(x)$ is a continuous fuzzy number throughout.) Thus $f(x)^* : \mathbb{R}^* \to [0,1]^*$ since $f((x)^*([(r_1, r_2, ...)\}]_U = [(f(x)(r_1), f(x)(r_2), ...)]_U$ and $f(x)(r_i) \in [0,1], i = 1, 2, ...$ Note that since $f(x)(r_i) \geq 0, [(f(x)(r_1), f(x)(r_2), ...)]_U \geq [(0, 0, ...)]_U = 0$.

Write $f^-(x)$ for $f^-(\_, x)$ and $f^+(x)$ for $f^+(\_, x)$. Then $f^-(x)$ and $f^+(x)$ map $[0,1]$ into $\mathbb{R}$. Let $f^-(x)^*$ and $f^+(x)^*$ be the natural extensions of $f^-(x)$ and $f^+(x)$ to mappings of $[0,1]^*$ to $\mathbb{R}^*$, respectively. Let $\alpha \in [0,1]$. Then $f(x)_\alpha = [f^-(\alpha, x), f^+(\alpha, x)]$. Since $f(x)$ is a fuzzy number, there exists $a, b \in R$ such that $a \leq b$ and $f(x)(y) = 0$ if $y \notin (a, b)$. (a and b are dependent on x.) Now $a \leq f^-(x)(\alpha) \leq f^+(x)(\alpha) \leq b$ for $\alpha > 0$.

Proposition 4.12. *Let $a, b, c \in \mathbb{R}^*$. Then $a \lessapprox c \lessapprox b$ if and only if $st(a) \leq st(c) \leq st(b)$.*

Proof. We have that $a \approx c \Leftrightarrow st(a) = st(c)$. Also, $a < c \Leftrightarrow st(a) < st(c)$ or ($st(a) = st(c)$ and $nst(a) < nst(c)$). Similar arguments hold for c and b. □

Proposition 4.13. *Let $x \in \mathbb{D}$. Let $f(x)$ be a fuzzy number and $f(x)^*$ its natural extension to $\mathbb{R}^*$. Then for all $\alpha \in [0,1], f^-(x)^*(\alpha^*) \lessapprox y \lessapprox f^+(x)^*(\alpha^*)$ if and only if $f^-(\alpha, x) \leq st(y) \leq f^+(\alpha, x)$.*

Proof. Since $f^-(x)$ and $f^+(x)$ are continuous on $[0,1]$, $f^-(x)^*$ and $f^+(x)^*$ are micro-continuous on $[0,1]^*$. Now $f^-(x)^*(\alpha^*) \approx f^-(x)(\alpha)$ and $f^+(x)^*(\alpha^*) \approx f^+(x)(\alpha)$. Also, $st(f^-(x)^*(\alpha^*)) = f^-(x)(\alpha)$ and $st(f^+(x)^*(\alpha^*)) = f^+(x)(\alpha)$.

For $x \in \mathbb{D}$ fixed, $f(x)$ gives the shape of the fuzzy number at x. □

5 Relativity

We next make a connection to the theory of relativity. Let $q : \mathbb{D} \times \mathbb{R} \to [0,1]$. For example, let $\mathbb{D}$ be a time interval $[a,b]$ and let r be distance, [3]. Then $q(t,r)$ is the intensity with which an object travels r units in time t. We next consider $q^* : \mathbb{D}^* \times \mathbb{R}^* \to [0,1]^*$. The following Proposition shows that q^* can be considered to be the natural extension of q to $(\mathbb{D} \times \mathbb{R})^*$.

Proposition 5.1. *Define f from $\mathbb{D}^* \times \mathbb{R}^*$ into $(\mathbb{D} \times \mathbb{R})^*$ by for all* $([(t_1, t_2, ...)]_U, [(r_1, r_2, ...)]_U) \in \mathbb{D}^* \times \mathbb{R}^*$,

$$f(([(t_1, t_2, ...)]_U, [(r_1, r_2, ...)]_U)) = [((t_1, r_1), (t_2, r_2), ...)]_U.$$

Then f is a one-to-one function of $\mathbb{D}^* \times \mathbb{R}^*$ onto $(\mathbb{D} \times \mathbb{R})^*$.

Proof. We have

$$\begin{aligned}
([(t_1, t_2, ...)]_U, [(r_1, r_2, ...)]_U) &= ([(t'_1, t'_2, ...)]_U, [(r'_1, r'_2, ...)]_U) \Leftrightarrow \\
[(t_1, t_2, ...)]_U &= [(t'_1, t'_2, ...)]_U \text{ and } [(r_1, r_2, ...)]_U = [(r'_1, r'_2, ...)]_U \Leftrightarrow \\
(t_1, t_2, ...) &\approx (t'_1, t'_2, ...) \text{ and } (r_1, r_2, ...) \approx (r'_1, r'_2, ...) \Leftrightarrow \\
\{i | t_i &= t'_i\} \in U \text{ and } \{j | r_j = r'_j\} \in U.
\end{aligned}$$

Now $\{k |\ t_k = t'_k \text{ and } r_k = r'_k\} = \{i |\ t_i = t'_i\} \cap \{j |\ r_j = r'_j\} \in U$.
Also,

$$\begin{aligned}
[((t_1, r_1), (t_2, r_2), ...)]_U &= [((t'_1, r'_1), (t'_2, r'_2), ...)]_U \Leftrightarrow \\
((t_1, r_1), (t_2, r_2), ...) &\approx ((t'_1, r'_1), (t'_2, r'_2), ...) \Leftrightarrow \\
\{k | (t_k, r_k) &= (t'_k, r'_k)\} \in U \Leftrightarrow \\
\{k | t_k &= t'_k \text{ and } r_k = r'_k\} \in U.
\end{aligned}$$

Thus f is a one-to-one function of $\mathbb{D}^* \times \mathbb{R}^*$ onto $(\mathbb{D} \times \mathbb{R})^*$.

Let $l : \mathbb{D} \times \mathbb{R} \to [0,1]$ and $l^* : \mathbb{D}^* \times \mathbb{R}^* \to [0,1]^*$, where $l(t,r)$ is the intensity of the velocity $r \in \mathbb{R}$ of a particle at time $t \in \mathbb{D}$. □

The following is from [3].

In [[3], p.10], it is stated that Newton's approach created a schism in the philosophy of mathematical modeling. One group of scientists believed that there exists actual real world entities that can be characterized in terms of infinitesimal measures of time, mass, volume, and charge. Another group assumed that such terms are auxiliary in character and do not correspond to objective reality. The mathematical model called *the nonstandard physical world* {i.e. NSP-world) uses the corrected theory of the infinitesimally small and infinitely large, with other techniques, along with a new physical language theory of correspondence.

In [[3], p.27], it is stated that experiments show that for small time intervals $[a,b]$ the Galilean theory of average velocities suffices to give accurate information relative to the compositions of such velocities. Let there be an internal function $q : [a,b]^* \to \mathbb{R}^*$ where q represents the NSP-world distance function. Also, let nonnegative and internal $l : [a,b]^* \to \mathbb{R}^*$ be a function that yields the NSP-world velocity of the electromagnetic propagation at a time $t \in [a,b]^*$. As usual $\mu(t)$ denotes the monad of standard time $\mu(t)$, where "t" is an absolute NSP-world "time" parameter.

The general and correct methods of infinitesimal modeling state that, within the internal portion of the NSP-worlds, two measures m_1 and m_2 are *indistinguishable* for dt (i.e., infinitely close of order one) (notation $m_1 \sim m_2$) if and only if $0 \neq dt \in \mu(0)$,

$$\frac{m_1}{dt} - \frac{m_2}{dt} \in \mu(0).$$

Intuitively, indistinguishable in this sense means that, although within the NSP-world the two measures are only equivalent and not necessarily equal, the *first level* (or first-order) effects these measures represent over dt are indistinguishable within the N-world (i.e., they appear to be equal.)

In [3], some continuity conditions are placed on q and l. It is argued that for each $t \in [a,b]$ and $t' \in \mu(t) \cap [a,b]^*$,

$$\frac{q(t')}{t'} - \frac{q(t)}{t} \in \mu(0) \text{ and } l(t') - l(t) \in \mu(0).$$

The above expressions give relations between nonstandard time $t' \in \mu(t)$ and the standard time t. Recall that if $x, y \in R^*$, then $x \approx y$ if and only if $x - y \in \mu(0)$. It thus follows that for each $dt \in \mu(0)$ such that $t + dt \in \mu(t) \cap [a,b]^*$,

$$\begin{aligned} \frac{q(t+dt)}{t+dt} &\approx \frac{q(t)}{t}, \\ l(t+dt) + \frac{q(t+dt)}{t+dt} &\approx l(t) + \frac{q(t)}{t}. \quad (4.1) \end{aligned}$$

Hence

$$(l(t+dt) + \frac{q(t+dt)}{t+dt})dt \sim (l(t) + \frac{q(t)}{t})dt$$

Thus, it follows [3] that

$$q(t+dt) - q(t) \sim (l(t+dt) + \frac{q(t+dt)}{t+dt})dt$$

and

$$q(t+dt) - q(t) \sim (l(t) + \frac{q(t)}{t})dt. \quad (4.2)$$

It is stated in [3] that Expression (4.2) is the basic result that will lead to conclusions relative to the Special Theory of relativity. In order to find out exactly what standard functions will satisfy (4.2), let arbitrary $t_1 \in [a,b]$ be the standard time at which electromagnetic propagation from position F_1. Next, the definition of $\sim$, yields

$$\frac{s^*(t+dt) - s(t)}{dt} \approx l(t) + \frac{s(t)}{t}. \quad (4.3)$$

Note that l is microcontinuous on $[a,b]^*$. For each $t \in [a,b]$, the value of $l(t)$ is limited. Hence let $st(l(t)) = v(t) \in \mathbb{R}$. From Theorem 1.1 in [3] or 7.6 in [11], v is continuous on $[a,b]$. Now (4.3) may be rewritten as

$$(\frac{d(s(t)/t)}{dt})^* = \frac{v^*(t)}{t}, \quad (4.4)$$

where all functions in (4.4) are $*$-continuous on $[a,b]^*$. Consequently, we may apply the *-integral to both sides of (4.4). Now (4.4) implies that for $t \in [a,b]$

$$\frac{s(t)}{t} =^* \int_{t_1}^{t} \frac{v^*(x)}{x} dx$$

for $t_1 \in [a,b]$, $s(t_1)$ has been initialized to be zero.

We next provide a possible extension of these results to nonstandard fuzzy numbers. We define a nonstandard fuzzy number to be the natural extension of a fuzzy number to $\mathbb{R}^*$ into $[0,1]^*$. Consider the function l above. Let l be a function of $[a,b]^*$ into the set of nonstandard fuzzy numbers. Then for all $t' \in [a,b]^*$ and $r' \in \mathbb{R}_{fin}, l(t',r') \in [0,1]^*$ is the intensity with the velocity is r' at time t'. Define (it is argued that) $l(t',r') \wedge l(t,r)$ to be the intensity with which $r' - r \in \mu(0)$. Note that if $l(t',r') = 1$ and $l(t,r) = 1$, then the intensity with which $r' - r \in \mu(0)$ equals 1. Similar, interpretations can be given to the other equations given above. For example, let $v(t) = q(t)/t$ and consider v be a function of $[a,b]^*$ into the set of nonstandard fuzzy numbers.

We next consider the sum in (4.1). Let $v : [a,b] \times \mathbb{R} \to [0,1]$ be such that for all $t \in [a,b], v(t)$ is a fuzzy number. Let v^* be the natural extension of v. Then $v^* : [a,b]^* \times \mathbb{R}^* \to [0,1]^*$. Consider two such v_1 and v_2. Define $v_1^* + v_2^* : [a,b]^* \times \mathbb{R}_{fin} \to [0,1]^*$ as follows: For all $t' \in [a,b]^*$ and $r' \in \mathbb{R}_{fin}$,

$$\begin{aligned}(v_1^* + v_2^*)(t_1')(r') &= \vee\{st(v_1(t', r_1') \wedge v_2(t', r_2'))|r' = r_1' + r_2'; r_1', r_2' \in \mathbb{R}_{fin}\} \\ &\quad + nst(r_1') \vee nst(r_2'),\end{aligned}$$

where $nst(r_i')$ denotes the nonstandard part of $r_i',\ i = 1, 2$.

Consider the definition of $m_1 \sim m_2$. Let $m_1 = [(1, \frac{1}{4}, \frac{1}{9}, ...)]_U$ and $m_2 = [(1, \frac{1}{2}, \frac{1}{3}, ...)]_U$. Then $m_1 \approx m_2$. Let $t = [(1, \frac{1}{2}, \frac{1}{3}, ...)]_U$. Then $t \in \mu(0)$ and $\frac{m_1}{t} = [(1, \frac{1}{2}, \frac{1}{3}, ...)]_U$ and $\frac{m_2}{t} = [(1, 1, 1, ...)]_U$. Thus it's not the case that $\frac{m_1}{t} \approx \frac{m_2}{t}$. Hence it is not the case that $m_1 \sim m_2$.

Recall that $s(t)$ is the distance traveled at time t so $s : [a,b] \times \mathbb{R} \to [0,1]$ gives the intensity that the distance traveled at time t is $r, s(t,r) \in [0,1]$.

Also $v(t)$ is the velocity of a particle at time t so $v : [a,b] \times \mathbb{R} \to [0,1]$ gives the intensity that a particle is traveling r at time $t., v(t,r) \in [0,1]$.

Thus $(s(t)/t)(r)$ is the intensity that the velocity at time t is r. Also $l(t)(r)$, is the intensity that the velocity is r at time t. Hence $l(t) + s(t)/t$ is the sum of two fuzzy numbers which we define as follows: Given $t \in [a,b]$,

$$(l(t) + \frac{s(t)}{t})(r) = \vee\{l(t)(r_1) \wedge \frac{s(t)}{t}(r_2)|r = r_1 + r_2, r_1, r_2 \in \mathbb{R}\}$$

for all $r \in \mathbb{R}$.

Let f be integrable on the interval $[a,b]$. For all $t \in [a,b]$, define the function F of $[a,b] \to \mathbb{R}$ by for all $t \in [a,b], F(t) = \int_a^t f(x)dx$. Let f^* and F^* be the natural extensions of f and F to $\mathbb{R}$, respectively. Define $\int_a^t f^*(x)dx$ to be $F^*(t)$.

6 Conclusion

In this paper, we laid a foundation for a new research area in fuzzy mathematics, namely the use of nonstandard analysis. This can be accomplished by extending the field of real numbers to the field of hyperreals $\mathbb{R}^*$. Then the closed interval $[0,1]$ can be replaced by its natural extension to $[0,1]^*$. We point out that many theoretical results in $\mathbb{R}^*$ will automatically hold by the transfer principle. The use of $]^-0, 1^+[$ instead of $[0,1]^*$ would be more general, but would be a little more difficult since $]^-0, 1^+[$ is not the natural extension of $[0,1]$. Along these lines, scholars should be aware of the work of Klement an Mesiar, [5], where it is shown that many results of certain variations of fuzzy sets automatically hold from results of ordinary fuzzy sets.

Conflict of Interest: The authors declare no conflict of interest.

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