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Optimal Shape Design for Heat Conduction and Convection Problems Using NURBS

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Abstract

This article presents an optimal shape design methodology for heat conduction and convection problems. In this study, the shape of the conductive and convective medium is parameterized by means of non-uniform rational B-spline (NURBS) surfaces, and their control points represent the design variables. The conductive and convective domain is discretized by choosing the parameters of NURBS surfaces as generalized curvilinear coordinates, and the heat conduction and convection equation is solved using the finite difference method. The simplified conjugate-gradient method (SCGM) is used as the optimization method to obtain the optimal shape and adjust the design variables intelligently. By optimizing the profile of a straight fin with the objective of enhancing heat transfer rate and reducing the fin mass the methodology is demonstrated for conduction problems and by optimizing the shape profile of a natural convective cavity with the objective of reducing the maximum wall temperature the methodology is shown for convection problems.

***Keywords*:** Conduction, Convection, Heat Transfer, Optimal shape

1. Introduction

Optimal shape design for heat transfer problems is of great importance, since using an optimal design reduces the consumption of energy, matter and time. The aim of optimal shape design for a heat transfer system is to improve the performance of the system or to meet some specific heat transfer requirements such as specified heat flux or temperature distribution.

Extensive work has been done in shape design problems, such as fin profile optimization [1-3], shape design for heat conduction problems [4,5], shape design of a cylinder with heat transfer [6], shape design of millimeter-scale air channels [7], geometric optimization of radiative enclosures [8], shape optimization of convective periodic channels [9], shape optimization of a heat exchanger [10] and optimization of steady fluid-thermal systems [11] and design optimization of an air-filled cavity [12].

In general, optimal shape design problems require a great amount of computation time and memory space. This paper is aimed at describing a robust and efficient method for shape optimization of heat conduction problems by reducing the computation time and improving the accuracy and the quality of the optimal design.

In the discussion that follows, a parametric representation of the conductive (convective) domain geometry is presented. The computational methods for solving the heat conduction equation and the conservation equations are then discussed. Subsequently, the simplified conjugate-gradient method (SCGM) is described as the optimization method. Finally the methodology is demonstrated by optimizing the profile of a straight fin with the objective of enhancing heat transfer rate and reducing the fin mass and also by optimizing the shape profile of a natural convective cavity with the objective of reducing the maximum wall temperature

2. Parametric representation of the conductive (convective) domain geometry

The first step in optimal shape design for heat conduction (convection) problems is to specify the conductive (convective) domain geometry. The shape profile of the conductive (convective) medium could be either represented parametrically or built by using a point-by-point approach [4,5]. Parametric representation of the shape profile reduces the overall number of design variables and consequently the computation time. However, the point-by-point approach gives a wider range of shape alternatives.

In the present study, the shape of the conductive (convective) medium is parameterized by means of non-uniform rational B-spline (NURBS) surfaces, and their control points represent the design variables. These parametric surfaces allow free-form representation with total geometry control over the surface. The number of control points, and hence the number of degrees of freedom (DOFs) of the shape profile could be increased, if a

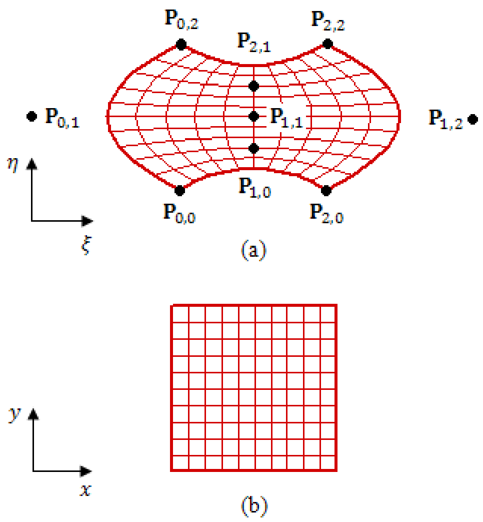

Fig. 1 Transformation of the physical domain (a) into the computational domain (b).

finer description of the shape and more flexibility in shape design are required.

A non-uniform rational B-spline (NURBS) surface is defined as

$$\mathbf{S}(\xi,\eta)=\frac{\sum_{i=0}^{n}\sum_{j=0}^{m}N_{i,p}(\xi)N_{j,q}(\eta)w_{i,j}\mathbf{P}_{i,j}}{\sum_{i=0}^{n}\sum_{j=0}^{m}N_{i,p}(\xi)N_{j,q}(\eta)} \qquad 0\le\xi,\eta\le 1 \tag{1}$$

where the $\mathbf{P}_{i,j}$ are the control points that form a bidirectional control net. The n and m are the number of control points in the ξ and η directions, respectively. The $w_{i,j}$ are the weights. The $N_{i,p}(\xi)$ and $N_{j,q}(\eta)$ are the non-rational B-spline basis functions of degree p and degree q, respectively, defined on the non-decreasing knot vectors

$$\Xi=\left\{\underbrace{0,\dots,0}_{p+1},\xi_{p+1},\dots,\xi_{r-p-1},\underbrace{1,\dots,1}_{p+1}\right\} \tag{2}$$

$$H=\left\{\underbrace{0,\dots,0}_{q+1},\eta_{q+1},\dots,\eta_{s-q-1},\underbrace{1,\dots,1}_{q+1}\right\} \tag{3}$$

where $r=n+p+1$ and $s=m+q+1$.

The i-th B-spline basis function of degree p, denoted by $N_{i,p}(\xi)$, is defined recursively by the Cox-De Boor formula as

$$N_{i,0}(\xi)=\begin{cases}1 & \text{if } \xi_i\le\xi<\xi_{i+1}\\ 0 & \text{otherwise}\end{cases}$$
$$N_{i,p}(\xi)=\frac{\xi-\xi_i}{\xi_{i+p}-\xi_i}N_{i,p-1}(\xi)+\frac{\xi_{i+p+1}-\xi}{\xi_{i+p+1}-\xi_{i+1}}N_{i+1,p-1}(\xi) \tag{4}$$

The shape of NURBS surfaces could be locally changed by moving the control points or modifying the weights. These surfaces have several unique properties that are effective and well suited for shape optimization. The surface interpolates the four corner control points, i.e. $\mathbf{S}(0,0)=\mathbf{P}_{0,0}$, $\mathbf{S}(1,0)=\mathbf{P}_{n,0}$, $\mathbf{S}(0,1)=\mathbf{P}_{0,m}$, and $\mathbf{S}(1,1)=\mathbf{P}_{n,m}$. Furthermore, the control points approximate the surface and the surface is contained in the convex hull of its control points. This property is very useful, especially in defining the geometric constraints. An example of a NURBS surface with its defining control points is depicted in Fig. 1. A complete description of NURBS surfaces can be found in [13].

3. Solution of the heat conduction equation

The second step in optimal shape design for heat conduction problems is to solve the heat conduction equation in the specified geometry. To this end, the conductive domain should be discretized first. Several methods are available for discretization and grid generation [14]. However, as in this study the conductive domain geometry is represented parametrically, through NURBS surfaces, it can be discretized by choosing the parameters of NURBS surfaces as generalized curvilinear coordinates. This method reduces the CPU time needed for grid generation significantly. Although this method is computationally efficient, the resulting grid could be highly skewed and non-uniform. When the grid becomes too distorted and the degree of non-uniformity becomes sever, the grid should be refined in order to

simulate heat transfer accurately. The use of generalized curvilinear coordinates transforms an irregular region in the physical domain into a rectangular region in the computational domain (Fig. 1).
In generalized coordinates, the steady-state heat conduction equation in a two-dimensional, homogeneous, isotropic conducting medium can be expressed as [15]

$$\begin{aligned}(y_\eta u - x_\eta v)\,\omega_\xi + (-y_\xi u + x_\xi v)\,\omega_\eta &= \nu[-\{(DELZI)\omega\}_\xi - \{(DELET)\omega\}_{\eta} + \{(GTT)\omega\}_{\xi\xi} \\ &+ \{(GWT)\omega\}_{\xi\eta} + \{(GWW)\omega\}_{\eta\eta}] + g\beta(y_\eta T_\xi - y_\xi T_\eta)\end{aligned} \tag{5}$$

where

$$\begin{aligned}-\{(DELZI)\psi\}_\xi - \{(DELET)\psi\}_\eta + \{(GTT)\psi\}_{\xi\xi} + \{(GWT)\psi\}_{\xi\eta} \\ + \{(GWW)\psi\}_{\eta\eta} = -(x_\xi y_\eta - x_\eta y_\xi)\omega\end{aligned} \tag{6}$$

$$\begin{aligned}(y_\eta u - x_\eta v)\,T_\xi + (-y_\xi u + x_\xi v)\,T_\eta \\ = \alpha[-\{(DELZI)T\}_\xi - \{(DELET)T\}_{\eta} + \{(GTT)T\}_{\xi\xi} \\ + \{(GWT)T\}_{\xi\eta} + \{(GWW)T\}_{\eta\eta}]\end{aligned} \tag{7}$$

where

$$\begin{aligned}
u &= \frac{-x_\eta\psi_\xi + x_\xi\psi_\eta}{x_\xi y_\eta - x_\eta y_\xi}, \\
v &= \frac{-y_\eta\psi_\xi - y_\xi\psi_\eta}{x_\xi y_\eta - x_\eta y_\xi}, \\
\mathrm{GTT} &\equiv \frac{x_\eta^2 + y_\eta^2}{x_\xi y_\eta - x_\eta y_\xi}, \\
\mathrm{GWT} &\equiv -\frac{2(x_\xi x_\eta - y_\xi y_\eta)}{x_\xi y_\eta - x_\eta y_\xi}, \\
\mathrm{GWW} &\equiv \frac{x_\xi^2 + y_\xi^2}{x_\xi y_\eta - x_\eta y_\xi}, \\
\mathrm{DELZI} &\equiv \frac{\mathrm{GTT}(x_\eta y_{\xi\xi} - y_\eta x_{\xi\xi})}{x_\xi y_\eta - x_\eta y_\xi} + \frac{\mathrm{GWT}(x_\eta y_{\xi\eta} - y_\eta x_{\xi\eta})}{x_\xi y_\eta - x_\eta y_\xi} + \frac{\mathrm{GWW}(x_\eta y_{\eta\eta} - y_\eta x_{\eta\eta})}{x_\xi y_\eta - x_\eta y_\xi}, \\
\mathrm{DELET} &\equiv \frac{\mathrm{GTT}(y_\xi x_{\xi\xi} - x_\xi y_{\xi\xi})}{x_\xi y_\eta - x_\eta y_\xi} + \frac{\mathrm{GWT}(y_\xi x_{\xi\eta} - x_\xi y_{\xi\eta})}{x_\xi y_\eta - x_\eta y_\xi} \\
&\quad + \frac{\mathrm{GWW}(y_\xi x_{\eta\eta} - x_\xi y_{\eta\eta})}{x_\xi y_\eta - x_\eta y_\xi}
\end{aligned} \tag{8}$$

The subscripts ξ and η represent partial derivatives with respect to ξ and η, respectively.
In the present study, the conservation equations with the imposed boundary conditions are discretized and solved iteratively using the finite difference method.

4. Optimization method

The last step in optimal shape design for heat conduction (convection) problems is to use an optimization method to adjust the design variables intelligently. The two most commonly used optimization methods for shape design problems are the genetic algorithms and the gradient-based optimization algorithms. The genetic algorithms are robust, and can be used for multi- objective problems. However, the main drawback to them is the reduced convergence rate. The gradient-based optimization algorithms are computationally efficient. But, the drawback of these methods is their tendency to get trapped in local optima. To remedy this problem, multiple optimizations should be performed, each starting from different values of design variables.
In this study, the simplified conjugate gradient method (SCGM), proposed by Cheng and Chang [16], is used as the optimization method. The SCGM is capable of dealing with various forms of the objective functions, and thus it is a well suited method for shape optimization. The iterative procedure of SCGM for finding the optimum design variables $\vec{\phi}$ and hence the optimal shape can be stated as follows:

(1) Define an objective function $f(\vec{\phi})$ that the minimum point of it corresponds with the optimal shape.

(2) Make an initial guess for $\vec{\phi}$ (as initial point). Set iteration number as $i = 1$.

(3) Solve the heat conduction equation and find the objective function $f(\vec{\phi})$ associated with the latest values of design variables.

(4) Compute the gradient of the objective function, $\vec{\nabla} f_i$, at the point $\vec{\phi}_i$, by means of the direct numerical sensitivity analysis [16].

(5) Compute the conjugate gradient coefficients β_i, and the search directions $\vec{S}_i$ as

$$\beta_i = \frac{\vec{\nabla} f_i^T \, \vec{\nabla} f_i}{\vec{\nabla} f_{i-1}^T \, \vec{\nabla} f_{i-1}} \tag{9}$$

$$\vec{S}_i = -\vec{\nabla} f_i + \beta_i \vec{S}_{i-1} \tag{10}$$

(6) Assign an appropriate fixed value to the step sizes λ_i^*, and update the design variables as

$$\vec{\phi}_{i+1} = \vec{\phi}_i + \lambda_i^* \vec{S}_i \tag{11}$$

(7) Test the new point $\vec{\phi}_{i+1}$ for optimality. If $\vec{\phi}_{i+1}$ is optimal, terminate the iteration process. Otherwise, set the new iteration number $i = i + 1$, and go to step (3).

5. Implementation for heat conduction problems

To demonstrate the performance of the methodology presented in this paper, the profile of a straight fin is optimized with the objective of enhancing heat transfer rate and reducing the fin mass. The fin geometry is shown in Fig. 2. For simplicity, the shape profile of the fin is presented parametrically, through a B-spline surface of degree three in the ξ direction and degree two in the η direction. B-spline surfaces

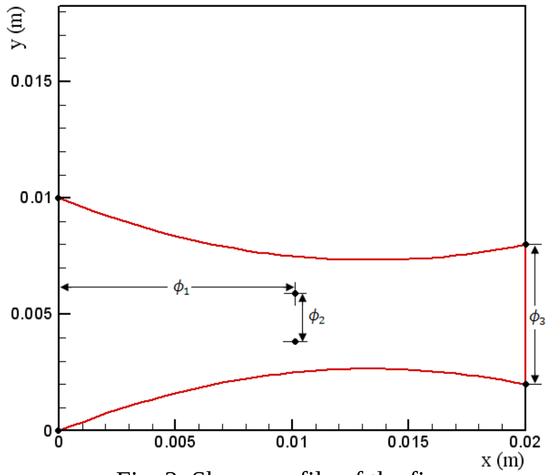

Fig. 2 Shape profile of the fin

are a special subclass of NURBS surfaces with $w_{i,j} = 1$ and the uniform knots distribution. The ambient temperature and the temperature of the base are set equal to 20°C and 70°C, respectively. The convective coefficient of the ambient h and the thermal conductivity of the fin k have been assumed constant and equal to 30 W/m^2 · K and 100 W/m K, respectively.

As shown in Fig. 2, the coordinates of selected control points represent the design variables $\vec{\phi}$. The following constraints have been imposed on the design variables $\vec{\phi}$ to restrict the fin dimensions and to prevent the grid from becoming too distorted and non-uniform.

$$\begin{aligned} &0.001 \le \phi_1 \le 0.019, \\ &0.001 \le \phi_2 \le 0.01, \\ &0.0001 \le \phi_3 \le 0.01 \end{aligned} \tag{12}$$

Now, in order to find vector of unknowns $\vec{\phi}$, an objective function $f(\vec{\phi})$ is defined as

$$f(\vec{\phi}) = \frac{1}{CQ + \frac{1}{M}} \tag{13}$$

where Q is the fin heat transfer rate and M is the fin mass. C is a constant that its value depends on the requirement of the design purpose. The minimum point of function f corresponds to the solution $\vec{\phi}$ of the problem. As explained previously, the computational method of the minimization procedure consists of two main modules; the direct problem solver and the search modules.

The temperature distribution in the optimal fin profile is shown in Fig. 3. Considering both the accuracy and the computational cost, the calculations were performed on a 40×40 grid system. Finer grids have been tested without finding any significant changes in the results.

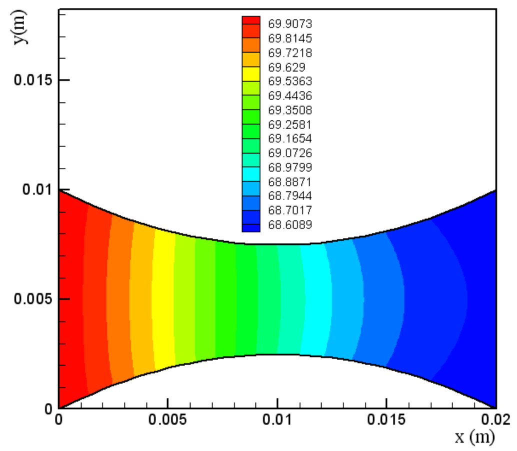

Fig. 3 Temperature distribution in the optimal fin profile (C=1000)

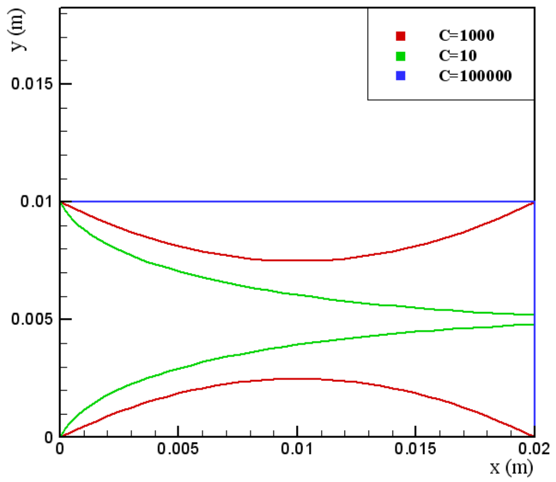

Fig. 4 Dependence of optimal shape on the value of C

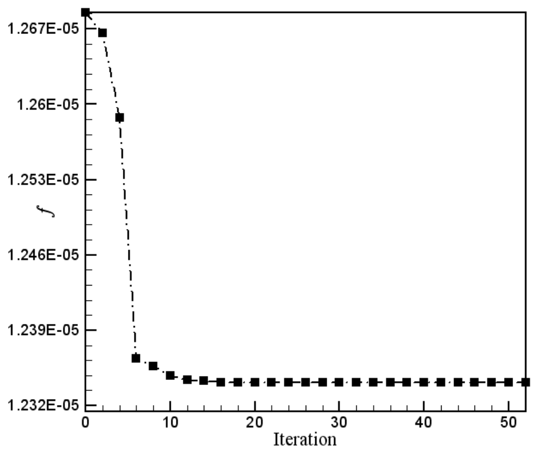

Fig. 5 Reduction history of the objective function per each cycle of the SCGM

A personal computer with a Pentium IV 3.2GHz processor has been used to perform the calculations. The CPU time required for the shape optimization problem is approximately 10-12 minutes. Fig. 4 shows the dependence of optimal shape on the value of C. The reduction history of the objective function f is shown in Fig. 5. The convergence criterion is set at $\|\nabla f\| \leq 10^{-10}$.

6. Implementation for heat convection problems

To demonstrate the performance of the methodology presented in this paper, the shape profile of a natural convective cavity is optimized with the objective of reducing the maximum wall temperature. The cavity geometry is shown in Fig. 6. For simplicity, the shape profile of the cavity is presented parametrically, through a B-spline surface of degree four in the ξ direction and degree two in the η direction. B-spline surfaces are a special subclass of NURBS surfaces with $w_{i,j} = 1$ and the uniform knots distribution. The cavity is filled with air and the top and bottom walls are insulated. The left vertical wall is subjected to a uniform heat flux of 200 W/m^2; while on the right vertical wall convection heat transfer takes place with the ambient. The ambient temperature and the convective coefficient of the ambient have been assumed constant and equal to 300 K and 20 W/m$^2 \cdot$ K, respectively.

As shown in Fig. 6, the coordinates of selected control points represent the design variables $\vec{\phi}$. The following constraints have been imposed on the design variables $\vec{\phi}$ to restrict the cavity dimensions and to prevent the grid from becoming too distorted and non-uniform.

$$\begin{aligned} &0.001 \leq \phi_1 \leq 0.5\phi_3 - 0.001, \\ &0.0 \leq \phi_2 \leq 0.02 \\ &0.01 \leq \phi_3 \leq 0.02 \end{aligned} \tag{14}$$

Now, in order to find vector of unknowns $\vec{\phi}$, an objective function $f(\vec{\phi})$ is defined as

$$f(\vec{\phi}) = T_{max} + CA \tag{15}$$

where T_{max} is the maximum wall temperature and A is the area of the cavity. C is a constant that its value depends on the requirement of the design purpose. The minimum point of function f corresponds to the solution $\vec{\phi}$ of the problem. As explained previously, the computational method of the minimization procedure consists of two main modules; the direct problem solver and the search modules.

The streamlines and isotherms of the optimal cavity profile are shown in Figs. 7 and 8, respectively. Considering both the accuracy and the computational cost, the calculations were performed on an 80×80 grid system. Finer grids have been tested without finding any significant changes in the results.

A personal computer with a Pentium IV 3.2GHz processor has been used to perform the calculations. The CPU time required for the shape optimization problem is approximately 7-10 hours. Fig. 9 shows the dependence of optimal shape on the value of C. The reduction history of the objective function f is shown in Fig. 10. The convergence criterion is set at $\|\nabla f\| \leq 10^{-10}$.

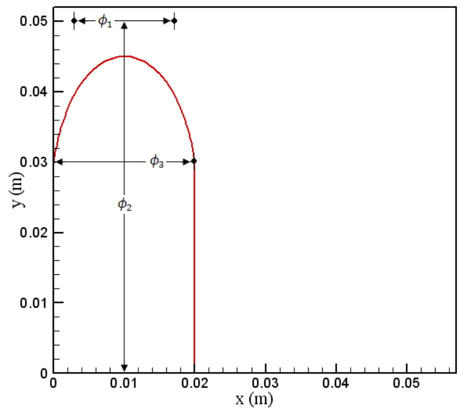

Fig. 6. Shape profile of the cavity

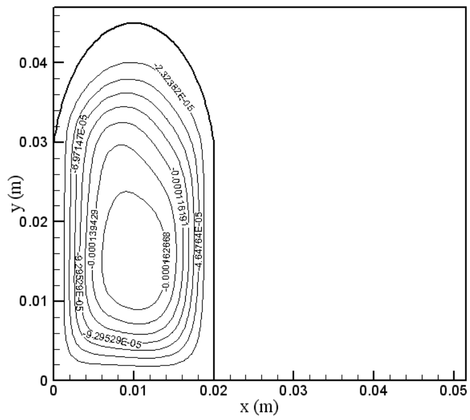

Fig. 7. Streamlines for the optimal cavity profile (C=1)

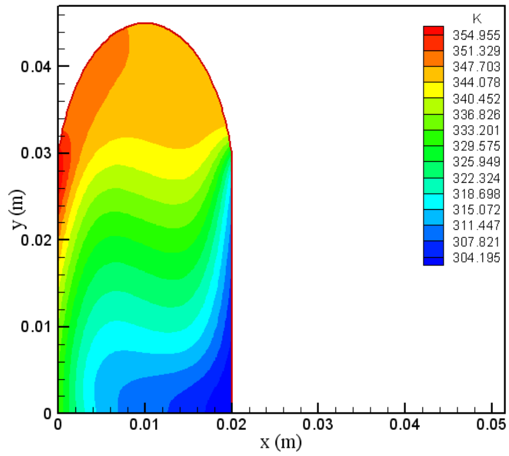

Fig. 8. Isotherms for the optimal cavity profile (C=1)

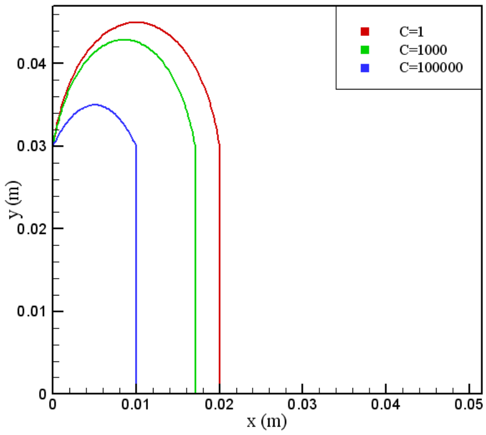

Fig. 9. Dependence of optimal shape on the value of C

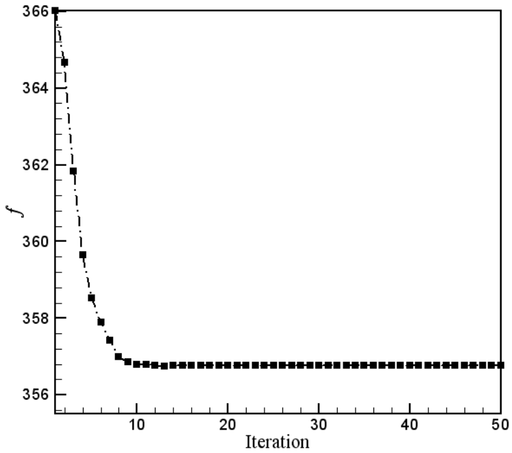

Fig. 10. Reduction history of the objective function per each cycle of the SCGM

7. Conclusions

In this paper a method is presented for shape optimization of heat conduction and convection problems. The shape of the conductive (convective) medium is represented parametrically, through non-uniform rational B-spline (NURBS) surfaces, and their control points represent the design variables. These parametric surfaces allow free-form representation with total geometry control over the surface. Moreover, parametric representation of the shape profile reduces the overall number of design variables and consequently the computation time. The simplified conjugate-gradient method (SCGM) is used as the optimization method to obtain the optimal shape and adjust the design variables intelligently. The SCGM is capable of dealing with various forms of the objective functions, and thus it is a well suited method for shape optimization. The performance of the proposed method is demonstrated by optimizing the shape profile of a straight fin with the objective of enhancing heat transfer rate and reducing the fin mass and also by optimizing the shape profile of a natural convective cavity with the objective of reducing the maximum wall temperature.

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