Two-stage Production Systems under Variable Returns to Scale Technology: A DEA Approach

Roza Azizi, Reza Kazemi Matin*

Islamic Azad University, Karaj Branch, Department of Mathematics, Karaj, Iran

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Abstract

Data envelopment analysis (DEA) is a non-parametric approach for performance analysis of decision making units (DMUs) which uses a set of inputs to produce a set of outputs without the need to consider internal operations of each unit. In recent years, there have been various studies dealt with two-stage production systems, i.e. systems which consume some inputs in their first stage to produce some intermediate outputs which are used as the inputs of the second stage in producing final outputs. One of these researches done by Kao and Hwang (2008) gives a decomposition of system efficiency score based on the efficiency of its sub-processes in the case of constant returns to scale (CRS) technology. This paper presents an extension of this approach for the technologies with variable returns to scale (VRS) and explains the results.

Keywords: Data envelopment analysis; two-stage systems; constant returns to scale; variable returns to scale

1. Introduction

Data envelopment analysis (DEA) introduced by Farrell 's [4] influential work is a non-parametric approach used for measuring the relative efficiency of a set of decision making units (DMUs) which convert the same inputs to the same outputs (Charnes, Cooper, & Rhodes [2]). DEA provides efficiency scores and efficient projections for inefficient units.

In recent years, various studies are carried out about the two-stage systems – systems which by consuming some inputs in their first stage produce some outputs used as the inputs of the second stage to produce final outputs. Some of these studies have used modified classical DEA models for the two-stage systems. Seiford and Zhu [9] dealt with the two-stage systems in the efficiency measurement of the US commercial banks. In their study, profitability is the first stage which uses labor and assets as inputs to produce profits and revenue as outputs. Marketability as the second stage uses profits and revenue as inputs while market value, returns and earnings per share are considered as the final outputs. Similarly, Lewis and Sexton [6] used the same method to study the performance of Major League Baseball. In another study, Schinnar et al. [8] used two Stage systems to evaluate the performance of mental health care programs. Chen and Zhu [3] introduced a new approach which uses unknown intermediate variables (first stage outputs used as inputs in second stage), and later Rho and An [7] completed Chen and Zhu's model by adding slacks to show weakly efficient units.

Our main focus in this paper is on Kao and Hwang's [5] relational model developed to measure the efficiency of a two-stage production system in which the total system efficiency can be decomposed into the product of two stage efficiencies. We can see through a simple numerical example that this approach only works under constant returns to scale assumption and some changes and modifications are needed in the model to get the same results for alternative returns to scale assumptions.

The rest of the paper is organized as follows: the next section introduces notation and terminology briefly. Section 3 presents Kao and Hwang’s [5] model for a two-stage production system under CRS assumption. Section 4 is devoted to an extension of the introduced model for the case of VRS technologies and presents the results. Finally, Section 5 includes the conclusions.

* Corresponding author E-mail: rkmatin@kiau.ac.ir.

2. Background

In DEA, it is most common to characterize each observed DMU by a pair of non-negative valued vectors $(X_j,Y_j) \in R_{+}^{m+s}$ with j∈{1,…,n} in which the input vector $X_j=(X_{1j},\dots,X_{mj})'$ is consumed to produce output vector $Y_j=(Y_{1j},\dots,Y_{sj})'$. With these notations, the radial DEA models presented by Charnes et al. [2] and Banker, Charnes, and Cooper [1] under CRS and VRS assumptions respectively could be shown by the following LP programs:

$$
\begin{aligned}
\theta_o = Max & \sum_r u_r y_{ro} \Big/ \sum_i v_i x_{io} \\
s.t \quad & \sum_r u_r y_{rj} \Big/ \sum_i v_i x_{ij} \le 1, \quad j=1,\dots,n \qquad (1)\\
& v_i, u_r \ge 0, \quad r=1,\dots,s,\ i=1,\dots,m
\end{aligned}
$$

$$
\begin{aligned}
\theta_o = Max & \sum_r u_r y_{ro} - u_o \Big/ \sum_i v_i x_{io} \\
s.t \quad & \sum_r u_r y_{rj} - u_o \Big/ \sum_i v_i x_{ij} \le 1, \quad j=1,\dots,n \qquad (2)\\
& v_i, u_r \ge 0, \quad r=1,\dots,s,\ i=1,\dots,m
\end{aligned}
$$

In the literature, the above models are called the fractional forms of the CCR and BCC, respectively and their optimal values yield a number between zero and one ($0<\theta_o \le 1$) as the efficiency score of the unit under evaluation. DMU_O is called efficient under these models if and only if its efficiency score is equal to one. We use the dual formulation of these models as the envelopment form of the CCR and BCC models in our ongoing discussion.

3. Review of Kao and Hwang's approach

A typical two-stage system is shown in Fig. 1 in which z_{dj} (d=1,…,D) are the outputs of the first stage and are consumed as inputs for the second stage (intermediate products).

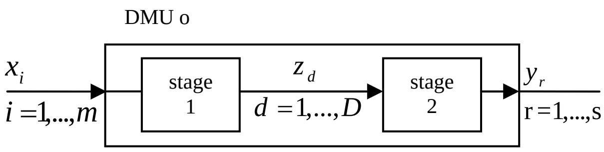

Fig. 1. A two-stage system

Kao and Hwang [5] use CCR model (1) for measuring the efficiency score of each stage with the following linear programs (LPs):

$$
\begin{aligned}
\theta_o^1 = Max & \sum_d w_d z_{do} \Big/ \sum_i v_i x_{io} \\
s.t \quad & \sum_d w_d z_{dj} \Big/ \sum_i v_i x_{ij} \le 1, \quad j=1,\dots,n \qquad (3)\\
& w_d, v_i \ge \circ \quad d=1,\dots,D \ \&\ i=1,\dots,m
\end{aligned}
$$

$$
\begin{aligned}
\theta_o^2 = Max & \sum_r u_r y_{ro} \Big/ \sum_d w_d z_{do} \\
s.t \quad & \sum_r u_r y_{rj} \Big/ \sum_d w_d z_{dj} \le 1, \quad j=1,\dots,n \qquad (4)\\
& w_d, u_r \ge \circ \quad d=1,\dots,D \ \&\ r=1,\dots,s
\end{aligned}
$$

Model (3) is used to assess the CCR efficiency score of the first stage with the pairs of (x, z) for input-output vectors and similarly, model (4) computes the CCR efficiency of the second stage with (z, y) data vectors. To link the two stages with the whole system, a model to describe this series relationship between the whole system and its two stages is needed.

In efficiency evaluation of the unit "o", if we denote the optimal weights of the above models by u^*, $\mathbf{v}^*$, and w^*, the multipliers that calculate the system efficiency θ_o and stage efficiencies θ_o^1 and θ_o^2 for this unit, we have:

$$
\begin{aligned}
\theta_o &= \sum_r u_r^* y_{ro} \Big/ \sum_i v_i^* x_{io} \le 1 \\
\theta_o^1 &= \sum_d w_d^* z_{do} \Big/ \sum_i v_i^* x_{io} \le 1 \\
\theta_o^2 &= \sum_r u_r^* y_{ro} \Big/ \sum_d w_d^* z_{do} \le 1
\end{aligned}
$$

With the assumption of equal weights for the intermediate products in both models (3) and (4), the following relation between the three efficiency scores could be interesting which leads the authors to give a decomposition of the whole system efficiency as the products of its sub-systems efficiencies, i.e. $\theta_o = \theta_o^1 \times \theta_o^2$.

Therefore, Kao and Hwang introduce a way to calculate the system efficiency θ_o, and obtain the above decomposition by taking into account the series relationship of the two stages in the production process. They consider the same weights for intermediate products of both stages to present their input-oriented model as follows:

$$
\begin{aligned}
\theta_o = Max & \sum_r u_r y_{ro} \Big/ \sum_i v_i x_{io} \\
s.t \quad & \sum_r u_r y_{rj} \Big/ \sum_i v_i x_{ij} \le 1, \ j=1,\dots,n \\
& \sum_d w_d z_{dj} \Big/ \sum_i v_i x_{ij} \le 1, \ j=1,\dots,n \qquad (5)\\
& \sum_r u_r y_{rj} \Big/ \sum_d w_d z_{dj} \le 1, \ j=1,\dots,n \\
& w_d, u_r, v_i \ge 0,\ d=1,\dots,D,\ i=1,\dots,m, r=1,\dots,s
\end{aligned}
$$

The efficiency score of this model is also between zero and one ($0<\theta_o \le 1$) and DMU_O is efficient in their model if and only if its computed efficiency score is equal to one. It should be noted that the second and third constraints of the model (5) entail the first one. So, the first constraint is redundant and can be omitted. The above fractional form of the model (5) can be transformed equivalently into the following LP:

$$\begin{aligned} \theta_o = Max & \sum_r u_r y_{ro} \\ s.t \quad & \sum_i v_i x_{io} = 1 \\ & \sum_d w_d z_{dj} - \sum_i v_i x_{ij} \le 0,\ j = 1,...,n \\ & \sum_r u_r y_{rj} - \sum_d w_d z_{dj} \le 0,\ j = 1,...,n \\ & w_d, u_r, v_i \ge 0,\ d = 1,...,D,\ i = 1,...,m, r = 1,...,s \end{aligned} \tag{6}$$

We also use the following form of the above model in the next section of the study.

$$\begin{aligned} & Min\ \theta \\ s.t \quad & \sum_j \lambda_j x_{ij} \le \theta x_{io},\ i = 1,...,m \\ & \sum_j \mu_j y_{rj} \ge y_{ro},\ r = 1,...,s \\ & \sum_j (\lambda_j - \mu_j) z_{dj} \ge 0,\ d = 1,...,D \\ & \lambda_j, \mu_j \ge 0,\ j = 1,...,n \end{aligned} \tag{7}$$

4. Kao and Hwang 's approach under variable return to scale (VRS) assumption

To adapt the results of Kao and Hwang's approach for the case of VRS technologies, we suggest the following input-oriented version of the model (5) which calculates the efficiency score of DMU_O.

$$\begin{aligned} \theta_o = Max \quad & \sum_r u_r y_{ro} - u_o - w_o \Big/ \sum_i v_i x_{io} \\ s.t \quad & \sum_d w_d z_{dj} - w_o \Big/ \sum_i v_i x_{ij} \le 1,\ j = 1,...,n \\ & \sum_r u_r y_{rj} - u_o \Big/ \sum_d w_d z_{dj} \le 1,\ j = 1,...,n \\ & w_d, u_r, v_i \ge 0,\ d = 1,...,D,\ i = 1,...,m, r = 1,...,s \\ & u_o, w_o\ free \end{aligned} \tag{8}$$

It is easy to verify that the optimal value of the model (7) is still between zero and one and could be interpreted as the efficiency score of the unit under evaluation. To use the results of duality theory in linear programs, we can use the following equivalent LP form of the model (8).

$$\begin{aligned} \theta_o = Max & \sum_r u_r y_{ro} - u_o - w_o \\ s.t \quad & \sum_i v_i x_{io} = 1 \\ & \sum_d w_d z_{dj} - \sum_i v_i x_{ij} - w_o \le 0,\ j = 1,...,n \\ & \sum_r u_r y_{rj} - \sum_d w_d z_{dj} - u_o \le 0,\ j = 1,...,n \\ & w_d, u_r, v_i \ge 0,\ d = 1,...,D,\ i = 1,...,m, r = 1,...,s \\ & u_o, w_o\ free \end{aligned} \tag{9}$$

If we write the dual form of the above model, we get the following model in VRS environment which has a structure similar to that of the model (7).

$$\begin{aligned} & Min\ \theta \\ s.t \quad & \sum_j \lambda_j x_{ij} \le \theta x_{io},\ i = 1,...,m \\ & \sum_j \mu_j y_{rj} \ge y_{ro},\ r = 1,...,s \\ & \sum_j (\lambda_j - \mu_j) z_{dj} \ge 0,\ d = 1,...,D \\ & \sum_j \lambda_j = 1, \sum_j \mu_j = 1 \\ & \lambda_j, \mu_j \ge 0,\ j = 1,...,n \end{aligned} \tag{10}$$

Now, based on the above modified version of Kao and Hwang's [5] model under VRS assumption, we have the following results for efficiency measurement of two-stage systems.

Theorem 1: If the efficiency score of the stage two is equal to 1, the system efficiency score could be represented as the product of the efficiencies of the two sub-processes.

Proof: If we write the BCC model (2) for the sub-processes, we have:

$$\begin{aligned} \theta_o^1 = Max & \sum_d w_d z_{do} - w_o \Big/ \sum_i v_i x_{io} \\ s.t \quad & \sum_d w_d z_{dj} - w_o \Big/ \sum_i v_i x_{ij} \le 1 \qquad j = 1,...,n \\ & w_d, v_i \ge \circ \qquad d = 1,...,D\ \&\ i = 1,...,m \\ & w_o\ free \end{aligned}$$

$$\begin{aligned} \theta_o^2 = Max & \sum_r u_r y_{ro} - u_o \Big/ \sum_d w_d z_{do} \\ s.t \quad & \sum_r u_r y_{rj} - u_o \Big/ \sum_d w_d z_{dj} \le 1 \qquad j = 1,...,n \\ & w_d, u_r \ge \circ \qquad d = 1,...,D\ \&\ r = 1,...,s \\ & u_o\ free \end{aligned}$$

The efficiency score of the stage two is 1, so we have a set of weights with

$$\frac{\sum_r u_r y_{ro} - u_o}{\sum_d w_d z_{do}} = 1 \Rightarrow \sum_r u_r y_{ro} - u_o = \sum_d w_d z_{do}$$

Therefore, using the same weights for the intermediate products, there exists a set of feasible weights for the stage 1 in which we have the following relation:

$$\frac{\sum_r u_r y_{ro} - u_o}{\sum_d w_d z_{do}} \times \frac{\sum_d w_d z_{do} - w_o}{\sum_i v_i x_{io}} = 1 \times \frac{\sum_d w_d z_{do} - w_o}{\sum_i v_i x_{io}} = \frac{\sum_r u_r y_{ro} - u_o - w_o}{\sum_i v_i x_{io}}$$

So, we can write the objective function of the model (10) as follows:

$$\frac{\sum_r u_r y_{ro} - u_o}{\sum_d w_d z_{do}} \times \frac{\sum_d w_d z_{do} - w_o}{\sum_i v_i x_{io}} = \frac{\sum_r u_r y_{ro} - u_o - w_o}{\sum_i v_i x_{io}}$$

This completes the proof.

As a direct result of the above theorem, we conclude that if the sub-processes operate in an efficient level, the system efficiency score computed by the model (10) is also 1.

Theorem 2: Under the VRS technology, the system efficiency score is not greater than the maximum efficiency score of its sub-processes.

Proof: Suppose θ shows the system efficiency score computed by the model (10) and let ρ and φ denote the optimal scores of the stages 1 and 2, respectively. We need to prove that $\theta \le \max\{\rho, \varphi\}$. To do this end, it is enough to show that the system efficiency score cannot exceed the efficiency score of the stage one.

The dual form of the model (2) for the stage 1 is as follows:

$$
\begin{aligned}
&Min\ \rho \\
&s.t\quad \sum_j \lambda_j x_{ij} \le \rho x_{io}, \quad i=1,\dots,m \\
&\qquad \sum_j \lambda_j z_{rj} \ge z_{do}, \quad r=1,\dots,s \qquad\qquad (11) \\
&\qquad \lambda_j \ge 0, \qquad j=1,\dots,n \\
&\qquad \sum_j \lambda_j = 1
\end{aligned}
$$

Also we can rewrite the model (10) as

$$
\begin{aligned}
&Min\ \theta \\
&s.t\quad \sum_j \lambda_j x_{ij} \le \theta x_{io}, \qquad i=1,\dots,m \qquad (a) \\
&\qquad \sum_j \mu_j y_{rj} \ge y_{ro}, \qquad r=1,\dots,s \qquad (b) \\
&\qquad \sum_j \lambda_j z_{dj} \ge \sum_j \mu_j z_{dj}, \quad d=1,\dots,D \qquad (c) \\
&\qquad \sum_j \lambda_j = 1, \qquad (d) \\
&\qquad \sum_j \mu_j = 1, \qquad (e) \\
&\qquad \lambda_j, \mu_j \ge 0, \qquad j=1,\dots,n \qquad (f)
\end{aligned}
$$

Suppose $(\rho, \boldsymbol{\lambda})$ denotes a feasible solution of the model (11), with $\mu_o = 1$ and $\mu_j = 0,\ (j \ne o)$ we get the following results for the other constraints of the model (10):

$$
\begin{aligned}
&\sum_j \lambda_j x_{ij} \le \rho x_{io}, \qquad i=1,\dots,m \\
&y_{ro} \ge y_{ro}, \qquad r=1,\dots,s \\
&\sum_j \lambda_j z_{dj} \ge z_{do}, \qquad d=1,\dots,D \\
&\sum_j \lambda_j = 1, \\
&\sum_j \mu_j = 1, \\
&\lambda_j, \mu_j \ge 0, \qquad j=1,\dots,n
\end{aligned}
$$

This means that $(\rho, \boldsymbol{\lambda}, \boldsymbol{\mu})$ is a feasible solution of the model (10) and hence in optimality $\theta \le \rho$.

The same result could be presented for the output-oriented version of Kao and Hwang's [5] approach under VRS assumption.

4.1. Illustration

Now, a simple numerical example is presented to gain further insights about the illustrated approach in VRS environments.

Example 1. Table 1 shows 7 hypothetical DMUs with one input (x), one intermediate product (z), and one output (y).

Table 1
Input/intermediate product/output data for 7 DMUs

DMU	*x*	*z*	*Y*
A	1	5	5
B	2	7	6
C	3	2	10
D	1	6	5
E	2	1	5
F	4	8	8
G	1.5	9	11

Figures 2 and 3 below show the sub-processes production set.

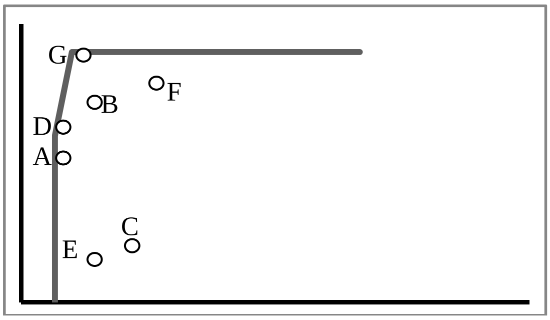

Fig. 2. Production possibility set (stage 1)

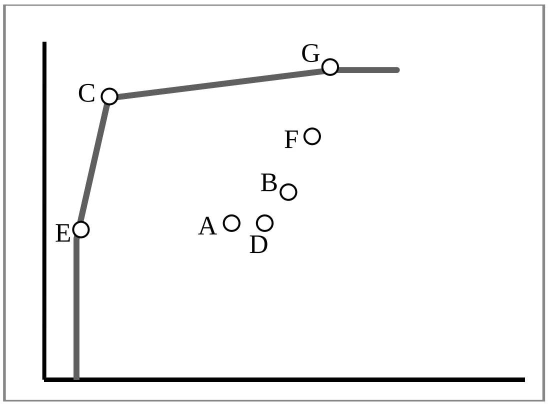

Fig. 3 Production possibility set (stage 2)

To do an efficiency analysis for this two-stage production system, we use the input-oriented BCC model (2) for the sub-processes and the model (10), as the adapted version of Kao and Hwang's model under VRS assumption, for the total system. The results are summarized in Table 2.

Table 2
Efficiency scores for sub-processes and the total system

DMU	Stage 1	Stage 2	System
A	1	0.2	1
B	0.5833	0.1714	0.5
C	0.3333	1	0.3333
D	1	0.1667	1
E	0.5	1	0.5
F	0.3333	0.2	0.25
G	1	1	1

For these data setting, the efficiency score of the stage 2 for the units C and D is 1. Therefore, as mentioned in the theorem 1, the system efficiency for these units is the product of the efficiencies computed in the sub-processes. DMU G is efficient in both stages. So, its efficiency score is 1. Besides, the system efficiency score does not exceed the maximum efficiency scores of the sub-processes in all units. Finally, the units A and D with complete system efficiency scores despite their bad performances in the stage 2 reveal a limitation in the introduced extension of Kao and Hwang's [5] approach in the model (8) for efficiency analysis of two-stage systems under VRS assumption.

5. Conclusions and further study

Conventional DEA models apply DMUs as a black box, that is, inputs enter and outputs exit. However, recently, there have been various approaches to deal with two-stage systems. One of these approaches suggested by Kao and Hwang [5] considers a decomposition of system efficiency score as a product of sub-processes efficiencies. It is presented under CRS assumption and is not applicable directly to VRS cases. In the present paper, we modified this model to present the validity cases under VRS assumption. A simple numerical example is used to illustrate the results and limitations more clearly. Developing models for performance analysis of two-stage systems to identify sub-processes and total system relation more properly seems to be an interesting challenge for the future studies.

References

[1] R. D. Banker, A. Charnes, W. W. Cooper, Models for estimation of technical and scale inefficiencies in data envelopment analysis. Management Science, 30, 1078-1092, 1984.

[2] A. Charnes, W. W. Cooper, E. Rhodes, Measuring the efficiency of decision making units. European Journal of Operational Research, 2(4), 429 – 444, 1978.

[3] Y. Chen, J. Zhu, Measuring information technology’s indirect impact on firm performance. Information Technology and Management, 5, 9–22, 2004.

[4] M. J. Farrell, The measurement of productive efficiency. Journal of Royal Statistical Society, 120(3), 253-281, 1957.

[5] C. Kao, S. N. Hwang, Efficiency decomposition in two-stage data envelopment analysis: An application to non-life insurance companies in Taiwan. European Journal of Operational Research, 185(1), 418-429, 2008.

[6] H. F. Lewis, T. R. Sexton, Network DEA: efficiency analysis of organizations with complex internal structure. Computers & Operations Research, 31, 1365–1410, 2004.

[7] S. Rho, J. An, Evaluating the efficiency of a two-stage production process using data envelopment analysis. International Transactions in Operational Research, 14, 395-410, 2007.

[8] A. P. Schinnar, A. B. Rothbard, R. Kanter, K. Adams, Crossing state lines of chronic mental illness. Hosp Community Psychiatry, 41, 756-760,1990.

[9] L. M. Seiford, J. Zhu, Profitability and marketability of the top 55 U.S. commercial banks. Management Science, 45, 1270-1288, 1999.