A New Algorithm for the Discrete Shortest Path Problem in a Network Based on Ideal Fuzzy Sets

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Abstract

A shortest path problem is a practical issue in networks for real-world situations. This paper addresses the fuzzy shortest path (FSP) problem to obtain the best fuzzy path among fuzzy paths sets. For this purpose, a new efficient algorithm is introduced based on a new definition of ideal fuzzy sets (IFSs) in order to determine the fuzzy shortest path. Moreover, this algorithm is developed for a fuzzy network problem including three criteria, namely time, cost and quality risk. Several numerical examples are provided and experimental results are then compared against the fuzzy minimum algorithm with reference to the multi-labeling algorithm based on the similarity degree in order to demonstrate the suitability of the proposed algorithm. The computational results and statistical analyses indicate that the proposed algorithm performs well compared to the fuzzy minimum algorithm.

Keywords: Shortest path problem; Single criterion networks; Multiple criteria networks; Fuzzy sets; Ideal fuzzy sets.

1. Introduction

A basic issue in a network problem is to determine the shortest path from source node to destination node in numerous applications, such as transportation, supply chain management, routing, scheduling and telecommunication (e.g., Chen and Hsueh, 2008; Keshavarz and Khorram, 2009; Liu, 2011; Deng et al., 2012). Because of conformity of the fuzzy shortest path (FSP) problem in real-world cases, many researchers have focused on this issue in the recent two decades. For instance, Dubois and Prade (1980) investigated the shortest path problem, and presented an algorithm to obtain the shortest path. They provided different solutions for the classical FSP problem by using extended sum and extended min and max operators. Klein (1991) developed a dynamic programming recursion algorithm in order to determine the path(s) related to the threshold of a membership degree set by the decision maker; however, the algorithm assumes that each arc considers an integer number for length between 1 and a fixed integer, in which the extended min and max operators for several fuzzy numbers cannot be any of these numbers. Chanas et al. (1994) extended an approach based on the α-cut concept. Furukawa (1995) proposed an approach based on parametric orders. Okada and Gen (1994) introduced a shortest path problem in a network with arcs represented as intervals on real line. They considered an order relation between intervals by using two parameters, and presented an algorithm based on the Dijkstra's method to solve the large-sized problems.

Okada and Gen (1993) took a shortest path problem into consideration with a new definition for order relation between intervals. They indicated a new parameter, called "a degree between partial and total order", for the problem solved by a modified Dijkstra's algorithm. Li et al. (1996) introduced the neural networks to solve fuzzy shortest path problems, in which the penalization of this approach was regarded after converting into the crisp shortest path model. Okada and Spore (2000) extended an algorithm based on the order relation for a fuzzy network problem with L-R fuzzy numbers. They considered non-dominated or Pareto optimal paths from the specified node to every other node. Okada (2004) developed the concept of degree of possibility that an arc was on the shortest path and considered a comparison index between the sums of fuzzy numbers, in which the interaction was regarded among fuzzy numbers. The approach may have a great dependence on the α-level and the lower degree of possibility on a network path, indicating a great number of non-dominated paths. Chuang and Kung (2005) introduced an algorithm to obtain the shortest path based on the idea of a minimum crisp number, if and only if any other number is greater than or equal to this number; they extended this idea to the FSP length. Then, they presented the area index between lengths minimum (Lmin) and the

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other lengths to calculate the degree of similarity between the fuzzy shortest path length and the other fuzzy paths lengths. Kung and Chuang (2005) developed an algorithm to solve the shortest path problem with discrete fuzzy arc lengths. They developed a FSP length procedure by a fuzzy minimum algorithm. Then, they evaluated the non-decreasing fuzzy shortest paths by similarity measure. Chuang and Kung (2006) presented the algorithm in a discrete mode and focused on an algorithm to determine the discrete fuzzy shortest length in a network.

Moazeni (2006) concentrated on a lexicographic order relation among fuzzy numbers. By applying multiple labeling and Dijkstra's shortest path algorithms, an algorithm was extended to obtain a set of non-dominated paths, which was related to the extension principle concept. Tajdin et al. (2010) introduced a method for the addition of various fuzzy numbers in a path using α-cuts, by presenting a linear least squares model to determine membership functions for the considered additions. Applying a proposed distance function for the comparison of fuzzy numbers, a dynamic programming method was proposed to obtain a shortest path in the network. Gao (2011) provided solutions to the α-shortest path and the most shortest path in an uncertain network. It is worth to note that there existed an equivalence relation between the α-shortest path in an uncertain network and the shortest path in a corresponding deterministic network, which resulted in an effective algorithm to determine the α-shortest path and the most shortest path. Dou et al. (2012) tried to choose the shortest path in multi-constrained network applying multi-criteria decision method based on vague similarity measure. Each arc length can describe multiple metrics. The multi-constraints were similar to the concept of multi-criteria based on vague sets. A similarity measure of vague sets was developed, in which the positive constraints and the negative constraints were considered. Deng et al. (2012) presented a generalized Dijkstra algorithm to solve the shortest path problem in an uncertain environment. Two key issues were addressed in the problem with fuzzy parameters. Hassanzadeh et al. (2013) considered the design of a model and presented an algorithm for computing the shortest path in a network by regarding different types of fuzzy arc lengths. A technique was extended for the addition of various fuzzy numbers in a path using α-cuts by proposing a least squares model to determine membership functions for the considered additions.

This paper presents a new algorithm to solve the FSP problems in networks for the real-life situations. The specification of arcs includes a number of criteria, such as time, cost, and quality risk of activities. The presented algorithm determines the best choice of the FSP in less computational steps based on the similarity degree indicating its efficiency and suitability in the networks. Moreover, the algorithm is developed to multi-criteria fuzzy networks, which are well suited to real-world cases. For this purpose, unlike the previous body of research, an ideal fuzzy set (IFS) is introduced to obtain the fuzzy shortest length between two paths with the set of discrete fuzzy numbers.

2. Preliminary Definitions

The IFS includes a set of shorter and longer lengths by considering the maximum and minimum degree of membership, respectively. Because it is interested in determining the shortest path, the path length is the most important factor in the given problem. Thus, the maximum and minimum membership degrees are assigned to shorter and longer lengths of the IFS, respectively. The set is an appropriate index to measure the available paths which results in shorter paths and differs from the fuzzy sets employed for the shortest path problems.

Computation of the ideal fuzzy set: If x1 is the shortest path length in the first member of the fuzzy path, based on the above definition of the IFS, the most possible membership is assigned to x1, where {μ(x1)=1}. Also, xn is the longest length in the last member of the fuzzy path, which has the lowest membership; however, the membership cannot be zero, where {μ(xn)>0}. Because this is a discrete set, zero membership is assigned to the next length (i.e., xn+1), where {μ(xn+1)=0}. Hence, xn gains the lowest feasible membership. Eq. (1) defines the membership function of the IFS.

$$\mu(x_i) = \frac{x_1 - x_i}{x_n - x_1 + 1} + 1 \qquad (1)$$

Fig. 1 depicts a discrete IFS where the membership of shorter lengths than x1 is set to 0. Because x1 is the shortest path length in the IFS and there is no shorter length than all lengths in this set.

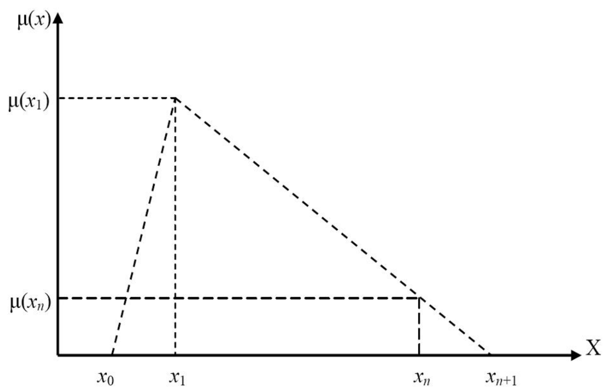

Fig. 1. Discrete ideal fuzzy set

Assume that $\tilde{L}_1, \tilde{L}_2, \ldots, \tilde{L}_k$ are fuzzy network paths defined as follows:

$$\tilde{L}_j = \{(x_i, \mu_{\tilde{L}_j}(x_i)) \mid x_i \in X_j\} \qquad \forall j = 1,2,\ldots,k \qquad (2)$$

where, X_j represents the set of lengths of path j. Following is the IFS of this network:

$$\widetilde{I} = \{\frac{1}{\alpha}, \frac{\mu(\alpha+1)}{\alpha+1}, ..., \frac{\mu(\gamma)}{\gamma}, \frac{0}{\gamma+1}\} \tag{3}$$

where,

$\gamma = \text{Max}\{\underset{j}{\text{Max}}\ \{x_i^j\}\}$, $\alpha = \text{Min}\{\underset{j}{\text{Min}}\ \{x_i^j\}\}$ and x_i^j is member i of set j. Its degree of membership is calculated by using Eq. (4).

$$\mu(x_i) = \frac{\alpha - x_i}{\gamma - \alpha + 1} + 1 \tag{4}$$

Distance $[\alpha, \gamma]$ is the reference set for the network paths, whose members are union of all members of network paths, and the shortest path of the network should be a subset of this distance.

Definition of the optimal set: The optimal set is a subset of the reference set of network paths, in which the members of the fuzzy shortest path are determined. The optimal set $[\alpha, \beta]$ is computed by Eq. (5).

$$\alpha = \text{Min}\{\underset{j}{\text{Min}}\ \{x_i^j\}\}$$
$$\beta = \text{Min}\{\underset{j}{\text{Max}}\ \{x_i^j\}\} \tag{5}$$

Paths L_1, L_2, …, L_k can be common in numerous distance points (i.e., $[\alpha, \beta]$). A point, say δ, can be two or more membership degrees, in which each membership degree belongs to a path. The aim is to find out which membership degree is desirable to define the indifferent point (x^*).

Indifferent point: An indifferent point belongs to the optimal set, $x^* \in [\alpha, \beta]$, which is not member of any shortest and longest length set. In other words, this point is neutral in relation to the previous and next points. For instance, if we want to divide the distance $[\alpha, \beta]$ to two equal parts of shortest and longest lengths, this distance will be $[\alpha, \frac{\alpha+\beta}{2}), (\frac{\alpha+\beta}{2}, \beta]$. Hence, point $\frac{\alpha+\beta}{2}$ does not belong to none of these two distances. If the distances are divided to $[\alpha, \frac{\alpha+\beta}{2}], [\frac{\alpha+\beta}{2}, \beta]$, because of the point $\frac{\alpha+\beta}{2}$ belonging to two sub-distances, then this amount is not regarded as the indifferent point. Based on the above definition, this point is the mean of optimal set, $x^* = \frac{\alpha+\beta}{2}$. Fig. 2 denotes that a higher membership is selected for shorter lengths (i.e., [α, X*)) and a lower membership for longer lengths (i.e., (X*, β]). In this figure, this change of membership occurs in X*, known as the indifferent point.

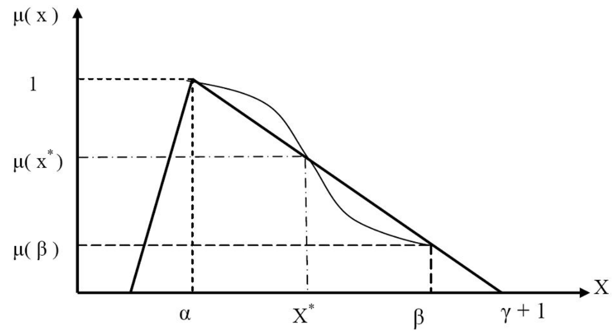

Fig. 2. Indifferent point

The indifferent point, X^*, is calculated by Eq. (6):

$$x^* = \alpha\, t + \beta(1-t) \tag{6}$$
$$0 \le t \le 1$$

If parameter t is set to 0.5, then $x^* = \frac{\alpha+\beta}{2}$. This indifferent point is the most likely one for the decision maker (DM) to select a shorter or longer length. If we consider $0.5< t \le 1$, then it is the optimistic form of the DM's point of view for selecting shorter lengths. If we consider $0 \le t < 0.5$, then it is the pessimistic form of the DM's point of view for selecting shorter lengths. Then, changing parameter t of position x^* impacts on selecting shorter and longer lengths and changing the position of x^* point can impact on finding the fuzzy shortest path. Fig. 3 illustrates changing parameter t along with the membership function of shorter lengths, $\mu_M(x)$, and longer lengths, $\mu_N(x)$. If $0.5< t \le 1$, then $\mu_M(x) > \mu_N(x)$. If $0 \le t < 0.5$, then $\mu_M(x) < \mu_N(x)$.

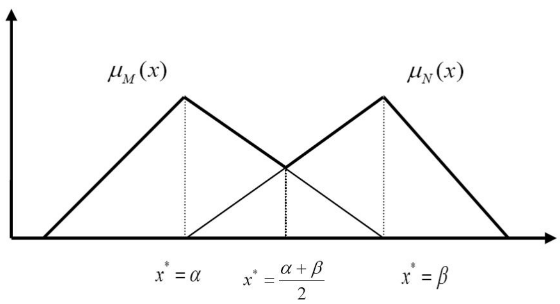

Fig. 3. Membership functions M and N

The membership degree of set lengths: By using proposition, the method is described to determine the membership degree of the shortest path set.

Definitions: Consider the fuzzy paths, $\widetilde{L}_1, \widetilde{L}_2, ..., \widetilde{L}_k$ where,

$\tilde{L}_j = \{(x_i\ ,\ \mu_{\tilde{L}_j}(x_i) \mid x_i \in X_j\};\ \forall j = 1,2,\ldots,k$.

Also, $\tilde{I}$ is regarded as the IFS.

x^*: Indifferent point

$\mu_{\tilde{I}}(x_i)$: Membership of ideal set at point i.

$\mu_{\tilde{L}_j}(x_i)$: Membership of path j at point i.

$\mu_{\tilde{L}^*}(x_i)$: Membership of the selected shortest path at point i.

$\delta_j^+(x_i)$: Positive deviation between membership of path j from the ideal set at point i, where ($x_i \le x^*$).

$\delta_j^-(x_i)$: Negative deviation between membership of path j from the ideal set at point i, where ($x_i \le x^*$).

$\varepsilon_j^+(x_i)$: Positive deviation between membership of path j from the ideal set at point i, where ($x_i > x^*$).

$\varepsilon_j^-(x_i)$: Negative deviation between membership of path j from the ideal set at point i, where ($x_i > x^*$).

$$\mu_{\tilde{L}_j}(X_i) - \mu_{\tilde{I}}(X_i) = \begin{cases} \delta_j^+(x_i) \ge 0 \\ \delta_j^-(x_i) < 0 \end{cases} ;\quad \forall j = 1,\ldots,k\ ;\ \forall i = 1,2,\ldots,n\ ;\ x \le x^*$$

$$\delta^+ = \operatorname*{Max}_i \{\delta_1^+,\delta_2^+,\ldots,\delta_k^+\}$$

$$\delta^- = \operatorname*{Max}_i \{\delta_1^-,\delta_2^-,\ldots,\delta_k^-\}$$

$$\mu_{\tilde{L}_j}(X_i) - \mu_{\tilde{I}}(X_i) = \begin{cases} \varepsilon_j^+(x_i) \ge 0 \\ \varepsilon_j^-(x_i) < 0 \end{cases} ;\quad \forall j = 1,\ldots,k\ ;\ \forall i = 1,2,\ldots,n\ ;\ x_i > x^*$$

$$\varepsilon^+ = \operatorname*{Min}_i \{\varepsilon_1^+,\varepsilon_2^+,\ldots,\varepsilon_k^+\}$$

$$\varepsilon^- = \operatorname*{Min}_i \{\varepsilon_1^-,\varepsilon_2^-,\ldots,\varepsilon_k^-\}$$

Proposition: If $x_i \le x^*$ and $\mu_{\tilde{L}^*}(x_i) = \operatorname*{Max}_{\tilde{L}_j}\{\mu_{\tilde{L}_j}(x_i)\}$, then

$\mu_{\tilde{L}^*}(x_i) - \mu_{\tilde{I}}(x_i) = \delta^+ + \delta^-$.

Otherwise, $x_i > x^*$ and $\mu_{\tilde{L}^*}(x_i) = \operatorname*{Min}_{\tilde{L}_j}\{\mu_{\tilde{L}_j}(x_i)\}$, then

$\mu_{L^*}(x_i) - \mu_I(x_i) = \varepsilon^+ + \varepsilon^-$.

Proof: (by contradiction) Assume that $\exists \mu_{\tilde{L}_m}(x_i) - \mu_{\tilde{I}}(x_i) = \eta_m^+ + \eta_m^- > \delta^+ + \delta^-$.

Then, we have $\mu_{\tilde{L}_m}(x_i) = \operatorname*{Max}_{\tilde{L}_j}\{\mu_{\tilde{L}_j}(x_i)\}$, which does not support the assumption. Then, the proposition is proved. Similarly, for $x_i > x^*$ we have $\exists \mu_{\tilde{L}_m}(x_i) - \mu_{\tilde{I}}(x_i) = \zeta_m^+ + \zeta_m^- < \varepsilon^+ + \varepsilon^-$.

Then, we have $\mu_{\tilde{L}_m}(x_i) = \operatorname*{Min}_{\tilde{L}_j}\{\mu_{\tilde{L}_j}(x_i)\}$, which does not support the assumption. Then, the proposition is proved.

3. Proposed Algorithm for Discrete Fuzzy Shortest Path Problem

The steps of the proposed algorithm are presented for fuzzy single-criterion networks as follows:

Step 1. Determine the optimal set (i.e., members of the shortest path set).

Step 2. Determine the indifferent point, $x^* = \dfrac{\alpha+\beta}{2}$.

Step 3. Determine the membership degree of the members of the shortest path set

$\forall i = 1,2,\ldots,n$

If $x_i \le x^* \Rightarrow \mu^*(x_i) =$

$\operatorname*{Max}_{\tilde{L}_j}\{\mu_{\tilde{L}_j}(x_i)\} \Rightarrow \tilde{S}_1 = \{(x_i, \mu^*(x_i)) \mid x_i \le x^*\}$

Else $x_i > x^* \Rightarrow \mu^*(x_i) =$

$\operatorname*{Min}_{\tilde{L}_j}\{\mu_{\tilde{L}_j}(x_i)\} \Rightarrow \tilde{S}_2 = \{(x_i, \mu^*(x_i)) \mid x_i > x^*\}$

Step 4. Find the shortest path in the fuzzy network, $\tilde{L}_{\min} = \tilde{S}_1 \cup \tilde{S}_2$.

The proposed algorithm can find a fuzzy shortest path that is not included among fuzzy paths in a network. Then, by applying the similarity degree relation, each pair of paths' length is calculated. The biggest similarity degree represents the fuzzy shortest path that is available in network paths. Also, the presented algorithm is introduced for a fuzzy multi-criteria network. As abovementioned, the fuzzy shortest path is selected from among all paths in a fuzzy network based on time criterion. Moreover, if this network is based on cost criterion, the fuzzy shortest path will be in terms of costs. The shortest path in the first network is not necessarily equal to the second one. The main aim of the proposed algorithm is to determine the shortest path in a network including multi-criteria, namely time, cost, risk and distance.

3.1. Weight of criteria for the network

The weight of every criterion (W_i) is evaluated in different conditions. Time criterion in a network is often more important than cost criterion and vice versa. The superiority of each criterion against each other is specified by the weight of each criterion. In a special case, these two criteria can be equal ($W_t = W_c$). It is pointed out that the sum of weights in the network is equal one ($\sum W_i = 1$). Also, the DMs may assign different weights for criteria.

3.2. Score of ranking for network paths

Path k is ranked in terms of time and cost criteria. The related ranks to the network paths based on time and cost criteria for the given Examples 1 and 2 are provided in Table 1. For instance, path L_5 is ranked as 1 and 2 for time and cost criteria, respectively. In other words, path L_5 is the shortest path in terms of time criterion in Example 1; however, path L_4 is the shortest path in terms of cost criterion in Example 2. Table 2 calculated the score of any rank, in which ranks are first inversed and then multiplied by the maximum rank ($w_r = 1/r \times \{\max r\}$), as shown bellow. For instance, the scores of rank for path L_4 in terms of cost and time criteria are 5 and 1.66, respectively.

Table 1
The ranking of criteria

Rank \ Criterion	Time	Cost
1	L_5	L_4
2	L_3	L_5
3	L_4	L_3
4	L_1	L_1
5	L_2	L_2

Table 2
The weight of rankings

Rank	Weight
1	$\frac{1}{1}\times 5 = 5$
2	$\frac{1}{2}\times 5 = 2.5$
3	$\frac{1}{3}\times 5 = 1.66$
4	$\frac{1}{4}\times 5 = 1.25$
5	$\frac{1}{5}\times 5 = 1$

3.3. Criterion matrix

Fig. 4 illustrates the criterion matrix n×n that is equal to the numbers of network paths (M^i = [mkr], k=r), as given below. In this case, each row and column has only one element of non-zero. Rows and columns of this matrix represent the network paths and their ranks, respectively. Every path in each criterion has only and only one rank. For instance, if path Lk in criterion i has rank r, Mkr = Wi × wr will be assumed, considering that every criterion of the network has one exclusive matrix. Following is the matrix of criterion i, where Wi is the weight of criterion i and wr is the score of rank r in criterion i:

$$M_i = \begin{bmatrix} W_i w_1 & 0 & 0 & . & . & . & 0 \\ 0 & W_i w_2 & 0 & . & . & . & 0 \\ . & . & . & & & & . \\ . & . & . & & & & . \\ . & . & . & & & & . \\ 0 & 0 & 0 & . & . & . & W_i w_n \end{bmatrix}$$

Fig. 4. Criterion matrix

Time and cost matrices with the same weight (i.e., Wt = Wc = 0.5) are given in Table 3. The network matrix is denoted in Table 4 that is equal to the sum of criteria matrices of the same network (i.e., $M_{Network} = \sum_{i=1}^{n} M_i$), where n is the number of criteria in the network.

Table 3
Matrix of time criterion

$$M_t = \begin{bmatrix} 0 & 0 & 0 & 0.5\times1.25 & 0 \\ 0 & 0 & 0 & 0 & 0.5\times1 \\ 0 & 0.5\times2.5 & 0 & 0 & 0 \\ 0 & 0 & 0.5\times1.66 & 0 & 0 \\ 0.5\times5 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M_c = \begin{bmatrix} 0 & 0 & 0 & 0.5\times1.25 & 0 \\ 0 & 0 & 0 & 0 & 0.5\times1 \\ 0 & 0 & 0.5\times1.66 & 0 & 0 \\ 0.5\times5 & 0 & 0 & 0 & 0 \\ 0 & 0.5\times2.5 & 0 & 0 & 0 \end{bmatrix}$$

Table 4
Network matrix

$$M_{network} = \begin{bmatrix} 0 & 0 & 0 & 1.25 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1.25 & 0.83 & 0 & 0 \\ 2.5 & 0 & 0.83 & 0 & 0 \\ 2.5 & 1.25 & 0 & 0 & 0 \end{bmatrix}$$

3.4. Comparison of network paths

Network paths are calculated by using the network matrix. The next step is to calculate the rank of paths in the network based on the sum of criteria scores. To specify the final rank of paths, the score of each path should be calculated. For instance, the score of path L_k is equal to the sum of the row arrays (i.e., $m_k = \sum_{r=1}^{n} m_{kr}$) related to path L_k. Hence, there is an order of paths priority from a path with more score to a path with the lowest score. In other words, a path with the highest score is the shortest path in the network.

$$\left.\begin{array}{l} m_1 = 1.25 \\ m_2 = 1 \\ m_3 = 1.25 + 0.83 = 2.08 \\ m_4 = 2.5 + 0.83 = 3.33 \\ m_5 = 2.5 + 1.25 = 3.75 \end{array}\right\} \Rightarrow$$

$$m_5 > m_4 > m_3 > m_1 > m_2$$

It is concluded that path L_5 is the shortest path, and the sequence of paths are as follows: $L_5 > L_4 > L_3 > L_1 > L_2$.

3.5. Steps of the proposed algorithm for fuzzy multiple criteria networks

The steps of the generalized algorithm are proposed in this section for the fuzzy shortest path in fuzzy multi-criteria networks. Notations and steps used in this algorithm are as follows:

$r_i(L_k)$ rank of path k in criterion i, where $i = 1, 2, \ldots, n$ and $k = 1, 2, \ldots, K$
W_i weight of criterion i
w_{ri} score of rank r in criterion i.
m_k=$m(L_k)$ score of path k

Step 0. Select the shortest path in the network for every criterion, according to Section 3.

Step 1. Determine the score of all ranks for paths (i.e., $w_r = \frac{1}{r} \times \{\max r\}$).

Step 2. Assign the weight for each criterion (W_i) in the network based on the DM's priority.

Step 3. Find the shortest path in the network for each criterion by the proposed algorithm, and determine the ranking of each network path for each criterion $r_i(L_k)$ by the use of the similarity degree.

Step 4. Calculate the score of each network path (i.e., $m_k = m(L_k) = \sum_{i=1}^{n} W_i w_{ri}(L_k)$).

Step 5. Rank the paths with more score indicating the shortest path.

3.6. Complexity of the algorithm

The FSP problem is NP-hard. For the given problem, the time complexity is O(k×m*), where k is the number of network paths and m* is the number of members of the fuzzy shortest path (Okad and Gen, 1994; Chuang and Kung, 2005).

4. Computational Results

Three examples are generated in this section for solving the fuzzy shortest path in both single and multi-criteria networks. The first two examples related to the single-criterion network are solved by the proposed algorithm. Moreover, the proposed algorithm is developed and generalized to solve the third example in the fuzzy multi-criteria networks. The computational results obtained by the algorithm are compared with two well-known algorithms taken from the literature. Finally, the statistical analysis is discussed to demonstrate the efficiency of the proposed algorithm for the discrete shortest path problem in the networks.

4.1. Example one

A classic network, namely Network 1 is depicted in Fig. 5, whose lengths are fuzzy.

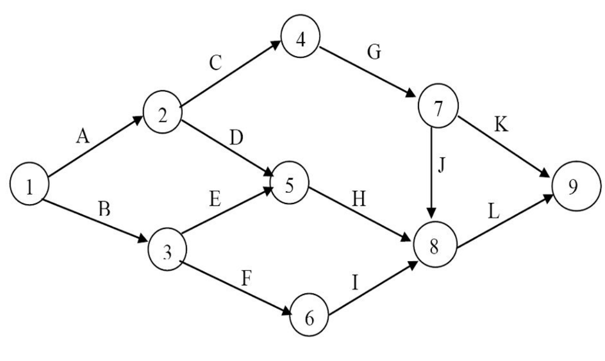

Fig. 5. Network one

According to Fig. 5, following are the length of arcs on the network one:

$A=\{\frac{0.7}{4},\frac{0.5}{5}\}$, $B=\{\frac{0.4}{3},\frac{0.7}{4},\frac{0.8}{5},\frac{0.6}{6}\}$,
$C=\{\frac{0.5}{5},\frac{0.8}{6},\frac{0.3}{7}\}$, $D=\{\frac{0.6}{5},\frac{0.9}{6},\frac{0.5}{7}\}$
$E=\{\frac{0.4}{5},\frac{0.7}{6}\}$, $F=\{\frac{0.4}{6},\frac{0.8}{7},\frac{0.5}{8}\}$,
$G=\{\frac{0.3}{5},\frac{0.7}{6},\frac{0.5}{7}\}$, $H=\{\frac{0.7}{6},\frac{0.5}{7}\}$
$I=\{\frac{0.8}{4},\frac{0.6}{5}\}$, $J=\{\frac{0.4}{2},\frac{0.7}{3},\frac{0.5}{4}\}$, $K=\{\frac{0.4}{4},\frac{0.6}{5}\}$,
$L=\{\frac{0.5}{2},\frac{0.7}{3},\frac{0.6}{4}\}$

The length of each arc shows the task duration on the network one. This network has five paths whose each length is given as follows:
L_1: ACGK , L_2: ACGJL , L_3: ADHL , L_4: BEHL , L_5: BFIL .

$$\tilde{L}_1=\{\frac{0.3}{18},\frac{0.4}{19},\frac{0.5}{20},\frac{0.6}{21},\frac{0.5}{22},\frac{0.5}{23},\frac{0.3}{24}\}$$
$$\tilde{L}_2=\{\frac{0.3}{18},\frac{0.5}{19},\frac{0.5}{20},\frac{0.7}{21},\frac{0.6}{22},\frac{0.5}{23},\frac{0.5}{24},\frac{0.5}{25},\frac{0.3}{26}\}$$
$$\tilde{L}_3=\{\frac{0.5}{17},\frac{0.6}{18},\frac{0.7}{19},\frac{0.6}{20},\frac{0.5}{21},\frac{0.5}{22},\frac{0.5}{23}\}$$
$$\tilde{L}_4=\{\frac{0.4}{16},\frac{0.4}{17},\frac{0.5}{18},\frac{0.7}{19},\frac{0.7}{20},\frac{0.6}{21},\frac{0.6}{22},\frac{0.5}{23}\}$$
$$\tilde{L}_5=\{\frac{0.4}{15},\frac{0.4}{16},\frac{0.5}{17},\frac{0.7}{18},\frac{0.7}{19},\frac{0.6}{20},\frac{0.6}{21},\frac{0.6}{22},\frac{0.5}{23}\}$$

By the use of the proposed algorithm, the fuzzy shortest path of this network is given as follows:

Step 1:
$$\left.\begin{array}{l}\alpha=\text{Min}\{18,18,17,16,15\}=15\\ \beta=\text{Min}\{24,26,23,23,23\}=23\end{array}\right.\Rightarrow \text{ Optimal set } = [15, 23]$$

Step 2: $x^*=\frac{15+23}{2}=19$

Step 3:
$$\left.\begin{array}{l}x_1=15<19\Rightarrow\mu(x_1)=\text{Max}\{0.4\}=0.4\\ x_2=16<19\Rightarrow\mu(x_2)=\text{Max}\{0.4,0.4\}=0.4\\ x_3=17<19\Rightarrow\mu(x_3)=\text{Max}\{0.5,0.4,0.5\}=0.5\\ x_4=18<19\Rightarrow\mu(x_4)=\text{Max}\{0.3,0.3,0.6,0.5,0.7\}=0.7\\ x_5=19\qquad\Rightarrow\mu(x_5)=\text{Max}\{0.4,0.5,0.7,0.7,0.7\}=0.7\end{array}\right\}\Rightarrow$$

$$\tilde{S}_2=\{\frac{0.5}{20},\frac{0.5}{21},\frac{0.5}{22},\frac{0.5}{23}\}$$
$$\left.\begin{array}{l}x_6=20>19\Rightarrow\mu(x_6)=\text{Min}\{0.5,0.5,0.6,0.7,0.6\}=0.5\\ x_7=21>19\Rightarrow\mu(x_7)=\text{Min}\{0.6,0.7,0.5,0.6,0.6\}=0.5\\ x_8=22>19\Rightarrow\mu(x_8)=\text{Min}\{0.5,0.6,0.5,0.6,0.6\}=0.5\\ x_9=23>19\Rightarrow\mu(x_9)=\text{Min}\{0.5,0.5,0.5,0.5,0.5\}=0.5\end{array}\right\}\Rightarrow$$

Step 4:
$$\tilde{L}_{\min}=\{\frac{0.4}{15},\frac{0.4}{16},\frac{0.5}{17},\frac{0.7}{18},\frac{0.7}{19},\frac{0.5}{20},\frac{0.5}{21},\frac{0.5}{22},\frac{0.5}{23}\}$$

The shortest path which is based on the time criterion is illustrated in Fig. 6 for the network one.

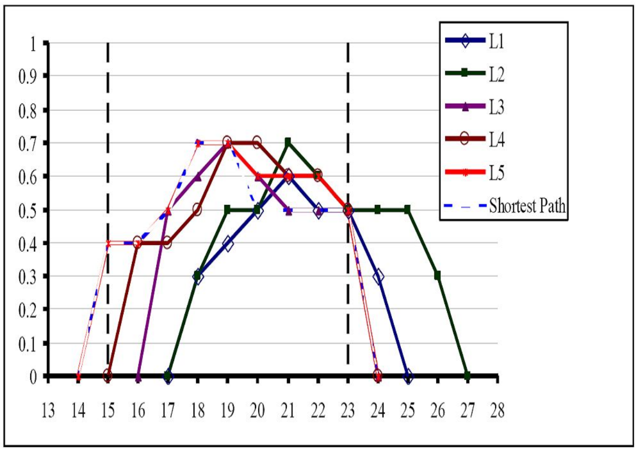

Fig. 6. Different fuzzy paths based on time criterion for network one

Following are three different methods to compute the similarity degree for two fuzzy sets that some of them are based on Eqs. (7) to (9).

Method I) Wang (1997) proposed the following equation to computes the similarity degree between two sets.

$$S(A,B)=\left\{\sum_{n=1}^{p}\left[\frac{m(\mu_A(x_n),\mu_B(x_n))}{M(\mu_A(x_n),\mu_B(x_n))}\right]\right\}\Big/p \tag{7}$$

S($\tilde{L}_5$, $\tilde{L}_{min}$)=0.9584, S($\tilde{L}_4$, $\tilde{L}_{min}$)=0.8246, S($\tilde{L}_3$, $\tilde{L}_{min}$) = 0.8045, S($\tilde{L}_2$, $\tilde{L}_{min}$) = 0.3908, S($\tilde{L}_1$, $\tilde{L}_{min}$) = 0.5694.

S($\tilde{L}_5$, $\tilde{L}_{min}$) > S($\tilde{L}_4$, $\tilde{L}_{min}$) > S($\tilde{L}_3$, $\tilde{L}_{min}$) > S($\tilde{L}_1$, $\tilde{L}_{min}$) > S($\tilde{L}_2$, $\tilde{L}_{min}$)

As a result $\tilde{L}_5$ is the shortest path in the network for Example 1.

Method II) Karokapilidis and Pappis (1993) recommended the following equation.

$$S(A,B)=\frac{\sum_{n=1}^{p} m(\mu_A(x_n),\mu_B(x_n))}{\sum_{n=1}^{p} M(\mu_A(x_n),\mu_B(x_n))} \tag{8}$$

By using this equation, we have: S($\tilde{L}_5$, $\tilde{L}_{min}$) = 0.94, S($\tilde{L}_4$, $\tilde{L}_{min}$) = 0.7843, S($\tilde{L}_3$, $\tilde{L}_{min}$) = 0.9717, S($\tilde{L}_2$, $\tilde{L}_{min}$) = 0.4444, S($\tilde{L}_1$, $\tilde{L}_{min}$) = 0.5294

As a result, we have: S($\tilde{L}_5$, $\tilde{L}_{min}$) > S($\tilde{L}_3$, $\tilde{L}_{min}$) > S($\tilde{L}_4$, $\tilde{L}_{min}$) > S($\tilde{L}_1$, $\tilde{L}_{min}$) > S($\tilde{L}_2$, $\tilde{L}_{min}$).

Method III) Karokapilidis & Pappis (1993) also suggested the following equation.

$$S(A,B)=1-\frac{\sum_{n=1}^{p}|\mu_A(x_n)-\mu_B(x_n)|}{\sum_{n=1}^{p}|\mu_A(x_n)+\mu_B(x_n)|} \tag{9}$$

S($\tilde{L}_5$, $\tilde{L}_{min}$)= 0.9691, S($\tilde{L}_4$, $\tilde{L}_{min}$)= 0.8791, S($\tilde{L}_3$, $\tilde{L}_{min}$)= 0.8837, S($\tilde{L}_2$, $\tilde{L}_{min}$)= 0.6154, S($\tilde{L}_1$, $\tilde{L}_{min}$)= 0.6923

As a result, we have: S($\tilde{L}_5$, $\tilde{L}_{min}$) > S($\tilde{L}_3$, $\tilde{L}_{min}$) > S($\tilde{L}_4$, $\tilde{L}_{min}$) > S($\tilde{L}_1$, $\tilde{L}_{min}$) > S($\tilde{L}_2$, $\tilde{L}_{min}$). So, the shortest path and other paths are given in order, as follows:

BFIL < ADHL < BEHL < ACGK < ACGJL

4.2. Example two

The network given in example one is considered, whose lengths of arcs indicate the cost of doing the related activity. These lengths are presented as follows:

$A=\{\frac{0.7}{40},\frac{0.6}{50},\frac{0.4}{60}\}$, $B=\{\frac{0.3}{30},\frac{0.8}{40},\frac{0.6}{50}\}$,

$C=\{\frac{0.7}{50},\frac{0.6}{60},\frac{0.3}{70}\}$, $D=\{\frac{0.6}{60},\frac{0.9}{70},\frac{0.5}{80}\}$,

$E=\{\frac{0.4}{60},\frac{0.7}{70}\}$, $F=\{\frac{0.6}{50},\frac{0.8}{60},\frac{0.7}{70},\frac{0.4}{80}\}$,

$G=\{\frac{0.4}{30},\frac{0.6}{40},\frac{0.5}{50}\}$, $H=\{\frac{0.3}{40},\frac{0.7}{50},\frac{0.5}{60}\}$,

$I=\{\frac{0.9}{50},\frac{0.8}{60},\frac{0.6}{70}\}$, $J=\{\frac{0.5}{30},\frac{0.8}{40}\}$, $K=\{\frac{0.7}{40},\frac{0.5}{50}\}$,

$L=\{\frac{0.8}{10},\frac{0.5}{20},\frac{0.7}{30}\}$

The network paths are the same as paths given in the example one. The lengths of arcs are listed as below:

$$\tilde{L}_1=\{\frac{0.4}{160},\frac{0.6}{170},\frac{0.6}{180},\frac{0.6}{190},\frac{0.5}{200},\frac{0.5}{210},\frac{0.4}{220},\frac{0.3}{230}\}$$

$$\tilde{L}_2=\{\frac{0.4}{160},\frac{0.5}{170},\frac{0.6}{180},\frac{0.6}{190},\frac{0.6}{200},\frac{0.5}{210},\frac{0.5}{220},\frac{0.4}{230},\frac{0.4}{240},\frac{0.3}{250}\}$$

$$\tilde{L}_3=\{\frac{0.3}{150},\frac{0.6}{160},\frac{0.7}{170},\frac{0.6}{180},\frac{0.5}{190},\frac{0.5}{200},\frac{0.5}{210},\frac{0.4}{220},\frac{0.4}{230}\}$$

$$\tilde{L}_4=\{\frac{0.3}{140},\frac{0.3}{150},\frac{0.4}{160},\frac{0.7}{170},\frac{0.6}{180},\frac{0.5}{190},\frac{0.5}{200},\frac{0.4}{210}\}$$

$$\tilde{L}_5=\{\frac{0.3}{140},\frac{0.6}{150},\frac{0.8}{160},\frac{0.8}{170},\frac{0.7}{180},\frac{0.6}{190},\frac{0.6}{200},\frac{0.5}{210},\frac{0.4}{220},\frac{0.4}{230}\}$$

By using the proposed algorithm, the shortest path can be found on this network, as given below:

$$\tilde{L}_{min}=\{\frac{0.3}{140},\frac{0.6}{150},\frac{0.8}{160},\frac{0.8}{170},\frac{0.6}{180},\frac{0.5}{190},\frac{0.5}{200},\frac{0.4}{210}\}$$

The shortest path is depicted in Fig. 7 based on cost criterion for the network one.

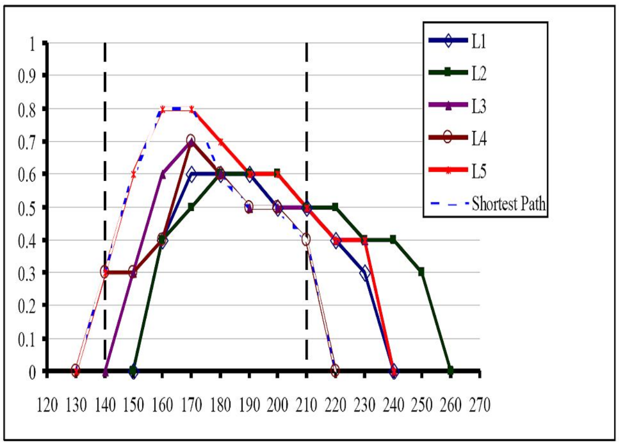

Fig. 7. Different fuzzy paths based on cost criterion for the network one

Also, three different methods are employed to compute the similarity degree for two fuzzy sets.

Method I) According to Eq. (7), the degree of similarity is calculated as:

$S(\tilde{L}_5, \tilde{L}_{min})=0.777$, $S(\tilde{L}_4, \tilde{L}_{min})=0.9063$, $S(\tilde{L}_3, \tilde{L}_{min}) = 0.6604$, $S(\tilde{L}_2, \tilde{L}_{min}) = 0.3826$, $S(\tilde{L}_1, \tilde{L}_{min}) = 0.5736$

As a result we have:

$S(\tilde{L}_4, \tilde{L}_{min}) > S(\tilde{L}_5, \tilde{L}_{min}) > S(\tilde{L}_3, \tilde{L}_{min}) > S(\tilde{L}_1, \tilde{L}_{min}) > S(\tilde{L}_2, \tilde{L}_{min})$.

Method II) According to Eq. (8), the degree of similarity is calculated as:

$S(\tilde{L}_5, \tilde{L}_{min})= 0.7895$, $S(\tilde{L}_4, \tilde{L}_{min})= 0.8222$, $S(\tilde{L}_3, \tilde{L}_{min})= 0.6667$, $S(\tilde{L}_2, \tilde{L}_{min})= 0.4531$,

$S(\tilde{L}_1, \tilde{L}_{min})= 06383$

As a result we have:

$S(\tilde{L}_4, \tilde{L}_{min}) > S(\tilde{L}_5, \tilde{L}_{min}) > S(\tilde{L}_3, \tilde{L}_{min}) > S(\tilde{L}_1, \tilde{L}_{min}) > S(\tilde{L}_2, \tilde{L}_{min})$.

Method III) According to Eq. (9), the degree of similarity is calculated as:

$S(\tilde{L}_5, \tilde{L}_{min})= 0.8823$, $S(\tilde{L}_4, \tilde{L}_{min})= 0.9024$, $S(\tilde{L}_3, \tilde{L}_{min})= 0.8$, $S(\tilde{L}_2, \tilde{L}_{min})= 0.6989$,

$S(\tilde{L}_1, \tilde{L}_{min})= 0.7792$

As a result we have:

$S(\tilde{L}_4, \tilde{L}_{min}) > S(\tilde{L}_5, \tilde{L}_{min}) > S(\tilde{L}_3, \tilde{L}_{min}) > S(\tilde{L}_1, \tilde{L}_{min}) > S(\tilde{L}_2, \tilde{L}_{min})$.

By using the proposed algorithm, the shortest path can be also found in this network, as given below.

BEHL < BFIL < ADHL < ACGK < ACGJL

4.3. Example three

A typical network, namely network two, is depicted in Fig. 8 in which the length of arcs indicates the risk value for example three.

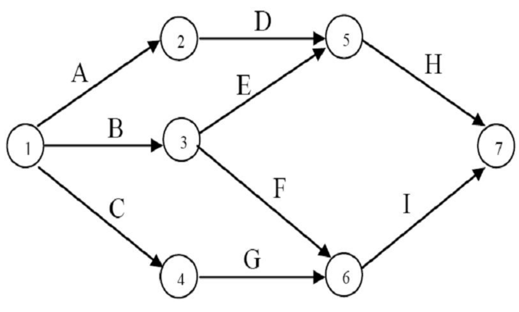

Fig. 8. Network two for given example three

$$A = \{\frac{0.6}{0.05},\frac{0.8}{0.1},\frac{0.7}{0.15},\frac{0.4}{0.2}\},\ B = \{\frac{0.5}{0.1},\frac{0.8}{0.15},\frac{0.3}{0.2}\},$$

$$C = \{\frac{0.7}{0.1},\frac{0.9}{0.15},\frac{0.6}{0.2},\frac{0.5}{0.25}\},$$

$$D = \{\frac{0.4}{0.1},\frac{0.7}{0.15},\frac{0.5}{0.2}\},\ E = \{\frac{0.7}{0.1},\frac{0.8}{0.15},\frac{0.7}{0.2},\frac{0.6}{0.25}\},$$

$$F = \{\frac{0.4}{0.05},\frac{0.9}{0.1},\frac{0.7}{0.15}\},$$

$$G = \{\frac{0.7}{0.1},\frac{0.8}{0.15}\},\ H = \{\frac{0.5}{0.05},\frac{0.9}{0.1},\frac{0.6}{0.15},\frac{0.3}{0.2}\},$$

$$I = \{\frac{0.3}{0.05},\frac{0.4}{0.1},\frac{0.8}{0.15},\frac{0.6}{0.2},\frac{0.5}{0.25}\}$$

$$\tilde{L}_1 = \{\frac{0.4}{0.2},\frac{0.5}{0.25},\frac{0.6}{0.3},\frac{0.7}{0.35},\frac{0.7}{0.4},\frac{0.6}{0.45},\frac{0.5}{0.5},\frac{0.4}{0.55},\frac{0.3}{0.6}\}$$

$$\tilde{L}_2 = \{\frac{0.5}{0.25},\frac{0.5}{0.3},\frac{0.7}{0.35},\frac{0.8}{0.4},\frac{0.7}{0.45},\frac{0.6}{0.5},\frac{0.6}{0.55},\frac{0.3}{0.6},\frac{0.3}{0.65}\}$$

$$\tilde{L}_3 = \{\frac{0.3}{0.2},\frac{0.3}{0.25},\frac{0.3}{0.3},\frac{0.5}{0.35},\frac{0.8}{0.4},\frac{0.7}{0.45},\frac{0.6}{0.5},\frac{0.5}{0.55},\frac{0.3}{0.6}\}$$

$$\tilde{L}_4 = \{\frac{0.3}{0.25},\frac{0.4}{0.3},\frac{0.7}{0.35},\frac{0.7}{0.4},\frac{0.8}{0.45},\frac{0.6}{0.5},\frac{0.6}{0.55},\frac{0.5}{0.6},\frac{0.5}{0.65}\}$$

The shortest network path by using the proposed algorithm is given as follows:

$$\tilde{L}_{min} = \{\frac{0.4}{0.2},\frac{0.5}{0.25},\frac{0.6}{0.3},\frac{0.7}{0.35},\frac{0.8}{0.4},\frac{0.6}{0.45},\frac{0.5}{0.5},\frac{0.4}{0.55},\frac{0.3}{0.6}\}$$

,

The sequence of paths is as follows:

ADH< BFI< BEH<CGI.

By using the proposed algorithm, the shortest fuzzy path as multi-criteria in the network can be specified. The shortest network path two for time, cost and quality risk of activity is provided below.

Step 1: Determine the score of ranks for paths.

$$w_1 = \frac{1}{1}\times 4 = 4 \qquad w_2 = \frac{1}{2}\times 4 = 2 \qquad w_3 = \frac{1}{3}\times 4 = 1.33$$

$$w_4 = \frac{1}{4}\times 4 = 1$$

Step 2: Assume the following weights of criteria arbitrarily.

W_t=0.45 , W_c=0.35 , W_r=0.2

Step 3: Rank paths in respect to three criteria as shown in Table 5.

Table 5
Paths-criteria matrix

Paths \ Criteria	Time rank $W_t = 0.45$	Cost rank $W_c = 0.35$	Rank of quality risk of activities $W_r = 0.2$
L_1	4	3	1
L_2	2	4	3
L_3	3	2	2
L_4	1	1	4

Step 4: Compute the score of paths.

$m_1 = m(L_1) = 0.45\times1+0.35\times1.33+0.2\times4 = 1.7155$

$m_2 = m(L_2) = 0.45\times2+0.35\times1+0.2\times1.33 = 1.516$

$m_3 = m(L_3) = 0.45\times1.33+0.35\times2+0.2\times2 = 1.6985$

$m_4 = m(L_4) = 0.45\times4+0.35\times4+0.2\times1 = 3.4$

Step5: Sort non-decreasing paths.

$$m_4 > m_1 > m_3 > m_2 \quad \rightarrow \quad L_4 < L_1 < L_3 < L_2$$

The order of paths can be concluded as follows: CGI < ADH < BFI < BEH. Thus, path is L_4 (i.e., CGI) in the shortest path network with three criteria.

4.4. Statistical analysis

32 networks are randomly generated, whose fuzzy arc lengths are discrete and random. Then, the similarity degrees are compared between the proposed algorithm and the labeling algorithm against the fuzzy minimum and the labeling algorithms. Regarding these networks, the cost of arc lengths and their memberships are randomly generated in [100,500] and (0,1) respectively. Also, the time of each activity and its membership are randomly generated in [10,99] and (0,1) respectively. Further, the quality risk of each activity and its membership are randomly generated in (0,1). Table 6 illustrates the computational results, in which the statistical hypotheses are as follows:

$$H_0 : \mu_1 - \mu_2 = 0$$

$$H_1 : \mu_1 - \mu_2 < 0$$

Where, parameters μ^1 and μ^2 are the mean similarity degrees for columns 1 and 2, respectively. It is clear that if μ^1 or μ^2 is equal one, then both algorithms given in column 1 or 2 are completely the same. The computational results of the hypothesis tests are provided below, in which the significant level of the test is 5% (i.e., $\alpha = 0.05$).

Difference = $\mu(c_1) - \mu(c_2)$

Estimated for difference: 0.00159

95% upper bound for difference: 0.019369

T-Test of difference = 0 (vs <): T – value = 0.15, P – Value = 0.559, DF = 61.

It can be concluded that H^0 is accepted and two means of μ^1 and μ^2 are not different significantly. Hence, the computational results indicate that if the proposed algorithm can be better than the fuzzy minimum algorithm then it is not worse than the fuzzy minimum algorithm. However, estimation reported by the MINITAB software package for the difference is 0.00159. It denotes that μ^1 is more than μ^2. Therefore, $\widetilde{L}_{\min}$ related to our proposed algorithm is close to the labeling algorithm against the fuzzy minimum algorithm.

Table 6
Numerical result

Network No.	Similarity degree* between our proposed algorithm and the labeling algorithm	Similarity degree between the labeling and the minimum fuzzy algorithms
1	0.972973	0.986667
2	0.968750	0.952381
3	0.968750	1.000000
4	0.985507	0.985507
5	0.831169	0.871795
6	0.927536	0.942857
7	1.000000	1.000000
8	0.974359	0.986667
9	0.973684	0.973684
10	0.880000	0.833333
11	0.986667	1.000000
12	0.968750	0.967742
13	0.935484	0.935448
14	1.000000	1.000000
15	1.000000	0.983051
16	0.983051	0.964286
17	0.906250	0.935484
18	0.974335	0.947368
19	0.965517	0.981818
20	0.967742	0.983051
21	0.915663	0.928571
22	1.000000	0986301
23	0.883117	0.921053
24	0.984615	0.984127
25	0.987013	0.973684
26	1.000000	0.987342
27	0.985075	0.969697
28	0.963855	0.950000
29	0.969072	0.958333
30	0.882353	0.836735
31	1.000000	0.976744
32	0.974359	0.961039

* Similarity degree is calculated by the Pappis's method

5. Conclusion

A new fuzzy algorithm was presented in this paper for the shortest path problem in the single criterion networks with discrete fuzzy arcs. The proposed algorithm was based on a new definition of the ideal fuzzy set in order to inherently evaluate shorter and longer lengths with the maximum and minimum membership, respectively. Moreover, this algorithm was developed for the fuzzy shortest path problem in the multiple criteria networks. Then, the similarity degrees area were calculated between the proposed algorithm and the labeling algorithm, and between the fuzzy minimum algorithm and the labeling algorithm in order to highlight the efficiency and applicability of the proposed algorithm according to the time complexity. The related statistical analysis demonstrated that the fuzzy shortest path provided by the proposed algorithm in respect to the labeling algorithm was closer than the fuzzy minimum algorithm. This paper generalized the presented algorithm to determine the fuzzy shortest path in the multi-criteria network by considering several conflicting criteria. The main advantage of this algorithm was to take the fuzzy multi-criteria network into consideration by ranking paths for each criterion and summing the weight of ranks in order to obtain the fuzzy shortest path.

6. References

[1] Chanas, S., Delgado, M., Verdegay, J. L. and Vila, M. A. (1994). Fuzzy optimal flow on imprecise structures. European Journal of Operation Research, 83, 568-580.

[2] Chen, S.-P. and Hsueh, Y.-J., (2008). A simple approach to fuzzy critical path analysis in project networks. Applied Mathematical Modelling, 32, 1289-1297.

[3] Chuang, T. N. and Kung, J.Y. (2005). The fuzzy shortest path length and the corresponding shortest path in a network. Computers & Operations Research, 32, 1409-1428.

[4] Chuang, T. N. and Kung, J.Y. (2006). A new algorithm for the discrete fuzzy shortest path problem in a network. Applied Mathematics & Computation, 174, 660-680.

[5] Deng, Y., Chen, Y., Zhang, Y. and Mahadevan, S. (2012). Fuzzy Dijkstra algorithm for shortest path problem under uncertain environment. Applied Soft Computing, 12, 1231-1237.

[6] Dou,Y., Zhu, L. and Wang, H.S. (2012). Solving the fuzzy shortest path problem using multi-criteria decision method based on vague similarity measure. Applied Soft Computing, 12, 1621-1631.

[7] Dubois, D. and Prade, H. (1980). Theory and applications, Academic Press, New York.

[8] Furukawa, N. (1995). A parametric total order on fuzzy numbers and a fuzzy shortest rout problem. Optimization, 30, 367-377.

[9] Gao, Y. (2011). Shortest path problem with uncertain arc lengths. Computers & Mathematics with Applications, 62, 2591–2600.

[10] Hassanzadeh, R., Mahdavi, I., Mahdavi-Amiri N. and Tajdin, A. (2013). A genetic algorithm for solving fuzzy shortest path problems with mixed fuzzy arc lengths. Mathematical and Computer Modelling, 57, 84-99.

[11] Hindi, K.S. (1995). Computationally efficient solution of the multi-item, capacitated lot-sizing problem. Computers & Industrial Engineering, 28(4), 709-719.

[12] Keshavarz, E. and Khorram, E., (2009). A fuzzy shortest path with the highest reliability. Journal of Computational and Applied Mathematics, 230, 204-212.

[13] Klein, C.M. (1991). Fuzzy shortest paths. Fuzzy Sets and Systems, 39, 27-41.

[14] Kung, J. Y. and Chuang, T. N. (2005). The shortest path problem with discrete fuzzy arc lengths. Computers & Mathematics with Applications, 49, 263-270

[15] Li, Y., Gen, M. and Ida, K. (1996). Solving fuzzy shortest path problems by neural networks. Computers & Industrial Engineering, 31(3-4), 861-865.

[16] Liu, S.-T. (2011). Fuzzy measures for profit maximization with fuzzy parameters. Journal of Computational and Applied Mathematics, 236, 1333-1342.

[17] Moazeni, S. (2006). Fuzzy shortest path problem with finite fuzzy quantities. Applied Mathematics and Computation, 183, 160-169.

[18] Okada, S. (2004). Fuzzy shortest path problems incorporating interactivity among paths. Fuzzy Sets and Systems, 142, 335-357.

[19] Okada, S. and Gen, M. (1993). Order relation between intervals and its application to shortest path problem. Computers & Industrial Engineering, 25(1-4), 147-150.

[20] Okada, S. and Gen, M. (1994). Fuzzy shortest path problem. Computers & Industrial Engineering, 27(1-4), 465-468.

[21] Okada, S. and Soper, T. (2000). A shortest path problem on a network with fuzzy arc length. Fuzzy Set and Systems, 109, 129-140.

[22] Pappis, C. P. and Karacapilidis, N. I. (1993). Similarity measure between fuzzy sets and between elements. Fuzzy Set and Systems, 56, 172-174.

[23] Tajdin, A., Mahdavi, I., Mahdavi-Amiri, N. and Sadeghpour-Gildeh, B. (2010). Computing a fuzzy shortest path in a network with mixed fuzzy arc lengths using α-cuts. Computers & Mathematics with Applications, 60, 989–1002.

[24] Vachajitpan, P. (1983). A computer program for solving job sequencing problems. Computers & Industrial Engineering, 7(3), 173-185.

[25] Wang, W. J. (1997). New similarity measures on fuzzy sets and on elements. Fuzzy Set and Systems, 85, 305-309.