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Research Article

Design of Accelerated Life Testing Plans for Products Exposed to Random Usage

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Abstract

Accelerated Life Testing (ALT) is very important in evaluating the reliability of highly reliable products. According to ALT procedure, products undergo higher stress levels than normal conditions to reduce the failure times. ALTs have been studied for various conditions and stresses. In addition to common stresses such as temperature and humidity, random usage can also be considered as another stress that can cause failure. Design of ALT plan for products which are exposed to random usage process has not been studied in the literature. Therefore, a procedure for designing ALT plan for these products is studied in this paper. To do so, hazard rate of products is formulated based on the random usage process and other stresses. Then, the variance of the hazard rate is estimated over a predetermined time period. Optimum stress levels and the number of units at every stress level are obtained by numerically minimizing the variance of the hazard rate estimate. Numerical example and sensitivity analysis are performed to show the application and robustness of the model to parameter deviations. The results show that the proposed procedure is robust to parameter changes and can be used for ALT planning of products under random usage.

Keywords**:** Accelerated life testing; Reliability; Hazard rate; ALT plans; Random usage

1. Introduction

Manufacturers constantly confront the challenge of rapid development of new products while striving to enhance product quality and reliability. Methods such as concurrent engineering and designed experiments are being used by engineers to design quality and reliability into products (Mostafaeipour (2016)). The need for higher reliability products has created the need for early testing of materials, components, and systems (Escobar and Meeker (2006)). This is consistent with the modern quality philosophy in Meeker and Hamada (1995) and Meeker and Escobar (2003) . They believe that "in order to produce high reliability products, the design and manufacturing processes should be improved and reliance on inspection should be avoided" (Escobar and Meeker (2006)).

There are difficulties in estimating failure time distribution and reliability measures of components in high reliability products. Today, most products are produced to function for years without any failure. Therefore, few units will fail in a test at normal conditions. For example, a satellite is designed to operate expectedly for 10 to 15 years, however the design and test process may only last eight months. In manufacturing industries Accelerated Tests (AT) are applied to measure reliability of components and subsystems. These tests are used to verify components and spot failure modes so that they can be fixed or the products can be provided from other manufacturers. Due to rapid technological changes, more complex products, higher expectations of customers, and rapid development of products, ATs have found great importance. However, accelerated testing of complex products that have multiple failure mechanisms involves different practical and statistical issues. Generally, in order to estimate life or long-term performance at normal conditions, a statistical model that is reasonable from physical viewpoint is used to extrapolate the information obtained from high levels of one or more accelerating variables such as humidity, temperature, voltage, use rate, power cycling, vibration, and mechanical loading to their normal levels (Escobar and Meeker (2006)).

Accelerated life testing is one of the testing programs that accelerates aging process of a product by exposing it to conditions in excess of its normal service parameters. These testing programs are applied for estimating the reliability by using an appropriate statistical model in less time than would typically be needed. For more information on ALTs readers can refer to Nelson (2009), Meeker and Escobar (2014), the references in Nelson (2005a), (2005b), Polson and Soyer (2017), Roy and Mukhopadhyay (2016), and Ahmadini and Coolen (2020).

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To obtain accurate estimates of reliability measures, an appropriate reliability model is required to find a relationship between failure times or failure rates at accelerated conditions to those at design conditions. Since these models usually predict the reliability in a long period of time, their accuracy is of great importance. Further, in order to achieve a greater accuracy in reliability prediction, ALT plans are used. In ALT plans, the proportion of units at each stress level, the optimum stress levels allocated to testing units, and the duration of the test are determined. Many researchers have worked on the design of ALT plans. Recently, Han (2020) studied a simple step- stress accelerated life test under progressive type I censoring and used D- optimality, C- optimality, A- optimality, and E- optimality to obtain an optimal plan. In another work, Han (2020) considered the same problem under the practical constraints that the test duration is pre-fixed and the total experimental cost does not exceed a pre-specified budget. The optimal design was investigated for exponential lifetimes with a single stress variable under several design criteria. Hakamipour (2020) proposed an approximated optimal design for a bivariate step-stress accelerated life test under type-I progressive censoring and generalized exponential distribution. By minimization of the asymptotic variance of the percentile life under the usual operating condition the optimum test plan was obtained. For more details on ALT plans readers can refer to the works of Nelson (2009), (2005a), (2005b), Zhu and Elsayed (2013b), (2013a), Yang and Pan (2013), Bai, Chung, and Chun (1993), Bai and Chung (1991), Han (2017), Lee et al. (2018), Wu and Huang (2019)). As it can be seen, previous studies on designing ALT plans have focused on products that are under one time-scale. In real world applications, systems often age because of two or more time-scales (Frickenstein and Whitaker (2003)). Sometimes several time-scales should be taken into consideration. Readers can refer to the works of Farewell and Cox (1979), Kordonsky and Gertsbakh (1997), Gertsbakh and Kordonsky (1998), Lawless, Crowder, and Lee (2009), Oakes (1995), Lawless and Crowder (2010), Duchesne and Lawless (2000), Noorossana and Sabri- Laghaie (2016), Sabri-Laghaie and Noorossana (2016), Asadi, Saidi-Mehrabad, and Fathi Aghdam (2019) to obtain more information about systems which are exposed to several time-scales. In this regard, Singpurwalla and Wilson (1998) developed probabilistic models for systems which are subjected to time-scales of time and usage. They utilized an additive hazards model and stochastic processes to relate the scales and describe the usage progress. Noorossana and Sabri- Laghaie (2016) investigated system reliability under competing risks and multiple time scales which follow independent Poisson processes. Finkelstein (2004) investigated the effect of a usage process on reliability performance of systems indexed by two scales. He discussed that the shape of the failure rate can be affected by random usage process. Sabri-Laghaie and Noorossana (2016) considered the effect of random usage on degradation processes and developed time-based and condition-based maintenance models to maximize availability of the system. Lawless and Crowder (2010) proposed models to predict failures of systems exposed to varying levels of usage when data are incomplete.

Despite the vast literature on optimal design of accelerated life tests, to the best of our knowledge, the optimal design of accelerated life tests for systems under multiple time-scales has not been addressed. For example, consider a system which may fail during time or due to a random usage process. Generally speaking, accelerated life tests developed for these systems don't take into account the effect of random usage process in the design and implementation of these tests. Therefore, in this paper an approach for implementing and a model for optimal design of accelerated life tests for such systems are proposed.

In the following section, we describe the problem of interest and its assumptions. In Section 3, mathematical formulations of the problem are provided. In Section 4, performance of the proposed model is presented through a numerical example and sensitivity analysis, respectively. Section 5 provides our concluding remarks.

2. Problem Formulation and Assumptions

As it was stated earlier, the purpose of this research is to propose an approach to implement accelerated life tests for systems under random usage. Therefore, we need to consider the effect of random usage process in applying an accelerated life test. Furthermore, we assume that a system is under the effect of a random usage according to a Poisson process. Hence, we are assuming that a random usage process affects the system as well as common stresses such as temperature, humidity, and electric field. This system should be randomly used while it is under a stress. Usage times are random and follow a specific probability distribution. Here we assume that time between two successive usages follows an exponential distribution. For example, suppose that we want to study the reliability of insulator coils used in electric motors. Under this condition, insulator coils are exposed to high temperature. Electric motors may be randomly used according to demand. In order to consider the effect of random usage in reliability tests we randomly use the motor while it is under a temperature stress. Therefore, each motor is simultaneously subjected to a random usage process and a temperature stress. It is clear that ignoring the effect of random usage with a specified time duration can distort the reliability analysis. Since reliability tests are implemented to predict the reliability of products, it is necessary to take into consideration the effect of random usage process.

In this paper, it is supposed that the product is under a random Poisson usage process. Therefore, a reliability accelerated life test which includes a random usage is considered. In addition, it is assumed that the product is exposed to another stress such as temperature or humidity too. Therefore, a model for designing accelerated life tests is developed by which the variance of the hazard rate estimate over a predetermined time period in design conditions is minimized and the optimal stress levels of test units and the number of units at every stress level are obtained.

In this problem following assumptions are adopted:

1- The product is exposed to a single constant stress *z*.
2- It is assumed that the cumulative number of usages till time *t*, *M(t)*, follows a Poisson process.
3- The product, while exposed to stress *z*, repeatedly goes operational (*M(t)* times) and each time stays operational for a pre-specified amount of time, say t_u .
4- In order to relate the reliability performance under different stress levels a hazard rate model, $h(t;z)$ is proposed.
5- The baseline hazard function $h_0(t)$ is:

$$h_0(t) = \gamma_0 + \gamma_1 t + \gamma_2 t^2$$

where, γ_0 , γ_1 , and γ_2 are unknown parameters and *t* is the base time-scale. Constant, increasing, and decreasing hazard rates can be modeled by this type of hazard function. Examples of this model are presented in Figure 1.

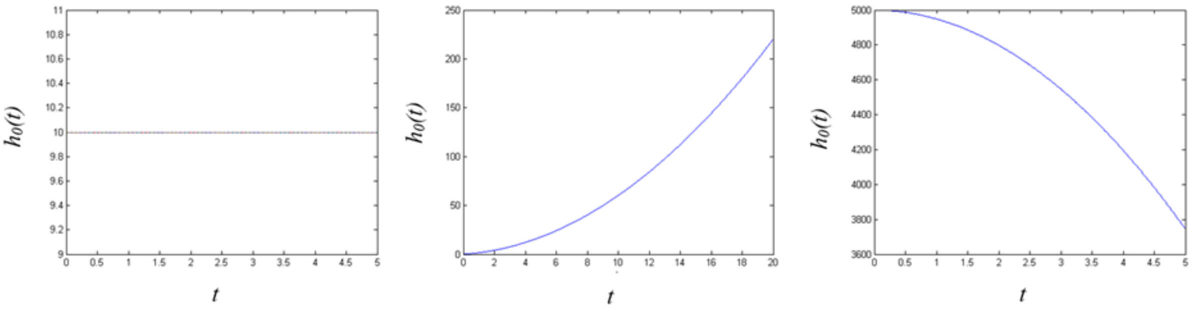

Fig. 1. Examples of baseline hazard function

6- The upper and lower bounds of stress are pre-specified. The upper bound is expressed as the maximum possible stress level which above that other failure modes may occur.
7- The total number of test units, *n*, is predetermined. The proportion of units allocated to the i^{th} level of stress *z* is represented by $p_i (i = 1, 2, ..., m)$ where *m* is the number of stress levels.
8- *m* or number of stress levels is pre-specified.
9- Test units have independent life times.
10- A predetermined censoring time, τ , is considered for test termination. Therefore, the product goes operational at most $M(\tau)$ times during the test. It is clear that

$$t_u \times M(\tau) \le \tau$$

11- The usage process is not accelerated.

3. Methodology

In this section, the methodology for designing ALT plans for products exposed to random usage is described. First, the hazard rate function is formulated based on the random usage process. Then, the parameter estimation procedure is specified. The toolbar and its menus.

3.1. Hazard rate function calculation by considering random usage

As mentioned, the baseline hazard rate of product is

$$h_0(t) = \gamma_0 + \gamma_1 t + \gamma_2 t^2 \quad (1)$$

Therefore, cumulative baseline hazard rate is given as

$$H_0(t) = \int_0^t h_0(s) ds = \gamma_0 t + \gamma_1 \frac{t^2}{2} + \gamma_2 \frac{t^3}{3} \quad (2)$$

Singpurwalla and Wilson (1998) studied a case when one additional time scale (random usage) that is time-dependent increases the hazard rate of the product based on a given function. Therefore, the hazard rate of the product can be defined as

$$h(t) = h_0(t) + \eta M(t) \quad (3)$$

where, *M(t)* is the cumulative usage process and η is the impact of additional time scale on hazard rate . Also, $h_0(t)$ is the base hazard rate of the product. Let *M(t)* follow a Poisson process as

$$\Pr\left(M(t) = u \mid \Lambda(t)\right) = \frac{\exp\left(-\Lambda(t)\right)\left(\Lambda(t)\right)^u}{u!} \qquad u = 0, 1, ...$$

where $\Lambda(t)$ is the mean of *M(t)*. Again, according to Singpurwalla and Wilson (1998), the reliability function is given as

$$R(t) = \exp\begin{pmatrix} -H_0(t) - \Lambda(t) + \\ \exp(-\eta t)\int_0^t \lambda(s)\exp(\eta s)\, ds \end{pmatrix} \quad (4)$$

where $\lambda(t) = d\Lambda(t)/dt$. $\Lambda(t)$ can be acquired based on the field data about the usage process of the product. In other words, one can fit a function on the usage count of the product over time and approximate $\Lambda(t)$. Therefore,

$$f(t) = -dR(t)/dt = \left\{h_0(t) + \int_0^t \eta\lambda(s)e^{\eta(s-t)}ds\right\} R(t) \tag{5}$$

The hazard rate function, $h(t)$, is finally given as

$$h(t) = f(t)/R(t) = h_0(t) + \int_0^t \eta\lambda(s)e^{\eta(s-t)}ds \tag{6}$$

We use relation (6) and propose the following model to relate hazard rates in different stress levels.

$$h(t;z) = h_0(t)e^{\beta_1 z} + e^{\beta_2 z}\int_0^t \eta\lambda(s)e^{e^{\beta_2 z}\eta(s-t)}ds \tag{7}$$

where, β_1 and β_2 are unknown parameters of the model. Based on relation (7) and $R(t;z) = \exp(-\int_0^t h(x;z)dx)$, reliability and failure probability distribution functions can be given as follows.

$$R(t;z) = \exp\begin{pmatrix} -\exp(\beta_1 z)H_0(t) - \Lambda(t) \\ +\int_0^t \lambda(s)\exp\left(\exp(\beta_2 z)\eta(s-t)\right)ds \end{pmatrix} \tag{8}$$

$$f(t;z) = \begin{Bmatrix} h_0(t)\exp(\beta_1 z) + \\ \int_0^t \exp(\beta_2 z)\eta\exp\begin{pmatrix}\exp(\beta_2 z)\\ \eta(s-t)\end{pmatrix}\lambda(s)ds \end{Bmatrix} \tag{9}$$

$$\times R(t;z) = Q(t;z)\times R(t;z)$$

where,

$$Q(t;z) = h_0(t)\exp(\beta_1 z)$$
$$+\int_0^t \exp(\beta_2 z)\eta\exp\begin{pmatrix}\exp(\beta_2 z)\\ \eta(s-t)\end{pmatrix}\lambda(s)ds \tag{10}$$

We utilize maximum likelihood estimation (MLE) procedure to obtain the values of the model parameters.

3.2. Parameter estimation

In this section, the log likelihood function of an observation *t* under stress *z* with type I censoring is specified. Here, the indicator function $I = I(t \le \tau)$ is defined in terms of censoring time *τ* as follows.

$$I = I(t \le \tau) = \begin{cases} 1 & \text{Failure observed before time } t \le \tau \\ 0 & \text{Censored at time } t > \tau \end{cases}$$

Log likelihood of a type I censored observation at stress *z* is given as

$$Ln\,L(t;z) = I\,Ln Q(t;z) - \exp(\beta_1 z)H_0(t) - \Lambda(t)$$
$$+\int_0^t \lambda(s)\exp\left(\exp(\beta_2 z)\eta(s-t)\right)ds \tag{11}$$

where, $Q(t;z)$ is given by equation (10).

Let i^{th} observation denoted as t_i correspond to stress level z_i where its log likelihood is given as l_i. The sample log likelihood l for *n* independent observations is as defined as

$$l = l_1 + l_2 + \ldots + l_n$$

For an individual observation, the partial derivatives with respect to the model parameters are

$$\frac{\partial Ln\,L(t;z)}{\partial\gamma_0} = \frac{I\exp(\beta_1 z)}{Q(t;z)} - t\exp(\beta_1 z) \tag{12}$$

$$\frac{\partial Ln\,L(t;z)}{\partial\gamma_1} = \frac{I\,t\exp(\beta_1 z)}{Q(t;z)} - \frac{t^2}{2}\exp(\beta_1 z) \tag{13}$$

$$\frac{\partial Ln\,L(t;z)}{\partial\gamma_2} = \frac{I\,t^2\exp(\beta_1 z)}{Q(t;z)} - \frac{t^3}{3}\exp(\beta_1 z) \tag{14}$$

$$\frac{\partial Ln\,L(t;z)}{\partial\beta_1} = \frac{I\,h_0(t)\,z\exp(\beta_1 z)}{Q(t;z)}$$
$$-H_0(t)\,z\exp(\beta_1 z) \tag{15}$$

$$\frac{\partial Ln\,L(t;z)}{\partial\beta_2} = \frac{I\int_0^t \lambda(s)\begin{Bmatrix} z\eta\exp\left(\exp(\beta_2 z)\eta(s-t)\right) \\ 1+\beta_2 z \\ \exp(\beta_2 z)\eta(s-t)\end{Bmatrix}ds}{Q(t;z)}$$
$$+\int_0^t \exp\begin{pmatrix}\exp(\beta_2 z)\eta(s-t)\\ +\beta_2 z\end{pmatrix}\lambda(s)\,z\,\eta(s-t)\,ds \tag{16}$$

$$\frac{\partial Ln\,L(t;z)}{\partial\eta} =$$
$$\frac{I\int_0^t \beta_2 z\exp\begin{pmatrix}\exp(\beta_2 z)\\ \eta(s-t)\end{pmatrix}\lambda(s)\begin{Bmatrix}1+\exp(\beta_2 z)\\ \eta(s-t)\end{Bmatrix}ds}{Q(t;z)}$$
$$+\int_0^t \exp\begin{pmatrix}\exp(\beta_2 z)\\ \eta(s-t)+\beta_2 z\end{pmatrix}\lambda(s)(s-t)ds \tag{17}$$

The maximum likelihood estimations of model parameters can be found by summing equations (12) to (17), equating them to zero, and simultaneously solving the resulting set of equations. Here, we just consider the correlation among parameters γ_0 , γ_1 , and γ_2 . Second partial derivatives are accessible through appendix I. The elements of the Fisher information matrix for an individual observation that are the negative expectations of second partial derivatives can also be found in appendix II. The Fisher information matrix *F* for all units is represented as

$$F = n\,p_0F_0 + n\,p_1F_1 + n\,p_2F_2 + \ldots + n\,p_mF_m$$

In which, *n* is the total number of units under test, p_0 is the proportion of test units allocated to stress level z_0 , p_1 is the proportion of test units allocated to stress level z_1 , etc., and p_m is the proportion of test units allocated to stress level z_m . In this regard, F_0 is the Fisher information matrix for a unit under test condition z_0 , F_1 is the Fisher information matrix for a unit under test condition z_1 , etc., and F_m is the Fisher information matrix for a unit under test condition z_m .

The Fisher information matrix is a function of γ_0, γ_1, γ_2, β_1, β_2, η, $z_i\,(i=1,\ldots,m)$, and $p_i\,(i=1,\ldots,m)$. The asymptotic variance-covariance matrix $\sum_{\sim}$ of maximum likelihood estimates of $\hat{\gamma}_0$, $\hat{\gamma}_1$, $\hat{\gamma}_2$, $\hat{\beta}_1$, $\hat{\beta}_2$, and $\hat{\eta}$ is equal to the inverse of the Fisher information matrix F.

$$\sum_{\sim}=\begin{bmatrix} Var(\hat{\gamma}_0) & Cov(\hat{\gamma}_0,\hat{\gamma}_1) & Cov(\hat{\gamma}_0,\hat{\gamma}_2) & Cov(\hat{\gamma}_0,\hat{\beta}_1) & Cov(\hat{\gamma}_0,\hat{\beta}_2) & Cov(\hat{\gamma}_0,\hat{\eta}) \\ Cov(\hat{\gamma}_0,\hat{\gamma}_1) & Var(\hat{\gamma}_1) & Cov(\hat{\gamma}_1,\hat{\gamma}_2) & Cov(\hat{\gamma}_1,\hat{\beta}_1) & Cov(\hat{\gamma}_1,\hat{\beta}_2) & Cov(\hat{\gamma}_1,\hat{\eta}) \\ Cov(\hat{\gamma}_0,\hat{\gamma}_2) & Cov(\hat{\gamma}_1,\hat{\gamma}_2) & Var(\hat{\gamma}_2) & Cov(\hat{\gamma}_2,\hat{\beta}_1) & Cov(\hat{\gamma}_2,\hat{\beta}_2) & Cov(\hat{\gamma}_2,\hat{\eta}) \\ Cov(\hat{\gamma}_0,\hat{\beta}_1) & Cov(\hat{\gamma}_1,\hat{\beta}_1) & Cov(\hat{\gamma}_2,\hat{\beta}_1) & Var(\hat{\beta}_1) & Cov(\hat{\beta}_1,\hat{\beta}_2) & Cov(\hat{\beta}_1,\hat{\eta}) \\ Cov(\hat{\gamma}_0,\hat{\beta}_2) & Cov(\hat{\gamma}_1,\hat{\beta}_2) & Cov(\hat{\gamma}_2,\hat{\beta}_2) & Cov(\hat{\beta}_1,\hat{\beta}_2) & Var(\hat{\beta}_2) & Cov(\hat{\beta}_2,\hat{\eta}) \\ Cov(\hat{\gamma}_0,\hat{\eta}) & Cov(\hat{\gamma}_1,\hat{\eta}) & Cov(\hat{\gamma}_2,\hat{\eta}) & Cov(\hat{\beta}_1,\hat{\eta}) & Cov(\hat{\beta}_2,\hat{\eta}) & Var(\hat{\eta}) \end{bmatrix}=F^{-1}$$

Each ALT plan is described by stress levels, proportion of allocated units to each stress level, total number of available test units, and censoring time. Here, it is supposed that maximum and design stress levels, total number of available test units, and censoring time are pre-determined. As Elsayed and Zhang (2007) proposed, the objective is to determine the stress levels and proportion of allocated test units to each level, such that the asymptotic variance of the MLE estimate of the hazard function at design conditions is minimized. This is a non-linear optimization problem which is stated as follows.

$$\text{Min} \int_0^T Var\begin{bmatrix}\left(\hat{\gamma}_0+\hat{\gamma}_1 t+\hat{\gamma}_2 t^2\right)e^{\hat{\beta}_1 z}+ \\ e^{\beta_2 z}\int_0^t \eta\lambda(s)e^{e^{\beta_2 z}\eta(s-t)}ds\end{bmatrix}\mathrm{d}t$$

s.t.

$$0<p_i<1, \qquad i=0,1,\ldots,m$$

$$\sum_{i=0}^{m} p_i = 1,$$

$$\sum_{\sim} = F^{-1},$$

$$z_L \le z_0 < z_1 < \cdots < z_{m-1} < z_m \le z_H,$$

$$np_i \ge \text{MNL}, \quad {}_{i=0,1,\ldots,m}$$

where, T is the time period that the hazard rate estimate is found over it and MNL is the least number of allocated units to stress levels. Stresses z_L and z_H are respectively the lowest and highest applicable stress levels under which other failure modes than the ones at normal conditions will not occur. In fact, z_L is the normal condition.

Here, Delta method is used to calculate

$$Var\left[h_0(t)e^{\hat{\beta}_1 z}+e^{\hat{\beta}_2 z}\int_0^t \hat{\eta}\lambda(s)e^{e^{\beta_2 z}\hat{\eta}(s-t)}\mathrm{d}s\right]$$

as below

$$Var\left[\left(\hat{\gamma}_0+\hat{\gamma}_1 t+\hat{\gamma}_2 t^2\right)e^{\hat{\beta}_1 z}+e^{\hat{\beta}_2 z}\int_0^t \hat{\eta}\lambda(s)e^{e^{\hat{\beta}_2 z}\hat{\eta}(s-t)}\mathrm{d}s\right]$$
$$=Var\left[\hat{\rho}\left(\hat{\gamma}_0,\hat{\gamma}_1,\hat{\gamma}_2,\hat{\beta}_1,\hat{\beta}_2,\hat{\eta}\right)\right]$$
$$=\left[\frac{\partial\hat{\rho}}{\partial\hat{\gamma}_0},\frac{\partial\hat{\rho}}{\partial\hat{\gamma}_1},\frac{\partial\hat{\rho}}{\partial\hat{\gamma}_2},\frac{\partial\hat{\rho}}{\partial\hat{\beta}_1},\frac{\partial\hat{\rho}}{\partial\hat{\beta}_2},\frac{\partial\hat{\rho}}{\partial\hat{\eta}}\right]\sum_{\sim}\left[\frac{\partial\hat{\rho}}{\partial\hat{\gamma}_0},\frac{\partial\hat{\rho}}{\partial\hat{\gamma}_1},\frac{\partial\hat{\rho}}{\partial\hat{\gamma}_2},\frac{\partial\hat{\rho}}{\partial\hat{\beta}_1},\frac{\partial\hat{\rho}}{\partial\hat{\beta}_2},\frac{\partial\hat{\rho}}{\partial\hat{\eta}}\right]'.$$

Numerical methods should be utilized to solve this non-linear optimization problem.

4. Results and Discussion

In this section, a procedure for executing ALT and collecting data is proposed. Then, a numerical example and sensitivity analysis on model parameters are given.

4.1. Data collection procedure

Here, a procedure for executing ALT and collecting failure data is proposed. Therefore, the following steps are considered:

1- Determine censoring time, τ, of the test.

2- According to the characteristics of the product and its usage conditions determine the usage time, t_u. In other words, specify the time that the product should stay operational when it is used.

3- According to a Poisson process with mean $\Lambda(\tau)$, generate *m* random numbers, M_i for $i=1,\ldots,m$. M_i is the maximum number of usage for stress level *i* during the test.

4- Based on the values of t_u and τ make a timetable for implementing the usage process of the products.

5- Put the units under stress levels and according to the timetable in step 4 implement the usage process on each unit.

6- Record failure times for failed units and censoring times for the units that have not failed at time τ. Failure times can be recorded by a device that detects failure or can be detected at some predetermined inspection intervals.

4.2. A numerical example

The type of testing in this article needs special conditions and has not already been considered in the literature. The testing equipment should be able to apply random usage to the units in addition to common stresses. Here, a set of failure times as represented in Table 1 is considered for the analysis. It is supposed that the failure data belong to a typical device and has been collected under accelerated life testing with a random usage. This product is exposed to three levels of temperature, 40, 60, and 80 degrees Celsius. The objective is to find if the product can meet the desired hazard rate in 5000 hours of testing and 30000 hours of functioning in ambient temperature of 10 degrees Celsius. In addition to the mentioned conditions it is also assumed that the products are subjected to a random usage process with rate $\Lambda(t)=t$. The results of this test are presented in Table 1.

Table 1
Accelerated life test data for a typical device

Hour	Condition	No. of units	Temperature ($^\circ$C)
5000	Censored	30	10
1298	Failed	1	40
1390	Failed	1	40
3187	Failed	1	40
3241	Failed	1	40
3261	Failed	1	40
3313	Failed	1	40
4501	Failed	1	40
4568	Failed	1	40
4841	Failed	1	40
4982	Failed	1	40
5000	Censored	90	40
581	Failed	1	60
925	Failed	1	60
1432	Failed	1	60
1586	Failed	1	60
2452	Failed	1	60
2734	Failed	1	60
2772	Failed	1	60
4106	Failed	1	60
4674	Failed	1	60
5000	Censored	11	60
283	Failed	1	80
361	Failed	1	80
515	Failed	1	80
638	Failed	1	80
854	Failed	1	80
1024	Failed	1	80
1030	Failed	1	80
1045	Failed	1	80
1767	Failed	1	80
1777	Failed	1	80
1856	Failed	1	80
1951	Failed	1	80
1964	Failed	1	80
2884	Failed	1	80
5000	Censored	1	80

According to data in Table 1, ML estimates of parameters are given as: $\hat{\gamma}_0 = 99.304$, $\hat{\gamma}_1 = 95.065$, $\hat{\gamma}_2 = 43.119$, $\hat{\beta}_1 = -2247.916$, $\hat{\beta}_2 = 0.2393$ and $\hat{\eta} = 36.646$. We have divided all data to 10000 and then estimated the parameters.

Now, we find the optimal test plan according to the estimated parameters. We aim to acquire the reliability estimates in design conditions and in a period of 30000 hours. Design condition is in temperature 10 $^\circ$C. Higher bound for temperature is 100 $^\circ$C. Allowed time for testing is 5000 hours and 200 units are under test. At least 10 units should be allocated to each stress level. The objective is to find optimal levels of temperature and proportion of units allocated to these levels. Here, three levels for temperature is considered. Therefore, test plan is determined as:

1- Based on Arrhenius model we use the following term as accelerating variable z in the ALT model

$$\frac{1}{\text{Absolute Temp.}}$$

2- Estimates of parameters are found as: $\hat{\gamma}_0 = 99.304$, $\hat{\gamma}_1 = 95.065$, $\hat{\gamma}_2 = 43.119$, $\hat{\beta}_1 = -2247.916$, $\hat{\beta}_2 = 0.2393$ and $\hat{\eta} = 36.646$.

3- Optimization problem is formulated as

$$\text{Min} \int_0^T Var\left[\begin{array}{l}\left(\hat{\gamma}_0 + \hat{\gamma}_1 t + \hat{\gamma}_2 t^2\right) e^{\hat{\beta}_1 z} \\ +e^{\hat{\beta}_2 z} \int_0^t \hat{\eta}\lambda(s) e^{\hat{\eta}(s-t)} \mathrm{d}s\end{array}\right] \mathrm{d}t$$

s.t.

$$0 < p_i < 1, \qquad i = 0,1,2$$

$$\sum_{i=0}^{2} p_i = 1,$$

$$\sum_{\sim} = F^{-1},$$

$$50\,^\circ\text{C} \le 1/z_0 - 273.15,$$

$$1/z_0 - 273.15 < 1/z_1 - 273.15,$$

$$1/z_1 - 273.15 < 1/z_2 - 273.15,$$

$$1/z_2 - 273.15 \le 100\,^\circ\text{C},$$

$$\underset{i=0,1,2}{np_i} \ge \text{MNL},$$

where, $\text{MNL} = 0.05 \times 200$, $T = 30000$, $n = 200$, $\tau = 5000$, and other parameters are as in step 2. In this problem $x = [z_0, z_1, z_2, p_0, p_1, p_2]$.

4- We apply numerical methods to solve this optimization problem. In this regard, optimization toolbox of Matlab software is used to optimize this problem. Since the optimization model is non-linear and we are not aware of the convexity of the objective function, we test different initial points to find the optimal solution.

5- The optimal solution which minimizes the objective function and satisfies the constraints is given as

$$T_0^* = 1/z_0 - 273.15 = 79.772\,^\circ\text{C},\ p_0^* = 0.37898$$

$$T_1^* = 1/z_1 - 273.15 = 84.773\,^\circ\text{C},\ p_1^* = 0.36896$$

$$T_2^* = 1/z_2 - 273.15 = 99.773\,^\circ\text{C},\ p_2^* = 0.25206$$

4.3. Sensitivity analysis

In order to solve the optimization model, we should first estimate the parameters. Since the estimations are point estimates we need to study the sensitivity of the model to changes in the important parameter estimates. Therefore, in this section the sensitivity of the results of the obtained ALT plan to parameter deviations is analyzed. If a slight change in a parameter results in a pretty big change in the results of the optimal ALT plan, that plan would be sensitive to that parameter. In addition, some parameters in the non-linear optimization plan are assigned arbitrary values or their values are determined based on engineering judgments, for example censoring time τ, minimum number of required units in each stress level MNL, and overall time period, T, are instances of these parameters. If the results of the optimal ALT plan are sensitive to the parameters then we should make changes in these parameters to have an accurate estimation of the reliability in design condition.

In order to analyse the sensitivity of the model to input parameters, we change the value of one parameter while the values of other parameters are fixed. Then, the non-linear optimization problem is solved to acquire the optimal ALT plan. If a slight change in a parameter results in a pretty big change in the optimization results, it is said that the ALT plan is not robust to that parameter. The results of the sensitivity analysis are shown in Table 2. As it can be seen, the results are robust to model parameters. However, the estimates of some parameters like $\hat{\gamma}_0$, $\hat{\gamma}_1$, $\hat{\gamma}_2$, $\hat{\beta}_1$, T, and λ show less robustness, therefore accurate estimation or appropriate determination of these parameters is necessary for having an optimal ALT plan.

5. Conclusion

In this paper, we designed an optimum ALT plan for products which are exposed to a random usage process. We supposed that random usage process affects hazard rate of the products. In the optimum ALT plan, the levels of stress and the number of allocated test units to each level are determined so that the variance of hazard rate estimate at normal operating conditions over a specified period of time is minimized. The proposed ALT plan is illustrated through a numerical example. A sensitivity analysis is also performed on the model and the results show that the ALT plan is not sensitive to parameters deviations. We also proposed a data collection procedure for ALT plan of products which are under a random usage process. For future research, one can develop other types of accelerating tests such as Accelerated Degradation Tests (ADTs) for products which are subjected to random usage.

Table 2
Sensitivity analysis: effect of uncertainty in model parameters on stress levels and allocated proportions

Parameter	Change (%)	T_0^*	T_1^*	T_2^*	p_0^*	p_1^*	p_2^*
$\hat{\gamma}_0 = 99.304$	-20%	80	89.881	100	0.37679	0.33879	0.28443
	+20%	79.748	89.063	99.748	0.4388	0.42878	0.13242
$\hat{\gamma}_1 = 95.065$	-20%	78.818	93.818	98.818	0.40006	0.27723	0.32271
	+20%	80	93.936	100	0.36036	0.28929	0.35035
$\hat{\gamma}_2 = 43.119$	-20%	80	86.373	100	0.37365	0.26272	0.36364
	+20%	80	95	100	0.34	0.33	0.33
$\hat{\beta}_1 = -2247.916$	-20%	80	94.98	100	0.34	0.33	0.33
	+20%	79.772	84.773	99.773	0.37898	0.36896	0.25206
$\hat{\beta}_2 = 0.2393$	-20%	79.772	84.773	99.773	0.37898	0.36896	0.25206
	+20%	79.772	84.773	99.773	0.37898	0.36896	0.25206
$\hat{\eta} = 36.646$	-20%	79.772	84.773	99.773	0.37898	0.36896	0.25206
	+20%	79.772	84.773	99.773	0.37898	0.36896	0.25206
$T = 30000$	-20%	79.772	84.773	99.773	0.37898	0.36896	0.25206
	+20%	79.748	89.063	99.748	0.4388	0.42878	0.13242
$\lambda = 1$	-20%	80	89.881	100	0.37679	0.33879	0.28443
	+20%	79.772	84.773	99.773	0.37898	0.36896	0.25206

Table 3
Sensitivity analysis: effect of censoring time uncertainty on stress levels and allocated proportions

τ	T_0^*	T_1^*	T_2^*	p_0^*	p_1^*	p_2^*
5000	79.772	84.773	99.773	0.37898	0.36896	0.25206
6000	79.8754	89.1203	99.7876	0.3847	0.3771	0.2382
7000	79.8963	90.3686	99.8142	0.4674	0.4236	0.1097
8000	79.9173	91.4731	99.8770	0.5551	0.2449	0.2007
9000	79.9521	93.8661	99.9887	0.5834	0.3030	0.1145
10000	80	94.9501	99.9965	0.6620	0.1646	0.1734

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Appendix I

Second partial derivatives of the log Likelihood function are given as follows.

$$\frac{\partial^2 Ln\,L(t;z)}{\partial \gamma_0^2} = -\frac{I\exp(2\beta_1 z)}{Q^2(t;z)} \tag{18}$$

$$\frac{\partial^2 Ln\,L(t;z)}{\partial \gamma_1^2} = -\frac{I\,t^2\exp(2\beta_1 z)}{Q^2(t;z)} \tag{19}$$

$$\frac{\partial^2 Ln\,L(t;z)}{\partial \gamma_2^2} = -\frac{I\,t^4\exp(2\beta_1 z)}{Q^2(t;z)} \tag{20}$$

$$\frac{\partial^2 Ln\,L(t;z)}{\partial \gamma_0 \partial \gamma_1} = -\frac{I\,t\exp(2\beta_1 z)}{Q^2(t;z)} \tag{21}$$

$$\frac{\partial^2 Ln\,L(t;z)}{\partial \gamma_0 \partial \gamma_2} = -\frac{I\,t^2\exp(2\beta_1 z)}{Q^2(t;z)} \tag{22}$$

$$\frac{\partial^2 Ln\,L(t;z)}{\partial \gamma_1 \partial \gamma_2} = -\frac{I\,t^3\exp(2\beta_1 z)}{Q^2(t;z)} \tag{23}$$

$$\frac{\partial^2 Ln\,L(t;z)}{\partial \beta_1^2} = \frac{I\,z\,h_0(t)\exp(\beta_1 z)\left(zQ(t;z) - z\,h_0(t)\exp(\beta_1 z)\right)}{Q^2(t;z)} - z^2 H_0(t)\exp(\beta_1 z) \tag{24}$$

$$\begin{aligned}\frac{\partial^2 Ln\,L(t;z)}{\partial \beta_2^2} &= \frac{I\,Q(t;z)\int_0^t z^2\eta^2(s-t)\exp\left(\exp(\beta_2 z)\eta(s-t)+\beta_2 z\right)\lambda(s)\left\{1+\beta_2 z\exp(\beta_2 z)\eta(s-t)\right\}ds}{Q^2(t;z)} \\ &+ \frac{I\,Q(t;z)\int_0^t z^2\eta^2(s-t)\exp\left(\exp(\beta_2 z)\eta(s-t)+\beta_2 z\right)\lambda(s)\left\{1+\beta_2 z\right\}ds - I\,T(t;z)^2}{Q^2(t;z)} \\ &+ \int_0^t z\eta(s-t)\lambda(s)\left(z+\eta(s-t)z\exp(\beta_2 z)\right)\exp\left(\exp(\beta_2 z)\eta(s-t)+\beta_2 z\right)ds\end{aligned} \tag{25}$$

where $T(t;z)$ is defined as

$$T(t;z) = \int_0^t z\eta\exp\left(\exp(\beta_2 z)\eta(s-t)\right)\lambda(s)\left\{1+\beta_2 z\exp(\beta_2 z)\eta(s-t)\right\}ds \tag{26}$$

Also, we have

$$\begin{aligned}\frac{\partial^2 Ln\,L(t;z)}{\partial \eta^2} &= \frac{I\,Q(t;z)\int_0^t \beta_2 z\exp(\beta_2 z)(s-t)\exp\left(\exp(\beta_2 z)\eta(s-t)\right)\lambda(s)\left\{1+\exp(\beta_2 z)\eta(s-t)\right\}ds}{Q^2(t;z)} \\ &+ \frac{I\,Q(t;z)\int_0^t \beta_2 z\exp(\beta_2 z)(s-t)\lambda(s)\exp\left(\exp(\beta_2 z)\eta(s-t)\right)ds - I\,J(t;z)^2}{Q^2(t;z)} \\ &+ \int_0^t \lambda(s)(s-t)^2\exp^2(\beta_2 z)\exp\left(\exp(\beta_2 z)\eta(s-t)\right)ds\end{aligned} \tag{27}$$

where, $J(t;z)$ is given as

$$J(t;z) = \int_0^t \beta_2 z\exp\left(\exp(\beta_2 z)\eta(s-t)\right)\lambda(s)\left\{1+\exp(\beta_2 z)\eta(s-t)\right\}ds \tag{28}$$

and, we have

$$\frac{\partial^2 Ln\,L(t;z)}{\partial\gamma_0\partial\eta}=-\frac{I\exp(\beta_1 z)\,J(t;z)}{Q^2(t;z)} \tag{29}$$

$$\frac{\partial^2 Ln\,L(t;z)}{\partial\gamma_1\partial\eta}=-\frac{I\,t\exp(\beta_1 z)\,J(t;z)}{Q^2(t;z)} \tag{30}$$

$$\frac{\partial^2 Ln\,L(t;z)}{\partial\gamma_2\partial\eta}=-\frac{I\,t^2\exp(\beta_1 z)\,J(t;z)}{Q^2(t;z)} \tag{31}$$

where $Q(t;z)$ and $J(t;z)$ are as defined by equations (10) and (28), respectively.

Appendix II

The elements of Fisher information matrix for an individual observation are as follows.

$$E\left[-\frac{\partial^2 Ln\,L(t;z)}{\partial\gamma_0^2}\right]=\int_0^\tau\frac{\exp(2\beta_1 z)}{Q(t;z)}\times R(t;z)\mathrm{d}t \tag{32}$$

$$E\left[-\frac{\partial^2 Ln\,L(t;z)}{\partial\gamma_1^2}\right]=\int_0^\tau\frac{t^2\exp(2\beta_1 z)}{Q(t;z)}\times R(t;z)\mathrm{d}t \tag{33}$$

$$E\left[-\frac{\partial^2 Ln\,L(t;z)}{\partial\gamma_2^2}\right]=\int_0^\tau\frac{t^4\exp(2\beta_1 z)}{Q(t;z)}\times R(t;z)\mathrm{d}t \tag{34}$$

$$E\left[-\frac{\partial^2 Ln\,L(t;z)}{\partial\gamma_0\partial\gamma_1}\right]=\int_0^\tau\frac{t\exp(2\beta_1 z)}{Q(t;z)}\times R(t;z)\mathrm{d}t \tag{35}$$

$$E\left[-\frac{\partial^2 Ln\,L(t;z)}{\partial\gamma_0\partial\gamma_2}\right]=\int_0^\tau\frac{t^2\exp(2\beta_1 z)}{Q(t;z)}\times R(t;z)\mathrm{d}t \tag{36}$$

$$E\left[-\frac{\partial^2 Ln\,L(t;z)}{\partial\gamma_1\partial\gamma_2}\right]=\int_0^\tau\frac{t^3\exp(2\beta_1 z)}{Q(t;z)}\times R(t;z)\mathrm{d}t \tag{37}$$

$$E\left[-\frac{\partial^2 Ln\,L(t;z)}{\partial\beta_1^2}\right]=-\int_0^\tau\frac{z\,h_0(t)\exp(\beta_1 z)\left(zQ(t;z)-z\,h_0(t)\exp(\beta_1 z)\right)}{Q(t;z)}\times R(t;z)\mathrm{d}t +\int_0^\infty z^2 H_0(t)\exp(\beta_1 z)\times Q(t;z)R(t;z)\mathrm{d}t \tag{38}$$

$$E\left[-\frac{\partial^2 Ln\,L(t;z)}{\partial\gamma_0\partial\eta}\right]=\int_0^\tau\frac{\exp(\beta_1 z)\,J(t;z)}{Q(t;z)}\times R(t;z)\mathrm{d}t \tag{39}$$

$$E\left[-\frac{\partial^2 Ln\,L(t;z)}{\partial\gamma_1\partial\eta}\right]=\int_0^\tau\frac{t\exp(\beta_1 z)\,J(t;z)}{Q(t;z)}\times R(t;z)\mathrm{d}t \tag{40}$$

$$E\left[-\frac{\partial^2 Ln\,L(t;z)}{\partial\gamma_2\partial\eta}\right]=\int_0^\tau\frac{t^2\exp(\beta_1 z)\,J(t;z)}{Q(t;z)}\times R(t;z)\mathrm{d}t \tag{41}$$

$$E\left[-\frac{\partial^2 Ln\,L(t;z)}{\partial\beta_2^2}\right]=$$
$$-\int_0^{\tau}\int_0^{t} z^2\eta^2(s-t)\exp\left(\exp\left(\beta_2 z\right)\eta\left(s-t\right)+\beta_2 z\right)\lambda(s)\left\{1+\beta_2 z\exp\left(\beta_2 z\right)\eta\left(s-t\right)\right\}\times R\left(t;z\right)\mathrm{d}s\mathrm{d}t$$
$$-\int_0^{\tau}\frac{Q\left(t;z\right)\int_0^{t} z^2\eta^2(s-t)\exp\left(\exp\left(\beta_2 z\right)\eta\left(s-t\right)+\beta_2 z\right)\lambda(s)\left\{1+\beta_2 z\right\}\mathrm{d}s-T(t;z)^2}{Q\left(t;z\right)}\times R\left(t;z\right)\mathrm{d}t \tag{42}$$
$$-\int_0^{\infty}\int_0^{t} z\eta(s-t)\lambda(s)\left(z+\eta(s-t)z\exp\left(\beta_2 z\right)\right)\exp\left(\exp\left(\beta_2 z\right)\eta\left(s-t\right)+\beta_2 z\right)\mathrm{d}s\mathrm{d}t$$

$$E\left[-\frac{\partial^2 Ln\,L(t;z)}{\partial\eta^2}\right]=$$
$$-\int_0^{\tau}\int_0^{t}\beta_2 z\exp\left(\beta_2 z\right)\left(s-t\right)\exp\left(\exp\left(\beta_2 z\right)\eta\left(s-t\right)\right)\lambda\left(s\right)\left\{1+\exp(\beta_2 z)\eta\left(s-t\right)\right\}\times R\left(t;z\right)\mathrm{d}s\mathrm{d}t$$
$$-\int_0^{\tau}\frac{I\,Q\left(t;z\right)\int_0^{t}\beta_2 z\exp\left(\beta_2 z\right)\left(s-t\right)\lambda\left(s\right)\exp\left(\exp\left(\beta_2 z\right)\eta\left(s-t\right)\right)ds-I\,J(t;z)^2}{Q\left(t;z\right)}\times R\left(t;z\right)\mathrm{d}t \tag{43}$$
$$-\int_0^{\infty}\int_0^{t}\lambda\left(s\right)\left(s-t\right)^2\exp^2\left(\beta_2 z\right)\exp\left(\exp\left(\beta_2 z\right)\eta\left(s-t\right)\right)\times Q\left(t;z\right)R\left(t;z\right)\mathrm{d}s\mathrm{d}t$$