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Effects of Probability Function on the Performance of Stochastic Programming

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Abstract

Stochastic programming is a valuable optimization tool where used when some or all of the design parameters of an optimization problem are defined by stochastic variables rather than by deterministic quantities. Depending on the nature of equations involved in the problem, a stochastic optimization problem is called a stochastic linear or nonlinear programming problem. In this paper, a stochastic optimization problem is transformed into an equivalent deterministic problem, which can be solved by any known classical methods (interior penalty method is applied here).The paper mainly focuses on investigating the effect of applying various probability functions distributions (normal, gamma, and exponential) for design variables. The following basic required equations to solve nonlinear stochastic problems with various probability functions for random variables are derived and sensitivity analyses to study the effects of distribution function types and input parameters on the optimum solution are presented as graphs and in tables by studying two considered test problems. It is concluded that the difference between probabilistic and deterministic solutions to a problem, when the normal distribution of random variables issued, is very different from the results when gamma and exponential distribution functions are used. Finally, it is shown that the rate of solution convergence tithe normal distribution is faster than the other distributions.

Keywords: Stochastic programming, Sensitivity Analysis, Linear programming, Nonlinear programming, Exponential, Gamma and normal probability functions.

1. Introduction

Stochastic or probabilistic programming deals with situations where some or all of the parameters of an optimization problem are defined by stochastic variables. Depending on the nature of equations involved in a problem, a stochastic optimization problem is called a stochastic linear or nonlinear programming problem. The basic idea used in stochastic programming is to convert a stochastic optimization problem into an equivalent deterministic optimization problem. The converted equivalent deterministic problem would be solved with constrained algorithms such as interior, exterior penalty, Rozen, Lagrangian methods.
Stability and sensitivity of some stochastic constrained methods reinvestigated Dentcheva (2006), Dentcheva and Ruszczy (2006), Dentcheva and Omisch (2013), Machi et al. (2014) and Kuo and Prasad (2000). The mathematical derivation of equations required for each method from a stochastic viewpoint toward the constrained optimization problems is taken into account.
Some new methods, such as Fuzzy Genetic Algorithm, have also been recently introduced by diverse researchers for system reliability optimization (Hennig, 2013; Mutingi and Mbohwa, 2014).
Berhan (2016) has considered a vehicle routing problem in urban public system and has developed it based on stochastic simultaneous pickup and delivery.
Pourbagheri and Akhavan (2015) considered a probabilistic demand model to model a single-vendor multiple-retailer supply chain that works according to vendor managing inventory policy.
In engineering problems, we usually encounter various random parameters. The sources of randomness of variables could be different. For example, the flexural or shear strength of concrete is a random variable in designing concrete structures, because the compressive strength of concrete samples is not the same as different samples. In addition, in designing mechanical systems, the actual length of a machined part is a random variable since this dimension may lie anywhere within a specified tolerance range. In stochastic solutions, fitting a distribution function that covers samples is essential. In choosing appropriate distribution function to cover sample points, we may face two or more choices for distribution functionsin a probabilistic engineering problem. For

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example, for annual rainfall, two distribution functions, exponential and gamma for $\alpha = 1$, can be applied appropriately. Although two or more distribution functions may fit with the same samples, it is shown in this research that choosing each one of them has its own effects on the rate of convergence and optimum solution.

To study the effects of probability functions used for distribution of variables in stochastic optimization, to the best of the authors' knowledge, a limited number of studies have been conducted. For reliability-based design optimization techniques, which try to reduce the failure probability of a certain objective function by reducing the area of its probability density function, Marchi et al. (2014) applied polynomial chaos expansion. It is a popular technique for the uncertainty quantification of stochastic processes. This technique is based on the theory of homogeneous chaos that is introduced to the description of stochastic Gaussian processes by means of an expansion based on Hermitian polynomials. Expansion of Hermitan polynomials, employed to solve finite-element and hydrodynamic problems nowadays, is used as a popular method for determination of stochastic properties of stochastic problems.

There are many probability distribution functions from a pure stochastic viewpoint, but some of these functions have more application in the modeling and solution of engineering problems. Among these distribution functions, three important ones are normal, gamma, and exponential functions used in this paper. Some of the applications of these considered distribution functions are explained as follows.

The most widely used probability distribution is the normal (Gaussian) distribution. For example, the compressive strength of similar concrete samples is a random variable in which the data follow a normal distribution (Rao, 2009).

Gamma distribution function is one of the waiting time distributions that may offer a good fit to time when collected data have gaps. This distribution is used for many important practical problems. For example, hydrologists usually use it because measuring, collecting, and storing hydrological data, such as rainfall in constant time intervals, is very difficult. The recording process both requires instrumental and personal supplies and there are gaps in the data. Since hydrological variables, such as rainfall and runoff, are positive values, the gamma distribution can be appropriate for them (AK soy, 2000; Krishnomorth et al., 2008; Khodabina and Ahmadabadi, 2010).

Another distribution function is the exponential function. This distribution does not require the previous data; therefore, it is well-suited to model the constant hazard rate portion of the bath-tub curve used in reliability theory (bath-tub curve is widely used in reliability engineering for describing a particular form of the hazard functions with different failure rates). The exponential distribution occurs naturally when describing the lengths of the inter-arrival times in the Poisson distribution. Exponential variables can also be used to model the situations where certain events occur with a constant probability per unit length, such as the distance between mutations on a DNA strand or between road kills on a given road (Aksoy, 2000; Krishnomorth et al., 2008; Khodabina and Ahmadabadi, 2010).

The focus of this paper is on studying the effects of various probability function distributions used to define design variables of the optimal solution to a problem. In the following section, the basic required equations are derived to solve nonlinear stochastic problems with various probability functions for variables. To investigate the effects of function type and input parameters on the solutions, sensitivity analysis is performed on two test problems. The results are presented as graphs and tables. This paper has shown that input parameters of objective function have significant effect on the optimum point value. Also, it is shown that the difference between probabilistic and deterministic solutions for the gamma distribution with $\alpha = 1$ is much more than that for other distribution functions, and so the results are not very sensitive to the probability values of normal and gamma distributions with $\alpha = 2$.

2. Formulations

In this section, first, some required equations for stochastic programming for various distribution functions, including normal, gamma and exponential formulations, are derived. In derivation of these equations, the probability of satisfying constraints is the main factor. Now, the required steps for the derivation of essential formulas are presented.

Assume that a random event is the measurement of quantityY, which takes on various values in the range of $-\infty$ to ∞. Such a quantity is called a random variable. In general, random variables are of two types: discrete and continuous. When some of the parameters involved in the objective function and the associated constraints vary their mean value or have a functional distribution as normal, gamma, or exponential distributions, a general optimization problem has to be formulated as a stochastic programming problem. The main difference between linear and nonlinear stochastic problems is the need of derivatives for nonlinear cases. A stochastic nonlinear programming problem can be stated in the standard form as in expression (1).

Find Y which minimizes f(Y)(1)

Subjected to Eqn. (2):

$P[g_j\ (Y) \geq 0] \geq p_j \quad j = 1,2,..,m$ (2)

whereY is the vector of Ndesired variables $y_1, y_2, ..., y_N$: some of them may be random in nature and some others as deterministic. What follows are basic equations (Eqs. 3 to 8) for a nonlinear stochastic optimization problem, Rao (2009).

2.1. Objective function

Objective function f(Y)can be expanded to the mean values of y_i as Taylor series (higher order derivative terms are omitted) in the form of Eq. (3).

$f(Y) = f(\overline{Y}) + \sum_{i=1}^{N}(\frac{\partial f}{\partial yi}|_{\overline{Y}})(y_i - \overline{y_i})$(3)

If the standard deviations of y_i, σ_{yi} be small, f(Y) can be approximated by the first two terms as Eq. (4).

$f(Y) \cong (\bar{Y}) - \sum_{i=1}^{N}(\frac{\partial f}{\partial yi}|_{\bar{Y}})(\bar{y}_i) + \sum_{i=1}^{N}(\frac{\partial f}{\partial yi}|_{\bar{Y}})(y_i) = \psi(Y)$ (4)

Assume that all y_i, i = 1, 2, . . . ,N follow the functional distribution ψ(Y), which is a linear function of Y, and follow the same distribution function. The mean and the variance of ψ are given by Eqs. (5) and (6) because all y_is are independent variables.

$\mu(\bar{Y}) = \bar{\psi}$ (5)

$Var(\psi) = \sigma_{\psi}^2 = \sum_{i=1}^{N}(\frac{\partial f}{\partial yi}|_{\bar{Y}})^2\sigma_{yi}^2$ (6)

For the purpose of optimization, a new objective function f(Y) can be constructed as Eq. (7).

$f(Y) = \bar{\psi}+\sigma_{\psi}$ (7)

Rao (2009) introduced two parameters $k_1 \geq 0$ and $k_2 \geq 0$ whose numerical values indicate the relative importance of ψ and σ_{ψ} for the minimization of an objective function, shown in Eq. (8). (Insection 3.2, the importance of these parameters k_1 and k_2 foran optimum solution to the problem is investigated).

$f(Y) = k_1\,\bar{\psi}+k_2\,\sigma_{\psi}$ (8)

2.2 Constraints

If some parameters are random in nature, the constraints will also be probabilistic. The constraint inequality can be written as in Eq. (9):

$\int_0^{\infty} f_{gj}\,(g_j)dg_j \geq p_j$ (9)

and noting that $\int_{-\infty}^{\infty} f(x)\quad dx = 1$

We have Eqn. (10):

$\int_0^{\infty} f_{g_j}(g_j)dg_j \geq \int_{-A}^{\infty} f(x)dx$(10)

Parameter A in Eq. (10) would be chosen in a way that it coincides with probability, where $f_{g_j}(g_j)$ is the probability density function of random variable g_j,whose range is assumed to be −∞ to ∞. Constraint function $g_j(Y)$ can be expanded to the vector of mean values of random variables Y in the form presented by Eq. (11):

$g_j(Y) \cong g_j(\bar{Y}) + \sum_{i-1}^{N}(\frac{\partial g_j}{\partial yi}|_{\bar{Y}})(y_i - \bar{y}_i)$(11)

From this equation, mean value $\bar{g}_j$ and standard deviation σ_{gj} of g_j can be obtained as Eqs. (12) and (13):

$\bar{g}_j = g_j(\bar{Y})$ (12)

$\sigma_{gj} = [\sum_{i=1}^{N}(\frac{\partial g_j}{\partial yi}|_{\bar{Y}})^2\sigma_{yi}^2]^{1/2}$ (13)

By introducing a new variable θ to Eq. (14) $\theta = \frac{g_j - \bar{g}_j}{\sigma_{gj}}$ (14) we can rewrite Eq. (10) in the form of Eq. (15):

$\int_{-\frac{\bar{g}_j}{\sigma_{gj}}}^{\infty} f(\theta)d\theta \geq \int_{-A_j}^{\infty} f(x)dx$(15)

So, Eq. (16) must be satisfied:

$\frac{\bar{g}_j}{\sigma_{gj}} \leq A_j$(16)

or can be written in the form of Eq. (17):

$-\bar{g}_j - \sigma_{gj}A_j \leq 0$ (17)

Equation (15) can be rewritten in the form of Eq. (18) by replacing Eq. (13) instead of σ_{gj}:

$-\bar{g}_j - A_j\left[\sum_{i=1}^{N}(\frac{\partial g_j}{\partial yi}|_{\bar{Y}})^2\sigma_{yi}^2\right]^{1/2} \leq 0$ (18)

2.3. Equivalent Deterministic Problem

Thus, the optimization problem can be stated in its equivalent deterministic form as Eqs. (19) and (20):

Find Y which minimizes f(Y)(19)

$f(Y) = k_1\,\bar{\psi}+k_2\,\sigma_{\psi}$ (20)

where:

$\psi(\bar{Y}) = \bar{\psi}$

and its variance is in the form of Eq. (21):

$Var(\psi) = \sigma_{\psi}^2 = \sum_{i=1}^{N}(\frac{\partial f}{\partial yi}|_{\bar{Y}})^2\sigma_{yi}^2$ (21)

Subjected to the probabilistic constraint in the form of Eq. (22):

$P[g_j\,(Y) \geq 0] \geq p_j \quad j = 1,2,..,m$ (22)

Similar to Eq. (18), we have Eq. (23):

$-\bar{g}_j - A_j\left[\sum_{i=1}^{N}(\frac{\partial g_j}{\partial yi}|_{\bar{Y}})^2\sigma_{yi}^2\right]^{1/2} \geq 0$ (23)

Depending on which probability function chosen and employed in Eq. (23), σ_{yi} and A_j in Eqs. (21) and (23) would be as follows:

2.4. Probability Functions

2.4.1.Exponential Distribution

The exponential distribution function is expressed in the form of Eq. (24). Its density function is shown in Fig. (1).

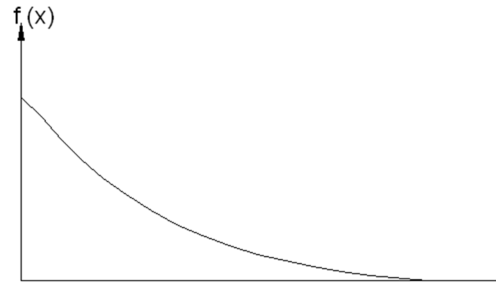


Fig. 1. Exponential density function.

$f(x;\lambda) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$(24)

Its mean and variance are in the form of Eq. (25).

$E(X) or \mu = \frac{1}{\lambda}$; $Var(X) or \sigma^2 = \frac{1}{\lambda^2}$(25)

In multi-dimensional optimization problem with constant α, we have Eq. (26):

$E(X) or \mu_i = \frac{1}{\lambda_i}$; $Var(X) or \sigma_i^2 = \frac{1}{\lambda_i^2}$(26)

Similar to Eq. (10), we. (27);

$\int_A^{\infty} \lambda e^{-\lambda x}\,dx = p_j \Rightarrow A = -\frac{\ln(p_j)}{\lambda}$(27)

In multi-dimensional optimization problem, A_i can be expressed as Eq. (28):

$\mu_i = \frac{1}{\lambda_i} \Rightarrow A_i = -\frac{\ln(p_j)}{\lambda_i}$(28)

2.4.2.Gamma Distribution

The gamma distribution function is in the form of Eq. (29) whose parameters α and θ would change this function. Its density function is shown in Fig. (2).

$$f(x) = \begin{cases} \frac{1}{\theta^{\alpha}\Gamma(\alpha)} x^{\alpha-1}\, e^{-x/\theta} & 0 \le x < \infty \\ 0 & x < 0 \end{cases} \quad (29)$$

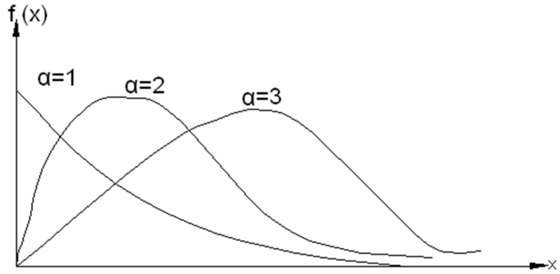


Fig. 2. Gamma density function**.**

Its mean and variance are in the form of Eq. (30).

$E(X) or \mu = \alpha\theta; Var(X) or \sigma^2 = \alpha\theta^2 (30)$

In multi-dimensional optimization problem with constant α, we haveEq. (31) for mean value and deviation:

$E(X) or \mu_i = \alpha\theta_i; \quad Var(X) or \sigma_i^2 = \alpha{\theta_i}^2 (31)$

Similar to Eq. (10), we have Eq. (32) for gamma distribution function for $\alpha = 1$:

$\alpha = 1 \Rightarrow$ wehave: $\int_{A_i}^{\infty} \frac{1}{\theta_i} e^{-x/\theta_i} dx = p_j (32)$

and A_i can be expressed as Eq. (33):

$A_i = -\theta_i \ln(p_j) \quad (33)$

For gamma distribution function for $\alpha = 2$, Eq. (10) can be written in the form of Eq. (34), and the solution to Eq. (35) to obtainA can be obtained through numerical methods.

$\alpha = 2 \Rightarrow$ wehave: $\int_{A_i}^{\infty} \frac{1}{{\theta_i}^2\Gamma(2)} x \quad e^{-x/\theta_i} dx = p_j (34)$

$e^{-A_i/\theta_i}[A_i + \theta_i] = \theta_i p_j (35)$

For gamma distribution function for $\alpha = 3$, Eq. (10) would be changed in the form of Eq. (36):

$\alpha = 3 \Rightarrow$ we have: $\int_{A}^{\infty} \frac{1}{{\theta_i}^2\Gamma(3)} x^2 \quad e^{-x/\theta_i} dx = p_j (36)$

${p_j}^2 e^{-A_i/\theta_i} + 2[\theta_i e^{-A_i/\theta_i}[A_i + \theta_i]] = 2{\theta_i}^2 p_j (37)$

Eq. (37) is a similar state to Eq. (35) and numerical methods are helpful.

2.4.3. Standard normal distribution

The normal distribution has a probability density function given by Eq. (38).Its density function is shown in Fig. (3).

$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-1/2\left[\frac{x-\mu_X}{\sigma_X}\right]^2} \quad -\infty < x < \infty \quad (38)$

where μ_X and σ_X are the parameters of the distribution, which are also the mean and standarddeviation of X, respectively. The distribution

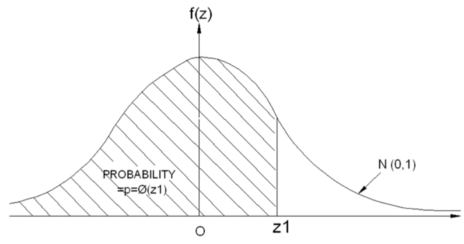


Fig. 3.Standard normal density function.

function of standard normal variable z is often designated as $\emptyset(z)$,and parameter p is the cumulative probability.

$\emptyset(z_1) = p_j and z_1 = \emptyset^{-1}(p_j)$ (39)Similar to Eq. (10) we. (40) for standard normal distribution:

$\int_{-\frac{\bar{g}_j}{\sigma_{gj}}}^{\infty} \frac{1}{\sqrt{2\pi}} \quad e^{-\theta^2/2} \; d\theta \ge \int_{A_j}^{\infty} \frac{1}{\sqrt{2\pi}} \quad e^{-t^2/2} \; dt (40)$

$A_j$is defined in Eq. (41)for normal distribution.

$A_j = -\emptyset_j(p_j) (41)$

In the next section, stochastic solutions to two considered test problems are numerically studied. The deterministic solution to a problem was studied by Rao(2009). Investigating the effects of various probability functions includes exponential, gamma, and normal distributions of random variables on the accuracy, and convergence of solution is the main subject of the following section.

3. Analysis

The objective function as given in Eq. (8) depends on two parametersk_1 and k_2. In addition, it is shown in Eq. (7) that mean value and standard deviation of every variable are important factors in the new objective function. Similar to the objective function, one can notice that in the constraint equations, standard deviation and probability of satisfying constraints are also important. To investigate the effects and importance of these parameters, separate codesare developed using MATLAB programming language. To solve an equivalent constrained optimization problem, interior penalty method is applied. The probability functions used to define variables are of three types; exponential, gamma with$\alpha = 1, \alpha = 2, \alpha = 3$, and normal distributions with various mean values and probabilities (90%, 93%, and 96%).Two test problems are considered, one of which is a general mathematical problem and the other one is a structural mechanical problem. The impact of initial point values, standard deviations, probability function type is mainly pointed out in this paper. These two problems are presented as follows.

Test problem 1; Rao, (2009):

It is a mathematical optimization problem whose objective function and constraints are nonlinear.

Objective function:$f(x) = 5y_1{x_1}^2 + \left(\frac{y_2y_3}{3}\right){x_2}^2 - \left(\frac{y_4}{10}\right)x_1 - y_1y_4x_2$

Subject to:

$g(1) = x_1 + y_1y_4x_2 - \left(\frac{y_3y_4}{40}\right) \le 0$

$$g(2) = \frac{y_4}{10}x_1 + y_2y_3x_2 - y_2y_4 \leq 0$$
$$g(3) = -x_1 \leq 0$$
$$g(4) = -x_2 \leq 0$$

where $\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$ are deterministic and $\begin{Bmatrix} y_1 \\ \vdots \\ y_4 \end{Bmatrix}$ are probabilistic (random) variables.

Test problem 2; Rao (2009):

It is a general and a rather relatively simple nonlinear optimization problem to minimize the following objective function, but it has been chosen due to its engineering feature. Its scheme is shown in Fig. (4).

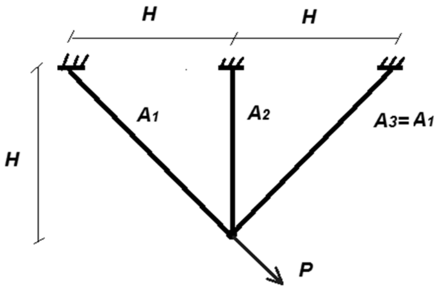


Fig. 4. Scheme of test problem no.2.

Objective function: $f(x) = 2\sqrt{2}x_1 + x_2$
subject to:
$$g(1) = \sigma_1(X) - \sigma^{(u)} \leq 0$$
$$g(2) = \sigma_2(X) - \sigma^{(u)} \leq 0$$
$$g(3) = \sigma_3(X) - \sigma^{(l)} \leq 0$$
$$g(4) = x^{(l)} - x_1 \leq 0$$
$$g(5) = x^{(l)} - x_2 \leq 0$$
$$g(6) = x_1 - x^{(u)} \leq 0$$
$$g(7) = x_2 - x^{(u)} \leq 0$$

where $\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$ are deterministic and $\begin{Bmatrix} x^{(l)} \\ x^{(u)} \end{Bmatrix}$ are probabilistic (random) variables and:

$$\sigma_1(X) = P\frac{\sqrt{2}x_1 + x_2}{\sqrt{2}{x_1}^2 + 2x_1x_2} \sigma_2(X) = P\frac{1}{\sqrt{2}x_1 + x_2} \sigma_3(X) = -P\frac{x_2}{\sqrt{2}{x_1}^2 + 2x_1x_2}$$

It is assumed that:
$P = 20 \ ; \sigma^{(l)} = 15; \sigma^{(u)} = 30$.

In sections 3.2, effects of k_1 and $k_2$coefficients of the objective function as defined in Eq. (20) and mean value are studied. In section 3.3, the solution to each test problem in deterministic and probabilistic forms with 100% probability of satisfying constraints is compared with each other. In section 3.4, the rate of convergence for various types of probability functions is studied. In all these sections, the type of the probability function is the focus in the presented results and derived conclusions.

3.1. Input parameters

In the first step, value of parameter Ain Eq. (23), which is a required quantity in the stochastic formulation, is derived. Its values are given in Table 1. Two initial values of the test problems are chosen based on trial-and-error method. Choosing a feasible initial point is a draw back to the interior penalty method. However, because of its stability and good precision, when compared with other optimization methods such as exterior or Rozen, it has been chosen here. This restriction limited us in assigning a wider range of probability values, so only probabilities of 90, 93, and 96% for constraints are considered. For these two considered test problems, because objective functions have no random variables, changing initial values and other input parameters do not affect the solution and only the type of the probability function affects the accuracy of the optimal point.

3.2. Effects of k_1 and k_2

To study the effects of k_1 and k_2 values (coefficients of objective function in Eq. 8) on the results, four cases as given in Table 2 are considered. The effects of probability function and probability values for test problem 1 are graphically shown in Fig. (5). Comparing the results shows that these parameters have significant effect on the optimum point value. General trend of all curves is the same as for all probabilities' functions.

Table 2
k_1 and k_2 values

Case Number	K_1	K_2
1	1	1
2	0.5	0.5
3	1	0
4	0	1

3.3. Comparing Probabilistic and Deterministic Solutions

In the deterministic solution, there are neither random variables nor standard deviations representing design variables. To compare the accuracy of the optimum points in deterministic and stochastic solutions, different mean values equal to 4 and 20 for 90%, 93%, and 96% probabilities are carried out. It is observed from the analyses that the

Table 1
Model parameters for exponential, gamma, and normal probability functions for test problem 1 with mean values equal to 4 and 20.

Function	**Probability**	**Function parameters**					
		$\mu = [4\ 4\ 4\ 4]$			$\mu = [20\ 20\ 20\ 20]$		
Exponential	90%	μ=4	$\lambda = 0.25$	A=0.42	μ=20	$\lambda = 0.05$	A=2.1072
Gamma		α=1	$\theta = 20$	A=0.42	α=1	$\theta = 20$	A=2.1072
		α=2	$\theta = 10$	A=1.04	α=2	$\theta = 10$	A=5.29
		α=3	$\theta = 6.66$	A=1.82	α=3	$\theta = 6.66$	A=7.35
Normal		$\sigma_y = 0.6$ *Mean value(y)		A=1.28	$\sigma_y = 0.6$ *Mean value(y)		A=1.28
Exponential	93%	μ=4	$\lambda = 0.25$	A=0.29	μ=20	$\lambda = 0.05$	A=1.45
Gamma		α=1	$\theta = 20$	A=0.29	α=1	$\theta = 20$	A=1.45
		α=2	$\theta = 10$	A=0.83	α=2	$\theta = 10$	A=4.28
		α=3	$\theta = 6.66$	A=1.58	α=3	$\theta = 6.66$	A=6.28
Normal		$\sigma_y = 0.6$ *Mean value(y)		A=1.48	$\sigma_y = 0.6$ *Mean value(y)		A=1.48
Exponential	96%	μ=4	$\lambda = 0.25$	A=0.163	μ=20	$\lambda = 0.05$	A=0.8164
Gamma		α=1	$\theta = 20$	A=0.163	α=1	$\theta = 20$	A=0.8164
		α=2	$\theta = 10$	A=0.59	α=2	$\theta = 10$	A=3.10
		α=3	$\theta = 6.66$	A=1.26	α=3	$\theta = 6.66$	A=4.67
Normal		$\sigma_y = 0.6$ *Mean value(y)		A=1.75	$\sigma_y = 0.6$ *Mean value(y)		A=1.75

optimum solution of the stochastic form with 100% probability of satisfying the constraints will not forms, two analyses for test problem 1 with converge to the similar results as deterministic solution converged. It is due to the simplifications applied to the derivation of the probabilistic form of the objective function and constraints (see section 2.1). Test problem 1 in the deterministic form for a mean value equal to 4 is defined as follows:

objective function:

$$f(x) = 100{x_1}^2 + 133{x_2}^2 - 2x_1 - 400x_2$$

$$g(1) = x_1 + 400x_2 - 10 \leq 0$$

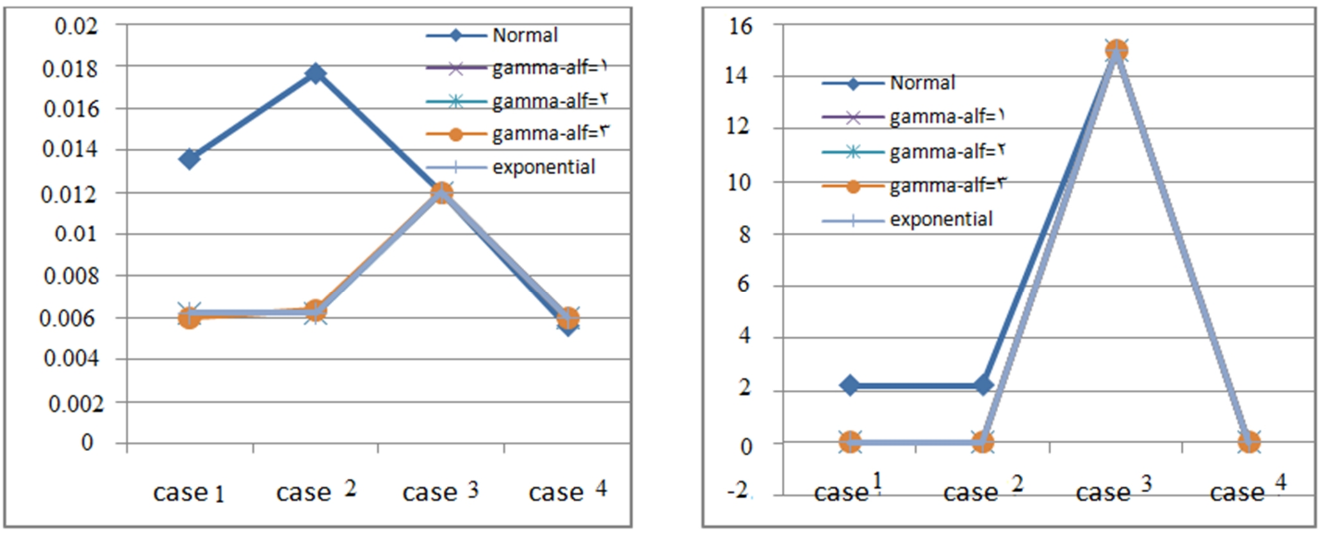
subject to:

$$g(2) = 2x_1 + 400x_2 - 400 \leq 0$$

$$g(3) = -x_1 \leq 0$$

$$g(4) = -x_2 \leq 0$$

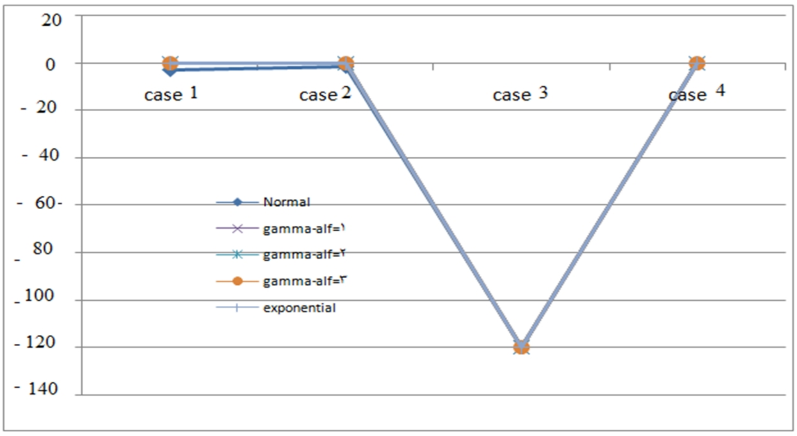
Its optimum point starting from feasible point $x_1 = \begin{Bmatrix} 4 \\ 4 \end{Bmatrix}$ is:$x^* = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0.0666 \\ 1.5 \end{Bmatrix}$ and with a starting feasible point $x_1 = \begin{Bmatrix} 20 \\ 20 \end{Bmatrix}$ is: $x^* = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0.1120 \\ 1.5009 \end{Bmatrix}$.



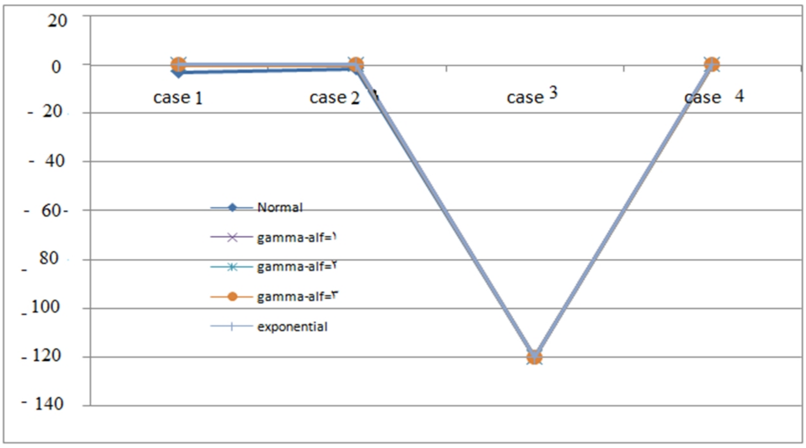
(a) **(b)**

Fig.5. Comparing the function values in the optimum point for exponential, gamma, and normal probability functions with (a):x_1, (b):x_2 and mean value equal to 4.

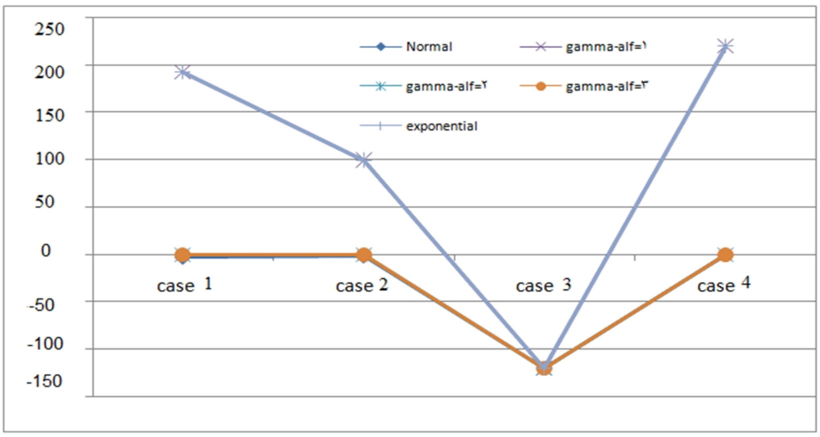
The difference between determinant and probabilistic solutions is presented as an error (it is not an error and is only a difference between the deterministic and stochastic optimum points in the normalized percentage form). Results of this comparison for test problem 1 with k_1 and $k_2$equal to one and mean values equal to 4 and 20 with normal, gamma, and exponential probability functions are summarized in Tables 3 and 4. By comparing the results, one can observe that the error for the gamma distribution with $\alpha = 1$ is much more than those for other distribution



(a)



(b)



(c)

Fig.6. Comparing Function values in the optimum point for exponential, gamma and normal probability functions with (a):90%, (b):93% and (c):96% probability and mean value equal to 4.

functions, and the results are not very sensitive to the probability values of normal and gamma distributions with $\alpha = 2$.

3.4. Comparing rate of convergence

Rate of convergence is measured by the speed of analysis. For a problem with numerous variables, the speed of analysis could be a decisive factor. Although choosing probability function depends on the nature of a problem, comparing the methods from the speed point of view is informative. For this reason, all analyses are carried out in a similar situation by a computer with a 2-core i-3 CPU.

Table3
Difference between deterministic and probabilistic solutions for test problem one with various distribution functions and probabilities for $\mu(X) = 4$.

Function		90% PRO.				93% PRO.				96% PRO.			
		x_1	x_2	Error-x_1%	Error-x_2%	x_1	x_2	Error-x_1%	Error-x_2%	x_1	x_2	Error-x_1%	Error-x_2%
Normal		0.0136	2.1749	40.2	8813.5	0.0138	2.1697	42.3	22268.0	0.0053	-0.0006	-45.4	-106.2
Gamma	**α=1**	0.0062	-0.0005	-36.1	-102.0	0.0062	-0.0005	-36.1	-105.2	0.2668	2.3869	2650.5	24507.2
	α=2	0.0062	-0.0004	-36.1	-101.6	0.0062	-0.0004	-36.1	-104.1	0.0057	-0.0004	-41.2	-104.1
	α=3	0.006	-0.0003	-38.1	-101.2	0.006	-0.0003	-38.1	-103.1	0.006	-0.0003	-38.1	-103.1
Exponential		0.0062	-0.0005	-36.1	-102.0	0.0062	-0.0005	-36.1	-105.2	0.2668	2.3869	2650.5	24507.2

Table 4
Difference between deterministic and probabilistic solutions for test problem one with various distribution functions and probabilities for $\mu(X) = 20$.

Function		90% PRO.				93% PRO.				96% PRO.			
		x_1	x_2	Error-x_1%	Error-x_2%	x_1	x_2	Error-x_1%	Error-x_2%	x_1	x_2	Error-x_1%	Error-x_2%
Normal		0.0104	2.1717	4.0	8621.7	0.0104	2.1717	4.0	21617.0	0.0104	2.1717	4.0	21617.0
Gamma	**α=1**	0.0084	-0.0005	-16.0	-102.0	0.0084	-0.0005	-16.0	-105.0	0.0084	-0.0005	-16.0	-105.0
	α=2	0.008	-0.0005	-20.0	-102.0	0.0138	-0.0005	38.0	-105.0	0.0146	-0.0005	46.0	-105.0
	α=3	0.0084	-0.0005	-16.0	-102.0	0.0081	-0.0005	-19.0	-105.0	-0.0128	-0.0004	-228.0	-104.0
Exponential		0.0084	-0.0005	-16.0	-102.0	0.0084	-0.0005	-16.0	-105.0	0.0084	-0.0005	-16.0	-105.0

Table 5
Comparing Speed of analyses for Normal, gamma, and exponential probability functions with mean values equal to 4 (test problem 1).

Function		x_1	x_2	Function value	Elapsed time (Min:Sec)
Normal		0.0136	2.1749	-2.6717	352
Gamma	**α=1**	0.0062	-0.0005	0.0439	592
	α=2	0.0062	-0.0004	0.0194	522
	α=3	0.006	-0.0003	0.0103	496
Exponential		0.0062	-0.0005	0.0439	582

Number of iterations for the convergence of interior penalty function algorithm in all cases is3. Results are summarized in Table 5. It is shown that normal distribution has a faster speed than other methods. Sorting other distribution functions is in the form of: gamma with $\alpha = 3$, exponential, gamma with α=2, and gamma with $\alpha = 1$. Although exponential and gamma with $\alpha = 1$ have similar optimum solutions, their rates of convergence are not the same.

4. Conclusion

The main concern of this paper is studying the effects of application of various probability functions used for a random variable distribution on the stochastic programming solution. To do so, normal, gamma, and exponential distributions are considered. In addition, the effects of inputs (initial point value, mean value, etc.) on the stochastic programming formulation are investigated. Toanalyze, two test problems are included and general conclusions are derived from results of the analyses. Main conclusions are:

1- Required parameters for stochastic programming for normal, gamma, and exponential probabilistic distribution functions can be regarded appropriately according to the derived formulations.

2- Generally, gamma distribution for$\alpha = 1$yields similar results to those of exponential function except for rate of convergence.

3- Importance coefficients (k_1 and k_2) in the probability formulation of an objective function have significant effect on the accuracy of the solution, especially when they are unequal.

4- Generally, the difference between probabilistic and deterministic solutions with an increase of probability increases for all distribution functions.

5-With the same expectancy (E(X)) and variance (Var(X)), optimum solution of the normal distribution function differs considerably from the other distributions (i.e., gamma and exponential distribution functions), while for the other ones, optimum solutions are very close to each other.

6-Generally, the rate of solution convergence for normal distribution is much faster than the other distributions and gamma for $\alpha = 3$, gamma for $\alpha = 2$, exponential and gamma for $\alpha = 1$are in thelower stairs, consequently.

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