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Capability Indices for Rayleigh Process

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Abstract. Monitoring, control and improvement of quality are important for companies. Process capability indices (PCIs) are tools widely used by the industries to determine the quality of their products and the performance of their manufacturing processes. Classic versions of these indices were constructed for processes whose quality characteristics have a normal distribution. But, many of these characteristics do not follow this distribution. In such a case, the classic PCIs must be modified to take into account the non-normality, because the effect of this non-normality can lead to misinterpretation of the process capability and ill-advised business decisions. A non-normal model is the Rayleigh distribution which is very useful. This paper proposes a Clements's method to estimate the PCIs for Rayleigh processes. Finally, an example to evaluate its performance is presented.

Keywords: Process Capability Indices, Manufacturing Processes, Rayleigh Distribution, Clements's Method.

1. Introduction

When the quality of products is being monitored, it is important to have information about whether their manufacturing processes are being capable of maintaining specification or tolerance limits. In such a sense, the concept of process capability provides a quantitative tool to establish how suitable a manufacturing process is. This concept corresponds to the ability that such a process has to generate a product that meets the specification limits established by the company so that it can be considered of good quality (Johnson, *et al.*, 1994 and Kane, 1986).

In statistical terms, a way to measure process capability is process capability indices (PCIs). These indices are defined as the ratio between the allowable variation (based on specification limits) and the natural variation of the production process (based on the data variability) due to non-assignable causes. PCIs were developed for processes whose quality characteristic to be monitored is normally distributed (Aslam, *et al.*, 2013). Let random variable X be quality characteristic which has $N(\mu,\sigma^2)$. In this case, the PCIs are defined as

$$C_p = \frac{USL - LSL}{6\sigma}, \qquad C_{pl} = \frac{\mu - LSL}{3\sigma},$$
$$C_{pu} = \frac{USL - \mu}{3\sigma}, \qquad C_{pk} = min\{C_{pl}, C_{pu}\},$$
$$C_{pm} = \frac{USL - LSL}{6\sqrt{E\left[(X-T)^2\right]}} = \frac{USL - LSL}{6\sqrt{(\sigma^2 + (\mu - T)^2)}},$$
$$C_{pmk} = \frac{USL - LSL - 2\,|\mu - M|}{6\sqrt{E\left[(X-T)^2\right]}} = \frac{USL - LSL - 2\,|\mu - M|}{6\sqrt{(\sigma^2 + (\mu - T)^2)}} \tag{1}$$

Which, USL and LSL are upper specification limit and a lower specification limit, respectively. T is the target value and M is $\frac{USL+LSL}{2}$. In the Table 1, presented the usage of PCIs .

In many cases, these quality characteristics follow non-normal distributions and then PCIs for the normal distribution should not be employed in such cases, because the obtained results using them could be inaccurate, misleading and unreliable (Kane, 1986).

Rayleigh distribution has wide applications, such as, in the field of acoustics (Rayleigh, 1880 and 1919), in communication engineering (Dyer and Whisenand, 1973a and 1973b), in life testing of electro vacuum devices (Polovko, 1968), so in this paper we consider the random variable has Rayleigh distribution and use a Clements's method to develop the PCIs .

Note that, for Rayleigh distribution, the lifetime performance index C_L been investigated [8]. But, since the PCIs very much used to evaluate the quality of products, we in this paper develop the PCIs for this distribution.

The rest of this paper is organized as follows. In Section 2, introduce the Clements's method. In section 3, introduce the Rayleigh distribution and in section 4, obtain the PCIs for this distribution. Finally in section 5, we illustrate the example for implement method by using the real data set.

2. Literature Review

Clements (1989) introduced a method for Pearson system, using their quintiles. Gilchrist (1993) proposed a quintile transformation similar to the Clements method, but based on a standardized distribution, instead of Pearson distributions. Johnson *et al.* (1994) applied the Clements method and obtained estimators for PCIs. Pearn and Chen (1996) derived a new method for PCIs that can be viewed as a modification of the Clements method, but with better results than those obtained applying this method. The idea of all these authors was to reproduce the property of the normal distribution to result in, at most, 0.27% of non-conforming products.

In Clements's method, the PCIs are defined as

$$\acute{C}_{p} = \frac{\text{USL} - \text{LSL}}{x(0.99865) - x(0.00135)}, \qquad \acute{C}_{pl} = \frac{2[x(0.5) - \text{LSL}]}{x(0.99865) - x(0.00135)},$$

$$\acute{C}_{pu} = \frac{2[\text{USL} - x(0.5)]}{x(0.99865) - x(0.00135)}, \qquad \acute{C}_{pk} = \min\{\acute{C}_{pl}, \acute{C}_{pu}\},$$

$$\acute{C}_{pm} = \frac{\text{USL} - LSL}{6\sqrt{\left[\frac{x(0.99865) - x(0.00135)}{6}\right]^2 + [x(0.5) - T]^2}}$$

$$\acute{C}_{pmk} = \frac{\text{USL} - LSL - 2\left|x(0.5) - M\right|}{6\sqrt{\left[\frac{x(0.99865) - x(0.00135)}{6}\right]^2 + [x(0.5) - T]^2}} \tag{2}$$

Table 1. Process Capability Indices

Index	Usage
C_p	It estimates what the process is capable of producing if the process mean were to be centered between the specification limits. Assumes process output is approximately normally distribute.
C_{pl}	It estimates process capability for specifications that consist of a lower limit only.Assumes process output is approximately normally distributed.
C_{pu}	It estimates process capability for specifications that consist of an upper limit only. Assumes process output is approximately normally distributed.
C_{pk}	It estimates what the process is capable of producing, considering that the process mean may not be centered between thespecification limits.
C_{pm}	It estimates process capability around a target T is always greater than zero. It assumes process output is approximately normally distributed. Cpm is also known as the Taguchi capability index.
C_{pmk}	It estimates process capability around a target T and accounts for an off-center process mean. Assumes process output is approximately normally distributed.

Which $x(q)$, $0<q<1$, is the quantile function of X, so $x(0.5)$ is the median. Note that 6σ replace by $R = x(0.99865) - x(0.00135)$. Because, R covers a 99.73% of the distribution of the monitored process data. The idea behind these substitutions is to mimic the normal distribution property, allowing the non-normal PCIs to cover the same percentage as the range 6σ under normality. Moreover in the method introduced by Clements, to compute the PCIs, replace the mean by the median, because the median is a robust measure of the central tendency of the process, particularly for skewed heavy-tailed distributions.

3. Methodology

A random variable X with Rayleigh distribution has one parameter. Which is shape parameter $(\lambda > 0)$. This is denoted by X ~ Rayleigh (λ).

If U~ $u(0,1)$ (Uniform distribution), then Rayleigh and Uniform RVs related by the transformation X = $\lambda\sqrt{-2\ln(U)}$. Thus, it is useful to generate data of Rayleigh distribution. In addition X^2 have the Exponential distribution with parameter $\frac{1}{2\lambda^2}$.

Let X ~ Rayleigh (λ). Then the probability density (PDF) function and the cumulative distribution function (CDF) of X are, respectively,

$$f(x;\lambda) = \frac{x}{\lambda^2}\, exp\left(-\frac{x^2}{2\lambda^2}\right), \qquad x > 0, \lambda > 0.$$
$$F(x;\lambda) = 1 - exp\left(-\frac{x^2}{2\lambda^2}\right), \qquad x > 0, \lambda > 0. \tag{3}$$

The $q \times 100$ th quantile or quantile function of x is

$$x\,(q\,,\lambda) = F^{-1}(q) = \lambda\sqrt{-2\,ln(1-q)}\,, \qquad 0 < q < 1. \tag{4}$$

Where $F^{-1}(.)$ is the inverse function of F(.). Hence, from Equation (4), $x(0.5) = \lambda\ \sqrt{2\ln 2}$, and so it is the median of the Rayleigh distribution, as mentioned. The mean and variance of X are, respectively, $E[X] = \lambda\sqrt{\pi/2}$ and Var[X] = $\lambda^2\ (2 - \frac{\pi}{2})$.

Let $X_1, \dots X_n$ be a random sample of size n from X ~ Rayleigh (λ), with observation $x_1, x_2, \dots x_n$. Then, the log-likelihood function for λ is

$$\ell(\lambda) = \sum_{i=1}^{n} \ln x_i - 2n ln\,\lambda - \frac{\sum_{i=1}^{n} {x_i}^2}{2\lambda^2} \tag{5}$$

Taking derivatives of Equation (5) with respect to the parameter λ and set them to zero, we obtain the maximum likelihood (ML) estimates of λ, $\hat{\lambda}$ say. Here,

$$\hat{\lambda} = \sqrt{\frac{\sum_{i=1}^{n} x_i^2}{2n}}$$

4. Findings

In this section, we develop PCIs when the quality characteristic to be monitored follows a Rayleigh distribution, which are denoted by C_p^R, C_{pl}^R , C_{pu}^R , C_{pk}^R , C_{pm}^R and C_{pmk}^R .

In this subsection, we propose a PCI for the Rayleigh distribution comparing the specification limits with a range that covers a high percentage of the distribution, $1-[q_1+\{1-q_2\}]$ say, which can be specified or obtained in an optimal way.

Consider the QF of the Rayleigh distribution provided in Equation (4) and the PCIs defined in Equation (2). Then, given LSL and USL, we propose a Rayleigh PCI by means of

$$\begin{aligned} C_{\mathrm{p}}^{\mathrm{R}} &= \frac{\mathrm{USL}-\mathrm{LSL}}{x(q_2)-x(q_1)} \\ &= \frac{\mathrm{USL}-\mathrm{LSL}}{\lambda\sqrt{-2\ln(1-q_2)}-\lambda\sqrt{-2\ln(1-q_1)}} \\ &= \frac{\mathrm{USL}-\mathrm{LSL}}{\lambda\left[\sqrt{-2\ln(1-q_2)}-\sqrt{-2\ln(1-q_1)}\right]} \end{aligned} \tag{6}$$

Now, to monitor a process only with a LSL, we use

$$\begin{aligned} C_{pl}^{R} &= \frac{2[x(0.5)-\mathrm{LSL}]}{x(q_2)-x(q_1)} \\ &= \frac{2\left[\lambda\sqrt{2\ln 2}-LSL\right]}{\lambda\left[\sqrt{-2\ln\left(1-q_2\right)}-\sqrt{-2\ln\left(1-q_1\right)}\right]} \\ &= \frac{2\left[\sqrt{2\ln 2}-LSL/\lambda\right]}{\left[\sqrt{-2\ln\left(1-q_2\right)}-\sqrt{-2\ln\left(1-q_1\right)}\right]} \end{aligned} \tag{7}$$

In addition, we have

$$C_{pu}^{R} = \frac{2[\mathrm{USL}-x(0.5)]}{x(q_2)-x(q_1)} = \frac{2\left[\frac{USL}{\lambda}-\sqrt{2\ln 2}\right]}{\left[\sqrt{-2\ln(1-q_2)}-\sqrt{-2\ln(1-q_1)}\right]} \tag{8}$$

$$\begin{aligned} C_{pm}^{R} &= \frac{\mathrm{USL}-LSL}{6\sqrt{\left[\frac{x(q_2)-x(q_1)}{6}\right]^2+[x(0.5)-T]^2}} \\ &= \frac{\mathrm{USL}-LSL}{6\sqrt{\lambda^2\left[\frac{\sqrt{-2\ln(1-q_2)}-\sqrt{-2\ln(1-q_1)}}{6}\right]^2+\left[\lambda\sqrt{2\ln(2)}-T\right]^2}} \end{aligned} \tag{9}$$

When the median moves away from the midpoint of the specification limits, we consider $C_{\mathrm{pk}}^{\mathrm{R}}=\min\{C_{pl}^{R}, C_{pu}^{R}\}$, where C_{pl}^{R} *and* C_{pu}^{R} are given in Equations (7) and (8), respectively. Similarly, the C_{pm}^{R} and C_{pmk}^{R} are as ollows

$$C_{pmk}^R = \frac{\text{USL} - LSL - 2|x(0.5) - M|}{6\sqrt{\left[\frac{x(q_2) - x(q_1)}{6}\right]^2 + [x(0.5) - T]^2}}$$
$$= \frac{\text{USL} - LSL - 2\left|\lambda\sqrt{2\ln(2)} - M\right|}{6\sqrt{\lambda^2\left[\frac{\sqrt{-2\ln(1-q_2)} - \sqrt{-2\ln(1-q_1)}}{6}\right]^2 + \left[\lambda\sqrt{2\ln(2)} - T\right]^2}} \tag{10}$$

Using the results on estimation provided in Section 3 and the invariance property of the ML estimators, we have that the ML estimator of the Rayleigh PCI given in section 4.1, can be obtained as

$$\hat{C}_p^R = \frac{\text{USL} - \text{LSL}}{\hat{\lambda}\left[\sqrt{-2\ln(1-q_2)} - \sqrt{-2\ln(1-q_1)}\right]}$$
$$= \frac{\text{USL} - \text{LSL}}{\left(\sqrt{\frac{\sum_{i=1}^n x_i^2}{2n}}\right)\left[\sqrt{-2\ln(1-q_2)} - \sqrt{-2\ln(1-q_1)}\right]} \tag{11}$$

From section 3, $\sum_{i=1}^n {X_i}^2 \sim Gamma\,(n, \frac{1}{2\lambda^2})$. Therefore,

$$\frac{\sum_{i=1}^n {X_i}^2}{4\lambda^2} \sim \chi_{2n}^2$$

And then we have

$$P\left(\chi_{2n,\alpha}^2 \le \frac{\sum_{i=1}^n {X_i}^2}{4\lambda^2} \le \chi_{2n,1-\alpha}^2\right) = P\left(\frac{4\,\chi_{2n,\alpha}^2}{\sum_{i=1}^n {X_i}^2} \le \frac{1}{\lambda^2} \le \frac{4\,\chi_{2n,1-\alpha}^2}{\sum_{i=1}^n {X_i}^2}\right) = 1-\alpha \tag{12}$$

From Equation (12), a $100 \times (1-\alpha)$ % CI for the $1/\lambda$ is

$$\left(2\sqrt{\frac{\chi_{2n,\alpha}^2}{\sum_{i=1}^n {X_i}^2}}, 2\sqrt{\frac{\chi_{2n,1-\alpha}^2}{\sum_{i=1}^n {X_i}^2}}\right)$$

Then, a 100 × (1-α) % CI for the C_p^R is

$$\left(\frac{2[USL - LSL]}{\left[\sqrt{-2\ln(1-q_2)} - \sqrt{-2\ln(1-q_1)}\right]}\sqrt{\frac{\chi_{2n,\alpha}^2}{\sum_{i=1}^n {X_i}^2}},\right.$$
$$\left.\frac{2[USL - LSL]}{\left[\sqrt{-2\ln(1-q_2)} - \sqrt{-2\ln(1-q_1)}\right]}\sqrt{\frac{\chi_{2n,1-\alpha}^2}{\sum_{i=1}^n {X_i}^2}}\right)$$

For estimate the C_{pk}^R, we have $C_{pk}^R = C_p^R(1-k)$ *which*

$$k = 2\,\frac{\left|x(0.5) - \frac{USL + LSL}{2}\right|}{USL - LSL}$$

(see Appendix, Part 1). So we have

$$\hat{C}_{pk}^{R} = \hat{C}_{p}^{R}\left(1 - 2\,\frac{\left|\hat{\lambda}\sqrt{2\ln 2} - \frac{\mathrm{USL}+\mathrm{LSL}}{2}\right|}{USL - LSL}\right)$$
$$= \hat{C}_{p}^{R}\left(1 - 2\,\frac{\left|\left(\sqrt{\frac{\sum_{i=1}^{n} x_i^2}{2n}}\right)\sqrt{2\ln 2} - \frac{\mathrm{USL}+\mathrm{LSL}}{2}\right|}{USL - LSL}\right)$$

The $\hat{C}_{p}^{R}$ is the Equation (11). The asymptotic distribution of $\hat{C}_{pk}^{R}$ is normal and given by

$$\hat{C}_{pk}^{R} \dot{\sim} N(C_{pk}^{R}, J(C_{pk}^{R})) \tag{13}$$

where J(C_{pk}^{R}) denotes its asymptotic variance provided in the Appendix (Part 2). From Equation (13), an approximate 100 × (1-α) % CI for the $\mathrm{C}_{\mathrm{pk}}^{\mathrm{R}}$ is

$$(\hat{C}_{pk}^{R} \pm z_{(1-\frac{\alpha}{2})}\,\widehat{SE}(\hat{C}_{pk}^{R})) \tag{14}$$

where SE($\hat{C}_{pk}^{R}$) is the standard error of $\hat{C}_{pk}^{R}$ and its estimate $\widehat{SE}(\hat{C}_{pk}^{R})$ can be calculated from $\mathrm{J}(C_{pk}^{R})^{1/2}$ and evaluated at $\hat{\lambda}$. Hypothesis $H_0\colon C_{pk}^{R} \le c_0$ (indicating that the process is not capable) against $H_1 : C_{pk}^{R} > c_0$ (indicating that the process is not capable) can be contrasted using the test statistic under H_0 given by $W = (\frac{\hat{C}_{pk}^{R} - c_0}{\widehat{SE}(\hat{C}_{pk}^{R})}) \dot{\sim} N(0,1)$, where c_0 is an established capability requirement.

Similarly, the ML estimator for C_{pm}^{R} and C_{pmk}^{R} denoted by $\hat{C}_{pm}^{R}$ and $\hat{C}_{pmk}^{R}$ respectively, which are as follows

$$\hat{C}_{pm}^{R} = \frac{\mathrm{USL} - LSL}{6\sqrt{(\widehat{\lambda})^2\left[\frac{\sqrt{-2\ln(1-q_2)} - \sqrt{-2\ln(1-q_1)}}{6}\right]^2 + \left[\hat{\lambda}\sqrt{2\ln(2)} - T\right]^2}}$$
$$= \frac{\mathrm{USL} - LSL}{6\sqrt{\left(\frac{\sum_{i=1}^{n} {x_i}^2}{2n}\right)\left[\frac{\sqrt{-2\ln(1-q_2)} - \sqrt{-2\ln(1-q_1)}}{6}\right]^2 + \left[\left(\sqrt{\frac{\sum_{i=1}^{n} x_i^2}{2n}}\right)\sqrt{2\ln(2)} - T\right]^2}}$$

The asymptotic distribution of $\hat{C}_{pm}^{R}$ is normal and given by

$$\hat{C}_{pm}^{R} \dot{\sim} N(C_{pm}^{R}, J(C_{pm}^{R})) \tag{15}$$

where J(C_{pm}^{R}) denotes its asymptotic variance that it will be obtained similar to J(C_{pk}^{R}). From Equation (15), an approximate 100 × (1-α) % CI for the $\mathrm{C}_{\mathrm{pm}}^{\mathrm{R}}$ is

$$(\hat{C}_{pm}^{R} \pm z_{\left(1-\frac{\alpha}{2}\right)} \widehat{SE}\left(\hat{C}_{pm}^{R}\right)) \tag{16}$$

where SE($\hat{C}_{pm}^{R}$) is the standard error of $\hat{C}_{pm}^{R}$ and its estimate $\widehat{SE}(\hat{C}_{pm}^{R})$ can be calculated from $J(C_{pm}^{R})^{1/2}$ and evaluated at $\hat{\lambda}$. Hypothesis test for C_{pm}^{R} is same to C_{pk}^{R} The difference is that replace the C_{pk}^{R} by C_{pm}^{R} . Moreover the test statistic under H_0 given by $W^{*} = \left(\frac{\hat{C}_{pm}^{R} - c_0}{\widehat{SE}(\hat{C}_{pm}^{R})}\right) \dot{\sim}$ N(0,1). For estimate C_{pkm}^{R} , method is like the above process. For example ML estimator for C_{pmk}^{R} as follows

$$\hat{C}_{pmk}^{R} = \frac{\text{USL} - LSL - 2\left|\hat{\lambda}\sqrt{2\ln(2)} - M\right|}{6\sqrt{(\hat{\lambda})^{2}[\frac{\sqrt{-2\ln(1-q_2)} - \sqrt{-2\ln(1-q_1)}}{6}]^{2} + [\hat{\lambda}\sqrt{2\ln(2)} - T]^{2})}}$$

Table 2 shows values recommended for the PCI by means of which we can decide whether the corresponding manufacturing processes are being capable or not (Montgomery, 2004).

Table 2. Minimum Values Recommended for PCIs

Type of process	Sigma level	Two-sided specifications	One-sided specifications
Existing	4.0	1.33	1.25
New	4.5	1.50	1.45
A	4.5	1.50	1.45
B	5.0	1.67	1.60
Six-sigma	6.0	2.00	2.00

A: Existing indluding safety strength or critical parameters.

B: New indluding safety, strength or critical parameters.

In order to implement the proposed method, we consider the data set in Table 3 (Leiva, *et al.*, 2014). With respect to Tables 4, we apply the Kolmogorov–Smirnov (KS) goodness-of-fit test and ML estimate for this data. More over in Fig 1 and Fig 2, illustrated the Q-Q plot and histogram for data set, respectively.

This results indicates that the data set follow a Rayleigh distribution because the KS test has a p-value = 0.41 .

Table 3 . Data Set

2.891	4.035	4.495	2.890	2.312	3.158	5.228	3.334	5.896	5.639
3.842	1.590	1.954	1.842	0.680	2.752	1.301	2.260	0.889	2.381
0.619	2.788	1.050	3.750	3.508	6.123	6.549	5.954	2.207	4.417
4.805	1.516	2.227	2.797	1.636	1.066	0.940	4.101	4.542	1.295
1.770	3.492	5.706	3.722	6.644	2.472	1.383	4.494	1.694	2.892
2.111	3.591	2.093	3.222	2.891	2.582	0.665	3.234	1.102	1.083
1.508	1.811	2.803	6.659	0.923	6.229	3.177	2.333	1.311	4.419
2.495	0.921	4.061	9.725	1.600	4.281	3.360	1.131	1.618	4.489
3.696	1.982	2.413	5.480	1.992	2.573	1.845	4.620	6.221	1.694
4.882	1.380	3.982	2.260	2.366	2.899	3.782	2.336	1.175	3.055

Table 4. Goodness-of-Fit Tests and ML Estimate for Indicated Data Set

Kolmogorov-Smirnov Test	Rayleigh
P – value	0.41
$\widehat{\lambda}$	3.37

Table 5, illustrate the descriptive statistics and values for quantiles for indicated data set, that includes: quantiles ($\mathrm{Q_q}\times 100, 0 < q < 1$), minimum and maximum ($x_{(1)}$ and $x_{(n)}$, respectively), mean (x), SD, usual ($R_g = x_{(n)} - x_{(1)}$) and interquartile ($IQR = \mathrm{Q}_{99.86} - \mathrm{Q}_{0.14}$) ranges, and coefficients of variation (CV), skewness or asymmetry (CS) and kurtosis (CK).

In order to study the capability of the production process, LSL and USL must be fixed; see these limits in Table 6.

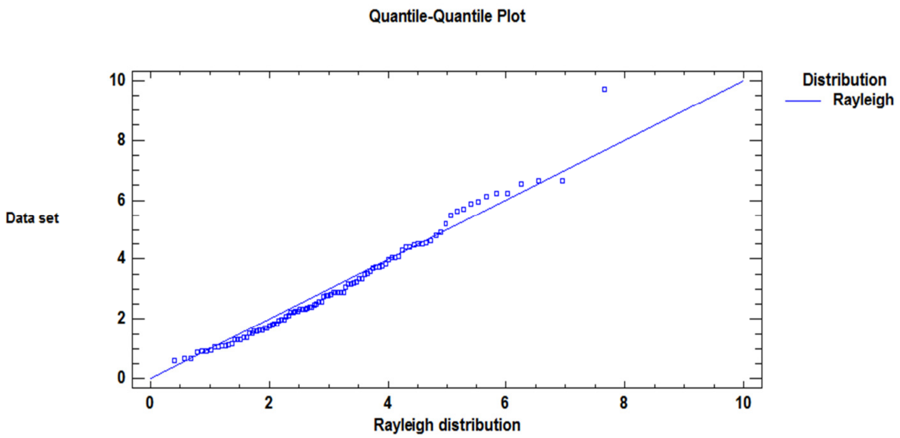

Fig. 1. Quantile-Quantile Plot for Data set

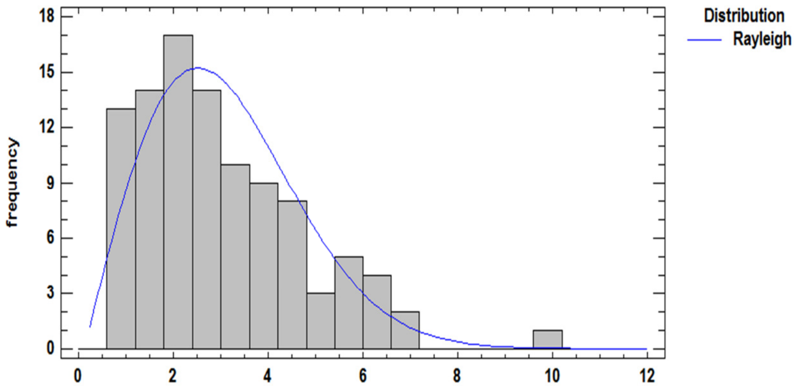

Fig. 2. Histogram for Data Set

Table 5. Descriptive Statistics of the Indicated Data Set

$x_{(1)}$	$Q_{0.14}$	$Q_{0.50}$	$\bar{x}$	$Q_{99.86}$	$x_{(n)}$	SD	CV	CS	CK	IQR	Rg
0.62	0.63	2.77	3.04	9.32	9.73	1.72	0.57	0.98	4.05	8.69	9.11

Table 6. LSL, USL

LSL	$\frac{[USL + LSL]}{2}$	USL	$x_{(0.5)}$	$\bar{x}$	$\hat{C}_p^R$	$\hat{C}_{pk}^R$
0.5	4.25	8	2.77	3.04	0.83	0.77

From Table 6, notice that production process is not centered in relation to the specification limits because $x(0.5) < (\mathrm{USL} + \mathrm{LSL})/2$. For this reason, we must consider $\hat{C}_{pk}^R = min\{\hat{C}_{pl}^R, \hat{C}_{pu}^R\}$. According to Table 2, which presents some recommended minimum values for the PCI, we can conclude the process not capable. For assurance, according to (14), the CI for C_{pk}^R is (0.7633, 0.7766) . As is clear, the upper limit of the CI for C_{pk}^R is smaller than 1.50. So according to the recommended minimum value for existing processes with two-sided specifications, we can conclude the process not capable.

Now, according to the Table 1 and importance of C_{pm}^R and C_{pmk}^R, let the manager use C_{pm}^R or C_{pmk}^R for study the process capability. So, we obtained the $\hat{C}_{pmk}^R$ and $\hat{C}_{pm}^R$ and their CIs for indicated data. The results are shown in Table 7, that we conclude the process not capable.

Table 7. Final Output

T	$\hat{C}^R_{pmk}$	$\hat{C}^R_{pm}$	CI for C^R_{pmk}	CI for C^R_{pm}
4	0.572	0.618	(0.562, 0.581)	(0.561, 0.679)

5. Conclusion

PCIs have become important tools for continuous improvement of the quality. Classically, process capability is evaluated by indices constructed assuming a normal distribution for the quality characteristic to be monitored. However, frequently, such characteristics do not follow this distribution. Ignoring the effect of the non-normality can lead to misinterpretation of process capability and ill-advised business decisions. Several approaches have been analyzed to address the problem of process capability for non-normally distributed data.We proposed a methodology to analyze capability by using indices based on Rayleigh processes, considering the interquartile range instead of dispersion measures of the process. To illustrate its development, we conducted application based on real-world case study using data set. These application showed convenience and potentiality of the new methodology.

References

[1] M. Aslam, C.W. Wu, M. Azam, and C.-H. Jun, *Variable sampling inspection for resubmitted lots based on process capability index* C_{pk} *for normally distributed items* , Appl. Math. Model. 37 (2013), 667–675.

[2] J.A. Clements, *Process capability calculations for non-normal distributions*, Qual. Prog. 22 (1989), 95–100.

[3] D.D. Dyer, C.W. Whisenand, *Best linear unbiased estimator of the parameter of the Rayleigh distribution—Part I: small sample theory for censored order statistics*, IEEE Transactions on Reliability 22 (1973) 27–34.

[4] D.D. Dyer, C.W. Whisenand, *Best linear unbiased estimator of the parameter of the Rayleigh distribution—Part II: optimum theory for selected order statistics*, IEEE Transactions on Reliability 22 (1973) 229–231.

[5] W. Gilchrist, *Modeling capability*, J. Oper. Res. Soc. 44 (1993), 909–923.

[6] N. Johnson, S. Kotz, and W.L. Pearn, *Flexible process capability indices*, Pak. J. Stat. 10 (1994), 23–31.

[7] V.E. Kane, *Process capability indices*, J. Qual. Technol. 18 (1986), 41–52.

[8] W.C. Lee, J.W. Wu, M.L. Hong, L.S. Lin, R.L. Chan, *Assessing the lifetime performance index of Rayleigh products based on the Bayesian estimation under progressive type II right censored samples*, J. of Comput and Appl Math. 235 (2011) 1676–1688 .

[9] V. Leiva, C. Marchant, H. Saulo, M. Aslam, F. Rojas, *Capability indices for Birnbaum Saunders processes applied to electronic and food industries*, J.Appl. Stat. (2014)Vol. 41, No. 9, 1881–1902 .

[10] D. Montgomery, *Introduction to Statistical Quality Control*, Wiley, New York, 2004.

[11] W.L. Pearn and K.S. Chen, *Estimating process capability indices for non-normal Pearson populations*, Qual. Reliab. Eng. Int. 11 (1995), 386–388.

[12] A.M. Polovko, *Fundamentals of Reliability Theory*, Academic Press, New York, 1968.

[13] J.W.S. Rayleigh, *On the resultant of a large number of vibrations of the same pitch and of arbitray phase*, Philosophical Magazine, Series 5 10 (1880) 73–78.

[14] J.W.S. Rayleigh, Philosophical Magazine, Series 6 37 (1919) 321–347.

Appendix

Part 1.

In this appendix we illustrate that $C'_{pk} = C'_p\,(1-k)$;

$$k = 2\,\frac{\left|x(0.5) - \dfrac{USL + LSL}{2}\right|}{USL - LSL}$$

$$C'_{pk} = \min\{C'_{pl}, C'_{pu}\}$$

$$= \min\left\{\frac{2[x(0.5) - LSL]}{x(q2) - x(q1)}, \frac{2[USL - x(0.5)]}{x(q2) - x(q1)}\right\}$$

$$= \frac{USL - LSL}{x(q2) - x(q1)} - \left|\frac{2\,x(0.5) - (USL + LSL)}{x(q2) - x(q1)}\right|$$

$$= \frac{USL - LSL}{x(q2) - x(q1)} - \left|\frac{x(0.5) - \dfrac{USL + LSL}{2}}{\dfrac{[x(q2) - x(q1)]}{2}}\right|$$

Let, $M = \frac{USL+LSL}{2}$, so we have

$$\frac{USL - LSL}{x(q2) - x(q1)} - \left|\frac{x(0.5) - M}{\dfrac{[x(q2) - x(q1)]}{2}}\right| = C'_p - \frac{2|x(0.5) - M|}{x(q2) - x(q1)}$$

$$= C'_p - \frac{(USL - LSL)(2|x(0.5) - M|)}{(USL - LSL)[x(q2) - x(q1)]}$$

$$= C'_p - C'_p\,\frac{2|x(0.5) - M|}{USL - LSL}$$

$$= C'_p\left(1 - \frac{|x(0.5) - M|}{(USL - LSL)/2}\right) = C'_p\,(1 - k)$$

Part 2.

by using the delta method, the asymptotic variance of the estimator of C^R_{pk} can be expressed as

$$J\left(C^R_{pk}\right) = \left[\frac{\partial\, C^R_{pk}}{\partial\,\lambda}\right]^2 J(\lambda)$$

Which, $J(\lambda)$ is the corresponding inverse expected Fisher information function for λ. For Rayleigh distribution, we have $J(\lambda) = \lambda^2/(4n)$, n is number of observations. $\partial\ C_{pk}^{R}/\partial\,\lambda$ is the Derivative of C_{pk}^{R} into λ, which equal to

$$\frac{\partial\ C_{pk}^{R}}{\partial\ \lambda}=\frac{LSL-USL}{\lambda^{2}\left[\sqrt{-2\ln(1-q_{2})}-\sqrt{-2\ln(1-\ q_{1})}\right]}\left(1-\frac{2\ \left|\ \lambda\sqrt{2\ ln2}\ -\ \frac{USL+LSL}{2}\ \right|}{USL-LSL}\right)$$

$$-\frac{2\sqrt{2\ ln2}\ \left(\lambda\sqrt{2\ln 2}\ -\ \frac{USL+LSL}{2}\right)}{\left|\lambda\sqrt{2\ ln2}\ -\ \frac{USL+LSL}{2}\right|\lambda\ \left[\sqrt{-2\ln(1-q_{2})}\ -\ \sqrt{-2\ln(1-\ q_{1})}\right]}$$