Adv. Math. Fin. App., 2024, **9**(4), P.1203-1212

Advances in Mathematical Finance & Applications
www.amfa.iau-arak.ac.ir
Print ISSN: 2538-5569
Online ISSN: 2645-4610
Doi:10.71716/amfa.2025.23091924

Original Research

Investigating Randomness By Walsh-Hadamard Transform in Financial Series

Seyed Jalal Tabatabaei*

Department of Financial Management, Payamenoor University, Tehran. Iran

ARTICLE INFO
Article history:
Received 2023-09-23
Accepted 2024-03-05

Keywords:
Walsh Transform
Randomness
Financial Series

ABSTRACT
The objective of the ongoing research is to introduce the initial, substantial, and practical implementation of the Walsh-Hadamard Transform in the realm of quantitative finance. It is worth noting that this particular tool, which has limited utility in the domain of digital signal processing, has demonstrated its effectiveness in evaluating the statistical significance of any binary sequence. Therefore, employing this approach in financial series would be exceptionally noteworthy. By employing five primary tests to assess the randomness of the series, including those pertaining to the Tehran Stock Exchange, as well as copper and gold, the outcomes reveal the presence of randomness in the transformed series in all aspects. Naturally, this randomness could be examined to identify any underlying trends.

1 Introduction

Randomness can be regarded as a magical phenomenon that occurs behind the scenes. The little black box, which is used in research and by quantitative researchers, is often seen as an incomprehensible tool. What is the rationale behind its speed? What causes it to be so random? Is randomness a byproduct of chaos and order? How reliable is it to incorporate the command for generating random numbers as a section of code in analysis programs? The output, in the form of a random number, is produced by a dedicated function. However, these numbers are not truly random, but rather pseudorandom. A typical pseudo-random number generator is designed for speed but is defined by the underlying algorithm. In most programming languages, the Mersenne Twister, which was developed in 1997, has become the standard. Interestingly, the Mersenne Twister is not flawless. Its use has been discouraged for generating cryptographic random numbers. After exploring the topic extensively to uncover uncharted areas, it is generally believed that the random walk hypothesis is applicable to financial markets, where returns are random .This theory dates back to the early 1800s, when Jules Regnault and Louis Bachelier observed the characteristics of randomness in the returns of stock options. The theory was later formalized by Maurice Kendall and popularized in 1965 by Eugene Fama in his seminal paper Random Walks in Stock Market Prices. Despite the entertainment value of these tests, they do not demonstrate that markets are random in any way. All they demonstrate is that, to the human eye, market returns, in the

* Corresponding author. Tel.: +98*9131519197*
E-mail address: *tabatabaei@pnu.ac.ir*

absence of any additional information, cannot be distinguished from random processes.This conclusion, by itself, does not provide any useful information regarding the random characteristics of markets. Several different tests have been proposed and examined in the field of market randomness, such as the Runs test, Gibbons, Dickinson, and winsor (2012), the , the Discrete Fourier Transform test by Song-Ju(2004), Umeno, and others. Moreover, the random walk hypothesis itself has certain limitations. This study presents the meaningful and practical application of the Walsh–Hadamard transform (WHT) in the field of quantitative finance. It is noteworthy that despite its limited utility in digital signal processing, this tool has proven to be highly effective in evaluating the statistical significance of any binary sequence in terms of randomness. The Walsh-Hadamard transform is a mathematical technique that has been extensively utilized in a variety of domains, such as signal processing, image compression, bioelectrical activity analysis, pattern recognition, and cryptography (Oczeretko et al, 2015, Antoniadis et al 2023, halko et al2012). However, its application in financial time series analysis has not been extensively explored. The primary utilization of WHT in financial return time series is through the randomness approach. Consequently, we present the introduction of the transform into the field of finance. We demonstrate a practical application of the WHT framework in the search for randomness in financial time series. We illustrate this through the example of three indices, namely the Tehran stock exchange, gold, and copper, and compare their respective results. The subsequent sections encompass the theoretical framework, research background, results, and recommendations.

2 Literature Review

2.1Theoritical background

In this section, we will present a succinct overview of the theoretical elements pertaining to the Walsh-Hadamard Transform (WHT). First, we will examine a discrete signal that possesses real-valued properties. X (t_i) where i = 0, 1. . . N 1. Its trimmed version, X (t_i), of the total length of $n = 2^M$ such that $2^M \leq (N-1)$ and $M \in \mathbb{Z}^+$ is considered as an input signal for the Walsh–Hadamard Transform, the latter defined as (1) :

$$WHT_n = \mathbf{x} \bigotimes_{i=1}^{M} \mathbf{H_2} \tag{1}$$

Where the Hadamard matrix of order $n = 2^M$ is obtainable recursively by (2):

$$\mathbf{H_{2^M}} = \begin{pmatrix} H_{2^{M-1}} & H_{2^{M-1}} \\ H_{2^{M-1}} & -H_{2^{M-1}} \end{pmatrix} \quad \text{therefore} \quad \mathbf{H_2} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{2}$$

and $\otimes$ denotes the Kronecker product between two matrices. Given that, WHT_n is the dot product between the signal (1D array; vector) and resultant Kronecker multiplications of H_2 times [6].

2.1.1. Walsh Functions

The Walsh-Hadamard transform uses the orthog- onal square-wave functions, $w_j(x)$, introduced by Walsh (1923, which have only two values 1 in the interval 0 < x < 1 and the value zero elsewhere. The original definition of the Walsh functions is based on the following re- cursive equations (3):

$$w_{2j}(x) = w_j(2x) + (-1)^j w_j(2x-1)$$
$$= w_{j-1}(2x) - (-1)^{j-1}$$
$$w_{j-1}(2x-1)\ for\ j = 1,2,\ldots \quad (3)$$

On the interval 0 < x < 1, and have zero value for all other values of x outside this interval. A comparison of both function classes looks as **Fig. 1**

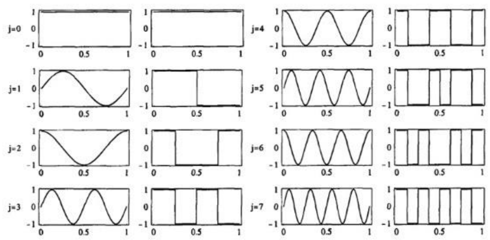


Fig. 1: comparison of both functions with different js

2.1.2. From Hadamard to Walsh Matrix

The Hadamard matrix of order 2^M is obtainable as each row of the matrix corresponds to a Walsh function. However, the ordering is different, known as Hadamard ordering. Therefore, in order to visually understand the shape of Walsh functions, it is necessary to rearrange their indexing. The resulting matrix is commonly referred to as the Walsh matrix.

2.1.3. Signal Transformations

In the initial stages, the Walsh-Hadamard Transform (WHT) possesses the capability to perform a signal transformation on any real-valued time-series. A crucial requirement for this transformation is that the signal must have a length of 2^M . When contemplating the WHT for a longer duration, one may comprehend its distinctiveness in comparison to the Fourier transform. Firstly, the waveforms exhibit much greater simplicity. Secondly, the computational complexity is significantly reduced. Finally, if the input signal is converted from its original form to only two discrete values, the aforementioned advantages become apparent ±1, It ends up with a bunch of trivial arithmetical calculations. If we consider the binary function $f: \mathbb{Z}_2^n \to \mathbb{Z}_2$ then the following transformation (4) is possible.

$$\bar{f}(\mathbf{x}) = 1 - 2f(\mathbf{x}) = (-1)^{f(\mathbf{x})} \quad (4)$$

Therefore $\bar{f}: \mathbb{Z}_2^n \to \{-1,1\}$ What does it do is it performs the following conversion, for instance:

$$\{0,1,0,1,1,0,1,0,0,0,1,\ldots\} \to \{-1,1,-1,1,1,-1,1,-1,-1,-1,1,\ldots\}$$

The transformation of the original binary time-series holds significant importance within the context of processing financial return-series as (5).

$$\overline{f}(\mathbf{x}) = \begin{cases} 1 & if\ f(x) \geq 0 \\ -1 & if\ f(x) < 0 \end{cases} \quad (5)$$

Given that, any return-series in the value interval [−1, 1] (real-valued) is transformed to the binary form of ±1. The utilization of this uncomplicated alteration of signals, in conjunction with the capabilities of the Walsh-Hadamard Transform, introduces novel prospects for scrutinizing the fundamental veritable signal. The Walsh- Hadamard Transform is entirely comprised of ± 1 values that lie dormant within its Hadamard matrices. When reaching close proximity with a signal of identical composition, this is the critical point at which two elements come together.

2.1.4 Random Sequences and Walsh-Hadamard Transform Statistical Test

The binary signal produced by the Walsh-Hadamard transform demonstrates a discernible pattern. In the case of a random signal, it is unlikely that any part of it will be repeatable. This is the underlying motivation for the creation of a pseudo-random generator, which aims to replicate the true randomness found in nature.

In 2009, Oprina et al proposed a statistical test that is based on results derived from a binary signal, where the signal can take on values of either +1 or -1. However, instead of analyzing the entire signal, they suggested dividing it into equally sized blocks for analysis. The statistical test they developed focuses on performing autocorrelation tests using a correlation mask derived from the rows of a Hadamard matrix. In addition to the methodology presented in Rukhin et al.'s work in 2010, which outlines 16 independent statistical tests for random and pseudorandom number generators used in cryptographic applications, Oprina proposed two additional methods based on confidence intervals. These methods can detect more general failures in random and pseudorandom generators. Ultimately, Oprina concludes with a total of five statistical tests. The underlying principle behind Oprina's Suite is a comprehensive test that can be applied to various purposes, including randomness testing, cryptographic design, crypto-analysis techniques, and steganographic detection.

2.2 Research Background

Mardan and Hamood [9] introduce an efficient algorithm for the Walsh-Hadamard-Hartley transform that combines the Walsh-Hadamard transform with the discrete Hartley transform, resulting in a single accelerated transform with a block diagonal arrangement. The algorithm is executed by means of a factorization approach using sparse matrices and the Kronecker product technique, which leads to a decrease in computational complexity compared to prior algorithms.Mazumder and et al. [11] introduce a parallel hardware design for the Walsh Hadamard Transform employing the Kronecker product method and Verilog simulation. This architecture holds potential for implementation in digital signal processing tasks. The suggested design boasts a rapid algorithm and parallel computational outcomes for both one dimensional and two dimensional transformations, effectively diminishing time complexity and optimizing re- source utilization. Broadbent and Maksik [2] examine the utilization of Walsh analysis, a methodology that applies rectangular functions to data, in order to discern cyclic elements within the data. The authors contrast Walsh analysis with Fourier analysis, expounding upon the advantages and drawbacks of each transformation. They proceed to present particular algorithms for the Walsh technique, thereby establishing its su- priority for data exhibiting significant discontinuities. Pan et al. [14] propose a unique layer that utilizes the fast Walsh-Hadamard transform and smooth-thresholding to replace convolution layers in deep neural networks. In the Walsh-Hadamard transform domain, they apply the new smooth- thresholding non-linearity to denoise the coefficients, which is a modified version of the well-known soft-thresholding operator. Additionally, they introduce a set of operators that are free from multiplication, based on the 2×2 Hadamard transform, to implement 3×3 depthwise separable convolution layers. Consequently, this approach offers improved computational efficiency compared to the 1×1 convolution layer. In the context of financial return series, the random walk behavior of Malaysia share returns has been examined. Multiple variance ratio tests have been conducted to assess the efficiency and randomness of the daily return series [10].The results suggest that during the financial crisis period, the movement of daily returns exhibited weak-form efficiency, indicating a departure from randomness.

3 Methodology

The ongoing investigation is classified as one of the research and development studies. It is regarded as an exploratory investigation based on the research question. Furthermore, the data collection process employed a survey methodology and extracted data from reliable databases, such as the Tehran Stock Exchange database and the Quandl website. The objective of the study is to execute the essential measures and analyses to evaluate the randomness of the data. This is accomplished by following a specific step-by-step approach.

3.1 Signal Pre-Processing

To provide a trimmed signal x(t) of the total length $n = 2^M$; choose a sequence (block) size of length 2^m; a significance level of α (rejection level); a probability p of occurrence of the digit 1. At first, $x(t_i)$ transform into $x(t_i)$ sequence. Second lower and upper rejection limits of the test $u_{\alpha/2}$ and $u_{1-\alpha/2}$ computed. Third, the number of sequences to be processed a = n/(2m) and split x(t) into an adjacent blocks (sequences) computed. 2D matrix of Xseq holding signal under investiga- tion is the starting point to its tests for randomness. Oprina test is based on computation of WHT for each row of Xseq and the t-statistics, t_{ij}, as a test function based on Walsh-Hadamard transformation of all sub-sequencies of x(t).
It is assumed that for any signal $y(t_i)$ where i = 0, 1, . . . the WHT returns a sequence $\{w_i\}$ and: (a) for w_0 the mean value is $m_0 = 2^m(1 - 2p)$; the variance is given by $\sigma^2 = 2^{m+2}$ p(1−p) and the distribution of $(w_0-m_0)/\sigma_i \sim N(0, 1)$ for m > 7; (b) for w_i ($i \geqslant 1$) the mean value is $m_i = 0$; the variance is $\sigma^2 = 2^{m+2}$ p(1 − p) and the distribution of $(w_i - m_i)/\sigma_i \sim N(0, 1)$ for m > 7 Recalling that p stands for probability of occurrence of the digit 1in xseq, for p = 0.5 (our desired test probability) the mean value of w_i is equal 0 for every i. In xseq array for every j = 0, 1. . . (a − 1) ,and for every i = 0, 1, . . . , (b − 1) ,t-statistic compute as (6)

$$t_{ij} = \frac{w_{ij} - m_i}{\sigma_i} \tag{6}$$

Where w_{ij} is the i-th Walsh-Hadamard transform component of the block j. In addition, all t_{ij} convert into p-values as (7)

$$p - \text{value} = P_{ij} = \Pr(X < t_{ij}) = 1 - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{\frac{-x^2}{2}} dx \tag{7}$$

3.1.1 Statistical Test Framework

In general. Binary signal x(t) for randomness will be tested. Therefore,
H0: x(t) is generated by a binary memory-less source i.e. the signal does not contain any predictable component;
H1: x(t) is not produced by a binary memory-less source, i.e. the signal contains a predictable component. The testing procedure that can be applied here is for a fixed value of α finding a confidence region for the test statistic and check if the statistical test value is in the confidence region. The confidence levels are computed using the quantiles $u_{\alpha/2}$ and $u_{1-\alpha/2}$ (otherwise, specified in the text of the test). Alternatively, if an arbitrary tstat is the value of the test statistics (test function) we may compare p−value = Pr(X < tstat) with α and decide on randomness when p-value ⩾ α.

3.1.2 Test 1(Crude Decision)

The first WHT test is a crude decision or majority decision. For chosen α and at u_α denoting the quantile of order α of the normal distribution, if $t_{ij} \notin [u_{\alpha/2}; u_{1-\alpha/2}]$
then reject the hypothesis of randomness regarding i-th test statistic of the signal x(t) at the significance level of α. Jot down both j and i corresponding to sequence number and sequence's element, respectively this test is suitable for small numbers of a < 1/α which is generally always fulfilled for our data. Decision on rejection of H_0 is too stiff. In the function the number of t_{ij}'s falling outside the test interval is calculated. If their number exceeds 1, we claim on the lack of evidence of randomness for x(t) as a whole.

3.13 Test 2(Proportion of Sequences Passing a Test)

For each row subsequence of x(t) and its elements, both t_{ij}'s and P_{ij} values computed. In this test, first, for every row of (re-shaped) t 2D array a number of p-values to be Pij < α is checked. If this number is greater than zero, reject j-th sub- sequence of x(t) at the significance level of α to pass the test. For all α sub-sequences count its total number of those which did not pass the test, n_2. If $n_2 \notin [a\alpha\sqrt{a\alpha(1-\alpha))}u_{\alpha/2}; a\alpha\sqrt{a\alpha(1-\alpha))}u_{1-\alpha/2}]$then there is evidence that signal x(t) is non-random.

3.1.4 Test 3(Uniformity of p-values)

In this test, the distribution of p-values is examined to ensure uniformity. This may be visually illustrated using a histogram, whereby, the interval between 0 and 1 is divided into k=10 sub-intervals, and the p-values, i.e. p_{ij}′ s, that lie within each sub interval are counted and displayed. Uniformity may also be determined via an application of χ^2 test and the determination of a p-value corresponding to the Goodness-of-Fit Distributional Test on the p-values obtained for an arbitrary statistical test (i.e., a p-value of the p-values). The computation of the test statistic is as (8)

$$\chi^2 = \sum_{i=1}^{K} \frac{(F_i - \frac{a}{K})^2}{\frac{a}{K}} \tag{8}$$

where F_i is the number of p_{ij} in the histogram's bin of i, and a is the number of sub-sequences of x(t).the hypothesis of randomness regarding i-th test statistic t_{ij} of x(t) at the significance level of α will be rejected if $\chi_i^2 \notin [0; \chi^2(\alpha, K-1)]$. Let χ2(α, K-1) be the quantile of order α of the distribution χ2(K-1) . If test value of Test 3, $\chi_i^2 \leq \chi^2(\alpha, K-1)$then i-th statistics will be counted to be not against randomness of x(t). This is an equivalent to testing i-th p-value of p_{ij} if p_{ij} ⩾ α .

3.1.5 Test 4(Maximum Value Decision)

This test is based on the confidence levels approach. Let $T_{ij} = \max_{j} t_{ij}$ then if $T_{ij} \notin [u_{(\frac{\alpha}{2})a^{-1}}; u_{(1-\frac{\alpha}{2})a^{-1}}]$. then reject the hypothesis of randomness (regarding i-th test function) of signal x(t) at the significance level of α. this test looks at the results derived based on WHTs. It is sensitive to the distribution of maximal values along i-th's elements of t-statistics.

3.1.6 Test 5(Sum of Square Decision)

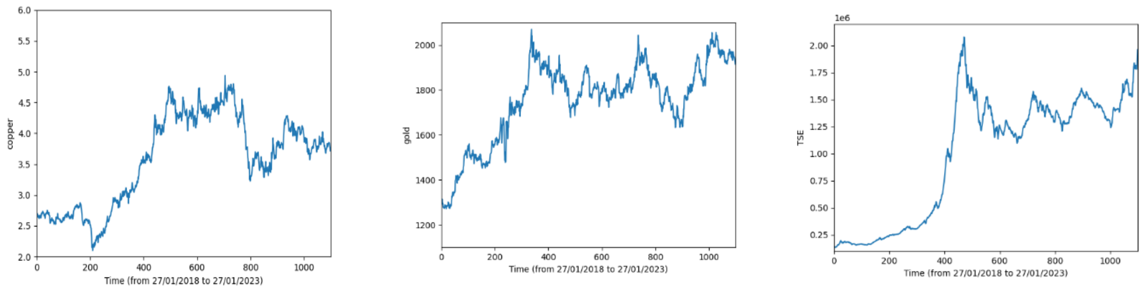
Final test makes use of the C- statistic designed as $C_i = \sum_{j=0}^{a-1} t_{ij}^2$ .If $C_i \notin [0; \chi^2(\alpha, a)]$the hypothesis of randomness of x(t) at the significance level of regarding i-th test function will be rejected.

3.1.7 The Overall Test for Randomness of Binary Signal.

signal x(t) will be accepted to be random if the average passing rate from all five WHT statistical tests is greater than 99%, i.e. 1% can be due to false negative results, at the significance level of α. fed by binary 1 signal of X. The last function return T variable storing 1 for the overall decision that x(t) is random, 0 otherwise. It can be used for a great number of repeated WHT tests for different signals in a loop, thus for determination of ratio of instances the WHT Statistical Test passed.

4 Findings

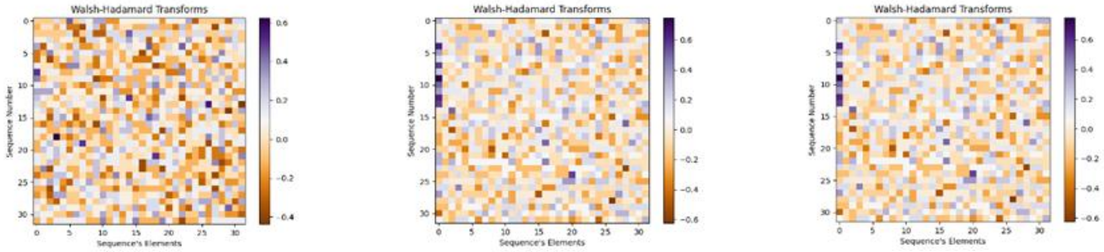
For the purposes of conducting an empirical analysis of the research, the daily data pertaining to the Tehran Stock Exchange index, as well as gold and copper, have been utilized for a duration of five years. It is worth mentioning that the aforementioned data has been procured from reputable databases. The research procedures were carried out using coding in a Python program and will be provided to the readers upon request. The price time series is depicted in **Fig. 2** in the context of the Tehran Stock Exchange, revealing a noticeable increase in the middle of 2020. However, it is crucial to acknowledge that fluctuations were present during other time periods. After the price-series has been converted into the return-series, the return-series is then further transformed into a binary signal.



(a) copper price series (b) Gold price series (c) TSE price series

Fig. 2: Data time series plot

Additionally, it is important to note that the signal is 1100 points in length and can be divided into 32 sub-sequences, each consisting of 32 points. Therefore, the variable n represents the trimmed signal of x(t) and is equal to 2^{10}. In order to provide clarity, the first 32 segments of the price-series, which are each 32 points in length, are plotted in **Fig. 3** and marked with vertical lines.



(a) copper (b) Gold (c) TSE

Fig. 3: Walsh-Hadamard Transform plot

From the comparison of both figures 2 and 3, one can comprehend the level of intricacy involved in deriving the WHT results. Notably, in the case of return-series, the WHT image exhibits a considerable lack of uniformity, indicating the random nature of the return-series. To ascertain the randomness of the return series, five statistical tests have been conducted. The results of these tests are shown thorough tables.

Table 1: Crude Decision Test

	copper		Gold		TSE	
Test NO.	statistic	pvalue	statistic	pvalue	statistic	pvalue
Test1	3.12	.995	4.5	.996	2.99	.990

Table 2: Proportion of Sequences Passing a Test

	copper		Gold		TSE	
Test NO.	number	pvalue	number	pvalue	number	pvalue
Test2	0	.999	1	.998	0	.990

Table 3: Uniformity of p-values Test

	copper		Gold		TSE	
Test NO.	statistic	pvalue	statistic	pvalue	statistic	pvalue
Test3	12.5	.999	9.50	.998	6.55	.990

Table 4: Maximum Value Decision Test

	copper		Gold		TSE	
Test NO.	statistic	pvalue	statistic	pvalue	statistic	pvalue
Test4	33.5	.999	29.50	.998	21.50	.990

Table 5: Sum of Square Decision Test

	copper		Gold		TSE	
Test NO.	statistic	pvalue	statistic	pvalue	statistic	pvalue
Test5	25.1	.999	9.2	.998	16.1	.990

As evidenced by the five administered tests, the presence of randomness in the transformed time series has been validated with a confidence level exceeding 99% in all instances. Furthermore, in the overall test, the random- ness of the transformed time series has been confirmed at a level surpassing 99%. in order to assess the robustness of the findings. The experiment involved varying values of the sequences and the distances between them, with subsequent verification of the outcomes.

5 Results and Recommendations

The objective of this article is to examine the implementation of financial time series utilizing the Walsh-Hadamard Transform method. This article endeavors to scrutinize the randomness of three distinct time series, specifically copper, gold, and the stock market index, through the utilization of five principal randomness tests. The findings unequivocally demonstrate the presence of randomness in the transformed data. The current article aims to offer an initial exposition of the subject matter under scrutiny, thereby necessitating further research in this domain, particularly in regard to the practicability of utilizing the transformed data. Specific patterns were identified within this field, with the results being presented as a viable pattern.

Referrences

[1] Antoniadis, J., Arumugam, P., Arumugam, S., Babak, S., Bagchi, M., Nielsen, A., The second data release from the European Pulsar Timing Array II, *Customised pulsar noise models for spatially correlated gravitational waves*. arXiv preprint arXiv:230616225. 2023.

[2] Broadbent, H.A., Maksik, Y.A., Analysis of periodic data using Walsh functions ,*Behavior Research Methods, Instruments, Computers,*1992; 24(2):238-47.

[3] Fama, E.F., The behavior of stock-market prices, *The journal of Business*, 1965; 38(1):34-105.

[4] Fox, M.D., Halko, M.A., Eldaief, M.C., Pascual-Leone, A., Measuring and manipulating brain connectivity with resting state functional connectivity magnetic resonance imaging (fcMRI) and transcranial magnetic stimulation (TMS), *Neuroimage.* 2012; 62(4):2232-43.

[5] Golubov, B., Efimov, A., Skvortsov, V., Walsh series and transforms: theory and applications: *Springer Science and Business Media;* 2012.

[6] Johnson, J., Puschel, M., editors. In search of the optimal Walsh-Hadamard transform. *2000 IEEE International Conference on Acoustics, Speech, and Signal Processing Proceedings (Cat No 00CH37100);* 2000: IEEE.

[7] Kendall, M.G., Hill, A.B., The analysis of economic time-series-part i: Prices, *Journal of the Royal Statistical Society Series A (General)*, 1953; 116(1):11-34.

[8] Kim, S-J., Umeno, K., Hasegawa, A., Corrections of the NIST statistical test suite for randomness, arXiv preprint nlin/0401040. 2004.

[9] Mardan, S.S., Hamood, M.T., New fast Walsh–Hadamard–Hartley transform algorithm, *International Journal of Electrical and Computer Engineering*, 2023;13(2):1533.

[10] Marsani, MF., Shabri, A., Random Walk Behaviour of Malaysia Share Return in Different Economic Circumstance, *MATEMATIKA: Malaysian Journal of Industrial and Applied Mathematics*, 2019.

[11] Mazumder, P., Chandra, S., Rana, S., Mukhopadhyay, M., Naskar, MK., Parallel Hardware Implementation of Walsh Hadamard Transform, *Journal of Scientific and Industrial Research,* 2022; 81(07):748-53.

[12] Oczeretko, E., Borowska, M., Brzozowska, E., Pawlinski, B., Borusiewicz, A., Gajewski, Z., editors. Walsh-Hadamard spectral analysis of signals representing bioelectrical activity of the reproductive tract in pigs, *IEEE 15th International Conference on Bioinformatics and Bioengineering (BIBE)*, 2015; IEEE.

[13] Oprina, A-G., Popescu, A., Simion, E., Simion, G., Walsh− Hadamard randomness testand new methods of test results integration, *Bulletin of the Transilvania University of Brasov Series III: Mathematics and Computer Science*, 2009:93-106.

[14] Pan, H., Badawi, D., Cetin, A.E., editors. Fast walsh-hadamard transform and smooth-thresholding based binary layers in deep neural networks, *Proceedings of the IEEE/CVF Conference on Computer Vision and Pattern Recognition*; 2021.

[15] Pratt, J.W., Gibbons, J.D., Concepts of nonparametric theory: *Springer Science & Business Media;* 2012.

[16] Walsh, J.L., A closed set of normal orthogonal functions, *American Journal of Mathematics*, 1923; 45(1):5-24.