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Original Research

Fixed Cost Allocation Based on DEA Cross Efficiency Considering Semi-Additive Production Technology: An Application to Bank Branches

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ABSTRACT

In the real world, situations frequently occur when we want to allocate a fixed cost between a set of decision-making units (DMUs) such as institutions, organizations. In this paper, we use the data envelopment analysis (DEA) technique to allocate fixed costs among DMUs. First, we introduce semi-additive production technology in DEA and present efficiency evaluation models in this technology. In estimating the frontier of this technology, in addition to the observed DMUs, the set of all aggregations of these DMUs are also used. In the following, we propose an interactive process for fixed cost allocation between DMUs in DEA based on the concept of cross-efficiency. We show that our proposed iterative approach is always feasible, and ensures that all DMUs become efficient after the fixed cost is allocated as an additional input measure. The cross-efficiency scores corresponding to all DMUs are improved at each stage of the interactive process. We also illustrate the proposed approach with a numerical example. The proposed approaches are demonstrated using an application of the fixed cost allocation problem for branches of commercial banks. Finally, we bring the results of the research.

1 Introduction

Efficiency analysis plays an important role in decision-making processes and is an important issue for improving the performance of organizations. Several different techniques have been proposed for efficiency analysis. One of these techniques is DEA and based on mathematical programming. At first, it was developed by Charnes et al. [5]. This method estimates the efficiency of a set of DMUs with multiple inputs and outputs. DEA obtain efficiency score from each DMU by making a set that called production possibility set (PPS). DEA accept the underlying assumptions for estimating the frontier of PPS, and consider the frontier of PPS as the efficiency frontier. DMUs located on this frontier are referred to as efficient DMUs and other DMUs are inefficient. By accepting different assumptions, different production technologies have been proposed to measure efficiency. Charnes et al. [5] accepted the property constant returns to scale (CRS) for reference technology, in order to estimate

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both technical and scale efficiencies. Banker et al. [3] introduced another production technology by accepting the property of variable returns-to-scale (VRS). This set based on the convex hull of individual units and strong disposability assumption. Fare et al. [11] used nonincreasing (decreasing) returns-to-scale (DRS) technology to estimate efficiency. In a different framework from DEA models, Koopmans [18] introduced another form of nonincreasing returns-to-scale technology. Grosskopf [16] showed Koopman's technology in DEA framework and pointed out that it includes the sum of the individual units and comprises Fare et al.'s technology. Deprins et al. [9] introduced Free Disposal Hall (FDH) technology, which is a non-convex PPS. This model is equivalent to the model of Banker et al. [3] if only binary intensity variables are considered. Green and Cook [15] presented a non-convex PPS known as free coordination hall (FCH). In order to evaluate the performance of DMUs, in addition to the observed units, they also used the aggregations DMUs corresponding to these DMUs. As mentioned, some of the production technologies in the DEA literature include aggregated DMUs. In some of the DEA literature, especially axiomatic-based work, inclusion of aggregated DMUs is referred to as the additivity assumption. This assumption states that if the two observed DMUs A and B can be produced, then the newly created DMU can also be produced as A + B. Another assumption in estimating the frontier of PPS is semi-additive assumption. Ghiyasi [13] used semi-additive assumption and introduced a new PPS that consider aggregations of the set of DMUs. He measured the performance of DMUs based on individually observed units and the aggregations DMUs corresponding to these DMUs. Hence, this technology creates a larger competitive environment than other technologies for DMUs to reach the efficiency frontier. The two general assumptions of additivity and semi-additive can be assumed to be distinct. In accordance with additivity assumption, new aggregated DMUs can be exact multiples of the original observed units, but not in semi-additive technology, and we have to use different units to create new aggregated DMUs. According to the additivity assumption, if unit A belongs to the PPS, then the new aggregated DMUs as 2A or 3A also belong to the PPS, but this will not be in accordance with semi-additive assumption, and if A and B are two distinct DMUs, then A + B also belongs to the PPS in accordance with semi-additive assumption. [13] We must note that all production technologies accepting general additivity presume specific returns to scale properties. In the real world, however, unless we have knowledge regarding the returns to scale of a technology, we cannot impose any specific such properties concerning that technology. One of the main contributions of the semi-additive technology to the literature is that it addresses the returns to scale problem. Namely, the semi- additive technology does not presume any returns to scale requirements. [14]

The main problem with semi-additive technology is the resulting computational complexity due to the large number of aggregated DMUs created based on the observed DMUs. However, in many real-world applications, we may not need to consider all the DMUs and be able to solve the above problem. But Ghiyasi and Cook [14] solved the above problem, and they proposed a new PPS that is equal to the original semi- additive technology based on the power set of all DMUs. They showed that the efficiency evaluation model based on the new semi- additive technology no longer needs to consider all the aggregated DMUs in the power set corresponding to the on individually observed units and proposed the semi- additive technology in a new form based only on the observed units. They showed that the new model proposed for calculating efficiency in semi- additive technology significantly reduces the amount of computation and can be easily used. [14] One of the important applications of DEA is the issue of fixed cost allocation among DMUs. In many real management applications, we have to allocate a total fixed cost between a set of competitive DMUs. Cook and Kress [7] first presented the issue of fixed cost allocation in the context of DEA. They hypothesized that fixed costs could be considered as a new input measure for all DMUs. The basis of two principles efficiency invariance and pareto-minimality presented a fair cost allocation scheme by solving several linear programming models. But in general, their approach was to find a unique efficient unit by cone ratio approach, thus avoiding a multiple cost fixed allocation. Tohidi and Tohidnia [23] measure the interval industry cost efficiency score in DEA. They extend the concept of "cost minimizing industry structure" and develop two DEA models for dealing with imprecise data. Also, they propose an approach to compute the industry cost efficiency measure in the presence of interval data. They show that the value obtained by the proposed approach is an interval value. The lower bound and upper bound of the interval industry cost efficiency measure are computed and then decomposed into three components to examine the relationship between them and the lower and upper bounds of the individual interval cost efficiency measures. They define the cost-efficient organization of the industry as a set of DMUs, which minimizes the total cost of producing the interval industry output vector. Mozaffari et al. [22] presents two strategies for allocating fixed costs with undesirable data. In the first strategy, DMU first determines the minimum and maximum shares that it can receive from the fixed resources while the efficiency of that DMU and other DMUs re-mains the same after receiving the fixed resources. Also, the decision maker chooses the fixed cost for each DMU between the minimum and maxi-mum cost values proposed. In the second strategy, the allocation of fixed costs is done using the CCR multiplicative model with undesirable data. The effectiveness of both methods is examined by an applied study on the commercial banks.

Beasly [4] proposed a nonlinear programming model to achieve a unique cost fixed allocation by maximizing the average efficiency across all DMUs. Cook and Zhu [8] presented a new approach of feasible cost fixed allocation (but not optimal) in the output oriented based on the efficiency invariance principle and the proposed approach by Cook and Kress [7]. Li et al [19] presented a new approach to the fixed cost allocation problem based on DEA and degree of satisfaction. They provided a unique cost fixed allocation, and at the end of the proposed algorithm, all DMUs were efficient by considering the amount of dedicated cost fixed allocation as an additional input for all DMUs. They showed that the approach presented by them is equivalent to the proportional sharing method in one dimension, and in the multidimensional state, cost fixed allocation may not be unique. Du et al. [10] presented the issue of fixed cost and resource allocation based on the concept of cross-efficiency. They proposed an interactive algorithm for establishing the concept of cross-efficiency to provide a fixed cost allocation scheme between all cost DMUs. They hypothesized that the amount of cross-efficiency proportional to each of the DMUs at each stage of the proposed algorithm would be non-decreasing, and that at the end of the algorithm all DMUs would be cross-efficient. They considered the fixed cost proportional to each of the DMUs as an additional input. They then presented the issue of resource allocation based on the concept of cross-efficiency and presented another interactive algorithm in this regard.

Li et al. [20] proposed a new data envelopment analysis-based approach for fixed cost allocation. They suggest a non-egoistic principle which states that each DMU should propose its allocation proposal in such a way that the maximal cost would be allocated to itself. Also, a preferred allocation scheme should assign each DMU at most its non-egoistic allocation and lead to efficiency scores at least as high as the efficiency scores based on non-egoistic allocations. They integrate a goal programming method with DEA methodology to propose a new model under a set of common weights. The final allocation scheme is determined in such a way that the efficiency scores are maximized for all DMUs through minimizing the total deviation to goal efficiencies. Li et al. [21] developed allocating a fixed cost based on a DEA-game cross efficiency approach. They approach the fixed cost allocation problem by explicitly considering both competition and cooperation relationships among DMUs. They integrate cooperative game theory and the cross-efficiency method to propose a DEA-game cross efficiency approach to generate a unique and fair allocation plan. Based on the proposed approach by them, each DMU is considered as a player and a super-additive characteristic function is defined for coalitions of DMUs. In the following, the Shapley value is calculated for each DMU and accordingly associated common weights are optimized to determine the final allocation plan. Since the cross-efficiency method considers peer appraisal and the cooperative game theory allows for equitable negotiations, all DMUs are supposed to reach a consensus on the equitable allocation scheme through their novel approach. An et al. [1] proposed fixed cost allocation for two-stage systems with cooperative relationship using DEA. They developed an approach for fixed-cost allocation issues of two-stage systems by considering a cooperative relationship among DMUs. They integrate cooperative game theory and the DEA methodology to generate a unique and fair allocation plan. The results confirm that each DMU can maximize its relative efficiency to one by a series of optimal variables after the fixed cost allocation. A unique nucleolus solution can be generated through a feasible computation algorithm. An et al. [2] proposed fixed cost allocation based on the principle of efficiency invariance in two-stage systems. They proposed a fixed cost allocation approach for basic two-stage systems based on the principle of efficiency invariance and then extend it to general two-stage systems. They developed a fixed cost allocation under the overall condition of efficiency invariance when the two stages have a cooperative relationship. Then, the model of fixed cost allocation under the divisional condition of efficiency invariance wherein the two stages have a noncooperative relationship is studied. Chu et al. [6] developed DEA-based fixed cost allocation in two-stage systems based on the leader-follower and satisfaction degree bargaining game approaches. They proposed a new fixed cost allocation approach for allocating a fixed cost among DMUs with two-stage structures under the framework of DEA. They give the set of possible fixed cost allocations, and prove that all DMUs can be overall efficient when evaluated by a common set of weights after fixed cost allocation. Secondly, from a centralized point of view, they consider the competition between the DMUs' two stages in fixed cost allocation and regard these two kinds of stages as two unions. They incorporate leader-follower models to propose a fixed cost allocation approach to handle the situation in which the two unions make decisions sequentially. Izadikhah [17] proposes a new two stage BAM model and further evaluates the banks and financial institutes in Tehran stock exchange by considering the financial ratios. Conventional DEA models consider each firm as black box and don't note into the inner activities. Two-stage data envelopment analysis has been researched by a number of authors that evaluate each firm by considering the inner operations. He proposes a new variant of two stage DEA models and further evaluates the banks and financial institutes in Tehran stock exchange by considering the financial ratios. In this paper, we present the issue of fixed cost allocation in semi-additive technology based on the concept of cross-efficiency. The proposed model for calculating cross-efficiency considers the fixed cost allocation as an additional input. According to the nonlinear form of the proposed model, we bring the necessary transformations for linearization of the model. To present an optimal fixed cost allocation plan, we propose an interactive algorithm. At each stage of algorithm, we improve the cross-efficiency scores corresponding to all DMUs, and when the algorithm terminates, we can obtain the optimal fixed cost allocation corresponding to each of the DMUs based on the concept of cross-efficiency. It can be said that the contribution of this paper is as follows. 1) In this paper, we present efficiency evaluation models in semi-additive technology based on the both envelopment and multiple models in DEA. 2) We present a cost fixed allocation scheme in semi-additive technology, based on the concept of cross-efficiency. 3) We apply the proposed approach in this paper for the data set of 18 branches of a bank in China.

The remainder of the paper is organized as follows. In the second section, we introduce semi-additive technology in theory and geometry. In the third second, we present the issue of fixed cost allocation in semi-additive technology. In the fourth section, we illustrate the proposed approach with a numerical example. In the fifth section, we show the application of the proposed approach for the set of bank branches in China, and in the sixth section, we bring the results of the research.

2 Semi-Additive Production Technology

Suppose that there are n $DMUs$ producing the same set of s outputs by consuming the same set of m inputs. Let DMU_j, $j = 1, \dots, n$, denote jth observed DMUs and its ith input and rth output from this DMU are denoted by x_{ij}, $i = 1, \dots, m$, $j = 1, \dots, n$, and y_{rj}, $r = 1, \dots, s$, $j = 1, \dots, n$, respectively. We define the possibility of general production as follows.

$$T = \{(x, y) | x \; Can \; produce \; y \}.$$

Ghiyasi [13] introduced semi-additive production technology. To introduce the above technology, we first introduce the following assumptions in creating production technology.

S1. Feasibility of observations.

This assumption implies that all observed DMUs belong to the production technology, i.e.

$$(x_j, y_j) \in T, j = 1, \dots, n.$$

S2. Free (strong) disposability.

This assumption states that if $(x_1, y_1) \in T$ and if a point (x_2, y_2) is such that, $x_2 \geq x_1$, $y_1 \geq y_2$, then $(x_2, y_2) \in T$.

S3. Convexity.

This assumption states that if $(x_1, y_1) \in T$, $(x_2, y_2) \in T$, then $\mu(x_1, y_1) + (1 - \mu)(x_2, y_2)$ for all $\mu \in (0,1)$.

S4. Radial rescaling.

This assumption states that if $(x, y) \in T$, then $\mu(x, y) \in T$, for all $\mu \geq 0$.

S5. Semi-additive.

This assumption implies that $(x_i, y_i) \in T$, $(x_j, y_j) \in T$, then $\left((x_i, y_i) + (x_j, y_j)\right) \in T$, that $i \neq j$.

S6. Minimum extrapolation.

This assumption states that the set T is the smallest set that holds in the above assumptions. In other words, the T set is the subscription of all sets of production technologies that have the above properties.

We now consider the power set corresponding to the set $J = \{1, \dots, n\}$ as the set $P(J)$. This set includes all subsets of the J set. Let set $J' = P(J)/\emptyset$. It should be noted that the set J' is the power set corresponding to the set J that includes all subsets of the set J except the empty set namely this production technology does not include the origin. $\emptyset$, denote empty set. The set J' has 2^n members.

By accepting the assumptions S1-S6, we can present the PPS in semi-additive technology as follows.

$$T_{SA} = \left\{(x, y) \middle| \sum_{j \in J'} \lambda_j x_j \leq x, \sum_{j \in J'} \lambda_j y_j \geq y, \sum_{j \in J'} \lambda_j = 1, \lambda_j \geq 0, j \in J' \right\} \tag{1}$$

The efficiency evaluation model of the unit under evaluation, i.e. $DMU_k = (x_k, y_k)$, in the input oriented based on semi-additive technology, will be as follows.

$$\min \{\theta^{SA} | (\theta^{SA} x_o, y_o) \in T^{SA}\}. \quad (2)$$

Now according to the definition of the set T_{SA}, model (2) will be as follows.

$$\begin{aligned} &\min \theta^{SA} \\ &S.t. \ \sum_{j \in J'} \lambda_j x_{ij} \le \theta^{SA} x_{io}, \ i = 1, \dots, m, \\ &\quad \sum_{j \in J'} \lambda_j y_{rj} \ge y_{ro}, \ r = 1, \dots, s, \\ &\quad \sum_{j \in J'} \lambda_j = 1, \ \lambda_j \ge 0, j = 1, \dots, n. \end{aligned} \quad (3)$$

As can be seen, model (3) has $m + s + 1$ constraints and 2^n variables. To solve model (3), we must form all the aggregated DMUs corresponding to the observed DMUs. As stated earlier. In this regard, we must create all aggregated DMUs from the set J' or the power set corresponding to the set J, which includes all subsets of the set J except the empty set. Therefore, it is very difficult to calculate the collective inputs and outputs for each member of the set J'. In this regard, Ghiyasi and Cook [14] showed that we can present the PPS under semi-additive condition, i.e. T^{SA}, as follows.

$$T_{new}^{SA} = \left\{(x, y) \middle| \sum_{j=1}^n \lambda_j x_j \le x, \sum_{j=1}^n \lambda_j y_j \ge y, \sum_{j=1}^n \lambda_j \ge 1, \ 0 \le \lambda_j \le 1, j = 1, \dots, n\right\} \quad (4)$$

The efficiency evaluation model of the unit under evaluation, i.e. $DMU_k = (x_k, y_k)$, in the input oriented based on semi-additive technology based on the the new PPs namely T_{new}^{SA} will be as follows.

$$\min \{\theta_{new}^{SA} | (\theta_{new}^{SA} x_o, y_o) \in T_{new}^{SA}\}. \quad (5)$$

Now according to the definition of the set T_{new}^{SA}, model (5) will be as follows.

$$\begin{aligned} &\theta_{new}^{SA\,*} = \min \theta_{new}^{SA} \\ &S.t. \ \sum_{j=1}^n \lambda_j x_{ij} \le \theta_{new}^{SA} x_{io}, \ i = 1, \dots, m, \\ &\quad \sum_{j=1}^n \lambda_j y_{rj} \ge y_{ro}, r = 1, \dots, s, \\ &\quad \sum_{j=1}^n \lambda_j \ge 1, \ 0 \le \lambda_j \le 1,, j = 1, \dots, n. \end{aligned} \quad (6)$$

Definition 1. DMU_o is called semi-additive efficiency in evaluation with model (6), if $\theta_{new}^{SA\,*} = 1$, otherwise it is called a semi-additive inefficient DMU.

As can be seen, model (6) has $m + s + n + 1$, constraints and only $n + 1$ variables. To solve model (3), we no longer need to create all the aggregated DMUs corresponding to the observed DMUs, and the efficiency is calculated only on the basis of the existing observed DMUs. Model (6) in evaluating the efficiency of the unit under evaluation i.e. $DMU_o = (x_o, y_o)$, significantly reduces the amount of calculations compared to model (1). Ghiyasi and Cook [14] showed that $T_{SA} = T_{new}^{SA}$.

For illustrate PPS under semi-additive technology, we consider three DMUs as follows.

$A = (2, 0.5)$ ،$B = (3, 2.5)$ ، and $C = (5, 3)$.

In this case, we can form aggregated DMUs as follows.

$D = A + B = (5, 3)$, $F = A + C = (7, 3.5)$, $G = B + C = (8, 5.5)$, $E = A + B + C = (10, 6)$.

In the state one input (Input-axis) and one output (Output-axis), and all three original DMUs are efficient. The PPS under constant returns to scale (CCR) property is the region restricted by the Input-axis and the right-hand side of the line starting from the origin and passing the B (the dash–dot line). CRS technology is the biggest technology that includes all the other technologies. The bounded region by the Input-axis starting from x_A and the segment A–B–C and the horizontal extension from B is

corresponding to the PPS of the BCC technology and has VRS property. The PPS under semi-additive assumption is bigger than the PPS of the BCC model and is bounded by the Input-axis starting from x_A passing the segment of A–B–G–E and horizontal extension from E.
Now, consider DMU C. This DMU is efficient in variable returns to scale technology, while it is inefficient in semi-additive technology. In order to evaluate the efficiency of this DMU, we can solve model (6). This model depicts DMU C at point C_1 on the efficiency frontier of the PPS corresponding to semi-additive technology. As shown in **Fig.** 1, the efficiency score is calculated as the ratio $\left|\frac{OC_1^X}{OC^X}\right| = 0.6$. C^X and C_1^X, represent the image of the points C and C_1 on the Input-axis, respectively.

3 Fixed Cost Allocation in Semi-Additive Technology

Suppose we have n decision units as $DMU_j = (X_j, Y_j), j = 1, \dots, n$. The input and output vectors corresponding to $DMU_j, j = 1, \dots, n$ are as $X_j = (x_{1j}, \dots, x_{mj})$ and $Y_j = (y_{1j}, \dots, y_{sj})$
respectively.
As stated in the second section, the efficiency score of the unit under evaluation, i.e. $DMU_k = (x_k, y_k)$ in semi-additive technology can be obtained using model (6).
Given that model (6) is a linear programming model, we can also get the efficiency score of the unit under evaluation based on the dual of this model as follows.

$$\begin{aligned} &\max \textstyle\sum_{r=1}^{s} u_r^k\, y_{rk} + u_k - \sum_{j=1}^{n} w_j^k \\ &s.t.\ \ \textstyle\sum_{r=1}^{s} u_r^k\, y_{rj} - \sum_{i=1}^{m} v_i^k x_{ij} + u_k - w_j^k \le 0, \qquad j = 1, .., n, \\ &\quad \textstyle\sum_{i=1}^{m} v_i^k x_{ik} = 1, \\ &\qquad v_i^k \ge 0, u_r^k \ge 0, \quad i = 1, \dots, m,\ r = 1, \dots, s, \\ &\qquad u_k \ge 0,\ w_j^k \ge 0, j = 1, .., n. \end{aligned} \tag{7}$$

In the above model, the multiples u_r^k, v_i^k are related to output, input constraints of model (6) respectively. Also, we consider the multiples u_k, w_j^k corresponding to constraints $\sum_{j=1}^{n} \lambda_j \ge 1$, $\lambda_j \le 1$, $j = 1, .., n$, in model (6), respectively.
Model (7) is presented in the fractional form as follows.

$$\begin{aligned} &\theta_k^{SAF^*} = \max \frac{\sum_{r=1}^{s} u_r^k y_{rk} + u_k - \sum_{j=1}^{n} w_j^k}{\sum_{i=1}^{m} v_i^k x_{ik}} \\ &\quad s.t.\ \frac{\sum_{r=1}^{s} u_r^k y_{rj} + u_k - w_j^k}{\sum_{i=1}^{m} v_i^k x_{ij}} \le 1, \qquad j = 1, .., n, \\ &\qquad v_i^k \ge \epsilon, u_r^k \ge \epsilon, \qquad i = 1, \dots, m,\ \ r = 1, \dots, s, \\ &\qquad u_k \ge \epsilon,\ w_j^k \ge \epsilon, \quad j = 1, .., n. \end{aligned} \tag{8}$$

Definition 2. DMU_k is called semi-additive efficiency in evaluation with model (7), if the optimal objective function score of model (7) is equal to one, otherwise it is called a semi-additive inefficient DMU.

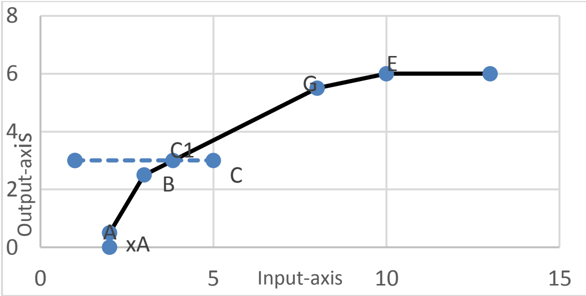

Fig. 1: Semi-Additive Technology.

Definition 3. DMU_k is called semi-additive efficiency in evaluation with model (8), if $\theta_k^{SAF^*} = 1$, otherwise it is called a semi-additive inefficient DMU.

Suppose that a fixed cost R is to be distributed among all $DMUs$. Then, we can allocate amount a cost $r_j, j = 1, \dots, n$, to each $DMU_j, j = 1, \dots, n$, in a way that $\sum_{j=1}^{n} r_j = R$. If the shared cost R is treated as a new input, then the efficiency for DMU_k, becomes

$$E_k^{SA} = \frac{\sum_{r=1}^{s} u_r^k\, y_{rk} + u_k - \sum_{j=1}^{n} w_j^k}{\sum_{i=1}^{m} v_i^k x_{ik} + v_{m+1}^k r_k^k}$$

For calculate the post-allocation efficiency score for DMU_k, we consider the following fractional mathematical program.

$$\begin{aligned} \theta_{k-co}^{SAF\ *} = \max\ & \frac{\sum_{r=1}^{s} u_r^k y_{rk} + u_k - \sum_{j=1}^{n} w_j^k}{\sum_{i=1}^{m} v_i^k x_{ik} + v_{m+1}^k r_k^k} \\ s.t.\ & E_j^{SA} \le \frac{\sum_{r=1}^{s} u_r^k y_{rj} + u_k - w_j^k}{\sum_{i=1}^{m} v_i^k x_{ij} + v_{m+1}^k r_j^k} \le 1, \quad j = 1, .., n, \\ & \sum_{j=1}^{n} r_j^k = R,\ r_j^k \ge 0, \quad j = 1, .., n, \\ & v_i^k \ge \epsilon,\ u_r^k \ge \epsilon, \quad i = 1, \dots, m, \quad r = 1, \dots, s, \\ & u_k \ge \epsilon,\ w_j^k \ge \epsilon, \quad j = 1, .., n. \end{aligned} \tag{9}$$

In model (9), $0 \le E_j^{SA} \le 1$, is a parameter. At first, for solving model (9), Initially, we consider the value E_j^{SA}, as the semi-additive efficiency score of DMU_j which is obtained by solving model (8). In other words, we can put $E_j^{SA} = \theta_{j-co}^{SAF\ *}$. $\theta_{j-co}^{SAF\ *}$ is the optimal objective function of model (8), when we evaluate DMU_j.

Note that model (9) is a non-linear program. In order to convert model (9) into a linear program, we use the Charnes–Cooper transformation and let $\hat{v}_{m+1} r_j^k$ be a new variable as $\tilde{r}_j^k$.

$$\begin{aligned} \max & \sum_{r=1}^{s} \hat{u}_r^k\, y_{rk} + \hat{u}_k - \sum_{j=1}^{n} \widehat{w}_j^k \\ s.t.\ & \sum_{r=1}^{s} \hat{u}_r^k\, y_{rj} - \left(\sum_{i=1}^{m} \hat{v}_i^k x_{ij} + \tilde{r}_j^k\right) + \hat{u}_k - \widehat{w}_j^k \le 0, \quad j = 1, .., n, \\ & \sum_{r=1}^{s} \hat{u}_r^k\, y_{rj} - E_j^{SA}\left(\sum_{i=1}^{m} \hat{v}_i^k x_{ij} + \tilde{r}_j^k\right) + \hat{u}_k - \widehat{w}_j^k \ge 0, \quad j = 1, .., n, \\ & \sum_{i=1}^{m} \hat{v}_i^k x_{ik} + \tilde{r}_k^k = 1, \\ & \sum_{j=1}^{n} \tilde{r}_j^k = \hat{v}_{m+1}^k R,\ r_j^k \ge 0, \quad j = 1, .., n, \\ & \hat{v}_i^k \ge \epsilon,\ \hat{u}_r^k \ge \epsilon, \quad i = 1, \dots, m, \ r = 1, \dots, s, \\ & \hat{u}_k \ge \epsilon,\ \hat{v}_{m+1} \ge \epsilon,\ \widehat{w}_j^k \ge \epsilon,\ j = 1, .., n. \end{aligned} \tag{10}$$

After solving model (10), suppose we obtain a set of optimal values of $\hat{u}_r^{k^*}$, $\hat{u}_k{}^*$, $w_j^{k^*}$, $\hat{v}_i^{k^*}$, $\hat{v}_{m+1}{}^*$, $r_j^{k^*}$, corresponding to DMU_k. Based on this set, the semi-additive k-cross efficiency for each DMU_j, $j = 1,..,n$, is calculated as follows.

$$E_j^{SA^k} = \frac{\sum_{r=1}^{s} \hat{u}_r^{k^*} y_{rj} + \hat{u}_k{}^* - \sum_{j=1}^{n} \hat{w}_j^{k^*}}{\sum_{i=1}^{m} \hat{v}_i^{k^*} x_{ij} + \tilde{r}_j^{k^*}} \tag{11}$$

Therefore, the semi-additive cross-efficiency score corresponding to DMU_j, $j = 1,..,n$, is calculated as the average of all of its the semi-additive k-cross efficiencies as follows.

$$E_j^{SA} = \frac{1}{n}\sum_{k=1}^{n} E_j^{SA^k} = \frac{1}{n}\sum_{k=1}^{n} \frac{\sum_{r=1}^{s} \hat{u}_r^{k^*} y_{rj} + \hat{u}_k{}^* - \sum_{j=1}^{n} \hat{w}_j^{k^*}}{\sum_{i=1}^{m} \hat{v}_i^{k^*} x_{ij} + \tilde{r}_j^{k^*}} \tag{12}$$

For the fixed cost allocation problem in the above strategy, we consider all DMUs. We determine the optimal solution of models (9) or (10) for each DMU_k, and its maximum efficiency provided that the semi-additive cross-efficiency score of all DMUs do not be less of their current level. Now, we present an interactive algorithm to obtain an optimal fixed cost allocation scheme. In this regard, we first obtain the semi-additive cross-efficiency score corresponding to each of the DMUs based on model (10) and the value of parameter E_j^{SA} in the first step of the proposed algorithm is equal to the optimal objective function score of model (8) in efficiency evaluation of DMU_j, $j = 1,..,n$.

Table 1: An Algorithm for the Fixed Cost Allocation Process in Semi-Additive Technology.

Step One: Determine the efficiency scores of all DMUs.

Solve model (8) and obtain a set of semi-additive efficiency scores $\theta_j^{SAF^*}$. Let p = 1, $E_j^{SA} = E_j^{SA}(1) = \theta_j^{SAF^{**}}$, and go to the second step.

Step Two: Determine the semi-additive cross-efficiency score of all DMUs.

Solve model (10) for each DMU_k. Let

$E_j^{SA}(p+1) = \frac{1}{n}\sum_{k=1}^{n} \frac{\sum_{r=1}^{s} \hat{u}_r^{k^*}(p) y_{rj} + \hat{u}_k{}^*(p) - \sum_{j=1}^{n} \hat{w}_j^{k^*}(p)}{\sum_{i=1}^{m} \hat{v}_i^{k^*}(p) x_{ij} + \tilde{r}_j^{k^*}(p)}$, that $\hat{u}_r^{k^*}(p)$, $\hat{u}_k{}^*(p)$, $\hat{w}_j^{k^*}(p)$, $\hat{v}_i^{k^*}(p)$, $\hat{v}_{m+1}^{k}{}^*(p)$, $\tilde{r}_j^{k^*}(p)$, represent optimal values for $\hat{u}_r^k$, $\hat{u}_k$, $\hat{w}_j^k$, $\hat{v}_i^k$, $\hat{v}_{m+1}^k$, $\tilde{r}_j^k$, when $E_j^{SA} = E_j^{SA}(p)$, respectively. Go to the third step.

Step Three: Check the termination condition.

If $\left|E_j^{SA}(p+1) - E_j^{SA}(p)\right| \geq \epsilon$, for some j, where ϵ is a specified small positive value, in this case, let $E_j^{SA} = E_j^{SA}(p+1)$, and go to step two. If $\left|E_j^{SA}(p+1) - E_j^{SA}(p)\right| < \epsilon$, for all j, then go to the fourth step.

Step Fourth: Determine the allocated fixed cost.

For each DMU_k, as the unit under evaluation, let the optimal allocation plan follows.

$r_j^k = \frac{\tilde{r}_j^{k^*}(p+1)}{\hat{v}_{m+1}^{k}{}^*(p+1)}$, $j = 1,..,n$, we select the average of this allocations, that is

$\bar{r}_j = \frac{1}{n}\sum_{k=1}^{n} \frac{\tilde{r}_j^{k^*}(p+1)}{\hat{v}_{m+1}^{k}{}^*(p+1)}$, $j = 1,..,n$, as the final amount of fixed cost R which is distributed to DMU_j, $j = 1,..,n$.

In the next steps, we solve the model (10) for DMU_k and get the semi-additive cross-efficiency score of all DMUs based on equation (12). This process continues until the termination condition of the algorithm is met. We consider the termination condition of the algorithm as $\left|E_j^{SA^{new}} - E_j^{SA^{old}}\right| \leq \epsilon$, this implies that the semi-additive cross-efficiency score for each of DMU_j, $j = 1,..,n$, does not improve. At the end of the proposed algorithm, we obtain a fair allocation of a fixed cost among all DMUs. We now present the proposed algorithm as follows.

Theorem 1. Model (9) is always feasible for each DMU_k**.**

Proof. At first, we show that when $p = 1$, or equivalently, when $E_j^{SA} = E_j^{SA}(1) = \theta_{j-co}^{SAF\,*}$, model (9) is feasible. Suppose $u_r^{k^*}$, $u_k{}^*$, $w_j^{k^*}$, $v_i^{k^*}$, $v_{m+1}^k{}^*$, $\tilde{r}_j^{k^*}$, be an optimal solution to the semi-additive fractional model (8) in evaluation DMU_k. In this case, we have

$\theta_k^{SAF^*} = \frac{\sum_{r=1}^s u_r^{k^*} y_{rj} + u_k{}^* - w_j^{k^*}}{\sum_{i=1}^m v_i^{k^*} x_{ij}}$, $k = 1,..,n$. Now we put $v_i^* = \min_{1\leq k\leq n}\{v_i^{k^*}\}$, $u_r^* = \max_{1\leq k\leq n}\{u_r^{k^*}\}$, $u^* = \max_{1\leq k\leq n}\{u_k{}^*\}$, $w_j^* = \min_{1\leq k\leq n}\{w_j^{k^*}\}$, Then we have $\frac{\sum_{r=1}^s u_r^* y_{rj} + u^* - w_j^*}{\sum_{i=1}^m v_i^* x_{ij}} \geq \theta_j^{SAF^*}$, therefore $\frac{1}{\theta_j^{SAF^*}}\left(\sum_{r=1}^s u_r^* y_{rj} + u^* - w_j^*\right) - \sum_{i=1}^m v_i^* x_{ij} \geq 0$, $j = 1,..,n$. If we put $r_j^k = \frac{R}{n}$, $v_i^k = v_i^*$, $u_r^k = u_r^*$, $w_j^k = w_j^*$, and v_{m+1}^k be any value between $\max\left\{\frac{n}{R}\max_{1\leq j\leq n}\left\{\left(\sum_{r=1}^s u_r^* y_{rj} + u^* - w_j^*\right), 0\right\}\right\}$, $\frac{n}{R}\min\left\{\frac{1}{\theta_j^{SAF^*}}\left(\sum_{r=1}^s u_r^* y_{rj} + u^* - w_j^*\right) - \sum_{i=1}^m v_i^* x_{ij}\right\}$, then

$\sum_{j=1}^n r_j^k = R$, $\theta_j^{SAF^*} \leq \frac{\sum_{r=1}^s u_r^k y_{rj} + u_k - w_j^k}{\sum_{i=1}^m v_i^k x_{ij} + v_{m+1}^k r_j^k} \leq 1$. This indicates that model (9) is feasible when $E_j^{SA} = E_j^{SA}(1)$. Now, we prove that if $E_j^{SA} = E_j^{SA}(p)$, and model (9) be feasible then when $E_j^{SA} = E_j^{SA}(p+1)$, model (9) is feasible also. Suppose $\{u_r^{k^*}(p), u_k{}^*(p), w_j^{k^*}(p), v_i^{k^*}(p), v_{m+1}^k{}^*(p), r_j^{k^*}(p)\}$ show an optimal solution to model (9) for DMU_k when $E_j^{SA} = E_j^{SA}(p)$, then we have $E_j^{SA}(p+1) = \frac{1}{n}\sum_{k=1}^n \frac{\sum_{r=1}^s u_r^{k^*}(p) y_{rj} + u_k{}^*(p) - \sum_{j=1}^n w_j^{k^*}(p)}{\sum_{i=1}^m v_i^{k^*}(p) x_{ij} + v_{m+1}^k{}^* r_j^{k^*}(p)}$. Put $v_i^* = \min_{1\leq k\leq n}\{v_i^{k^*}(p)\}$, $u_r^* = \max_{1\leq k\leq n}\{u_r^{k^*}(p)\}$, $u^* = \max_{1\leq k\leq n}\{u_k{}^*(p)\}$, $w_j^* = \min_{1\leq k\leq n}\{w_j^{k^*}(p)\}$. Then we have $\frac{\sum_{r=1}^s u_r^* y_{rj} + u^* - w_j^*}{\sum_{i=1}^m v_i^* x_{ij}} \geq E_j^{SA}(p+1)$, therefore

$\frac{1}{E_j^{SA}(p+1)}\left(\sum_{r=1}^s u_r^* y_{rj} + u^* - w_j^*\right) - \sum_{i=1}^m v_i^* x_{ij} \geq 0$, $j = 1,..,n$. If we put $r_j^k = \frac{R}{n}$, $v_i^k = v_i^*$, $u_r^k = u_r^*$, $u_k = u^*$, $w_j^k = w_j^*$, and v_{m+1}^k be any value between

$\max\left\{\frac{n}{R}\max_{1\leq j\leq n}\left\{\left(\sum_{r=1}^s u_r^* y_{rj} + u^* - w_j^*\right), 0\right\}\right\}$, $\frac{n}{R}\min\left\{\frac{1}{E_j^{SA}(p+1)}\left(\sum_{r=1}^s u_r^* y_{rj} + u^* - w_j^*\right) - \sum_{i=1}^m v_i^* x_{ij}\right\}$, then $\sum_{j=1}^n r_j^k = R$, $E_j^{SA}(p+1) \leq \frac{\sum_{r=1}^s u_r^k y_{rj} + u_k - w_j^k}{\sum_{i=1}^m v_i^k x_{ij} + v_{m+1}^k r_j^k} \leq 1$, $j = 1,..,n$, it show that

$\{u_r^k, u_k, w_j^k, v_i^k, v_{m+1}^k, r_j^k\}$, is a feasible solution to model (9) when $E_j^{SA} = E_j^{SA}(p+1)$. Therefore, model (9) is always feasible for any DMU and the proof is complete. ■

Theorem 2. For each, $j = 1,..,n$, then $E_j^{SA}(p)$ are non-decreasing for $p = 1,..,n$, and $\theta_j^{SAF^*} \leq E_j^{SA}(p) \leq 1$, that $\theta_j^{SAF^*}$ is semi-additive efficiency score corresponding to DMU_j, $j = 1,..,n$.

Proof. Suppose $\{u_r^{k^*}(p), u_k{}^*(p), w_j^{k^*}(p), v_i^{k^*}(p), v_{m+1}^k{}^*(p), r_j^{k^*}(p)\}$ show an optimal solution to model (9) for DMU_k when $E_j^{SA} = E_j^{SA}(p)$, then, we have

$E_j^{SA}(p) \leq E_j^{SA}(p+1) = \frac{1}{n}\sum_{k=1}^{n} \frac{\sum_{r=1}^{s} u_r^{k^*}(p) y_{rj} + u_k^*(p) - \sum_{j=1}^{n} w_j^{k^*}(p)}{\sum_{i=1}^{m} v_i^{k^*}(p) x_{ij} + v_{m+1}^{k}{}^{*} r_j^{k^*}(p)} \leq 1$. Thus, for any $j = 1,..,n$, $E_j^{SA}(p)$, $p = 1,..,n$, are non-decreasing, and satisfy $\theta_j^{SAF^*} \leq E_j^{SA}(p) \leq 1$, and the proof is complete. ■

Theorem 3. If we consider the fixed cost as an additional input, then for all $p \geq 1$, the self-evaluated efficiency, or the optimal objective of model (9) for each DMU_k, $k = 1,..,n$, is equals one, namely, $\theta_{k-co}^{SAF\,*} = 1$.

Proof. We consider the following program

$$
\begin{aligned}
E_k^{SA} = \max\ & \frac{\sum_{r=1}^{s} u_r^k y_{rk} + u_k - \sum_{j=1}^{n} w_j^k}{\sum_{i=1}^{m} v_i^k x_{ik}} \\
s.t.\ \ & E_j^{SA} \leq \frac{\sum_{r=1}^{s} u_r^k y_{rj} + u_k - w_j^k}{\sum_{i=1}^{m} v_i^k x_{ij}} \leq 1, \quad j = 1,..,n, \\
& v_i^k \geq \epsilon, u_r^k \geq \epsilon, \quad i = 1,\dots,m, \ \ r = 1,\dots,s, \\
& u_k \geq \epsilon,\ w_j^k \geq \epsilon, \quad j = 1,..,n.
\end{aligned} \tag{13}
$$

Suppose $u_r^{k^*}$, $u_k{}^*$, $w_j^{k^*}$, $v_i^{k^*}$, be an optimal solution to model (13) in evaluation DMU_k. Then we have $E_k^{SA} = \frac{\sum_{r=1}^{s} u_r^{k^*} y_{rj} + u_k{}^* - \sum_{j=1}^{n} w_j^{k^*}}{\sum_{i=1}^{m} v_i^{k^*} x_{ij}} \leq 1$, and $E_j^{SA} \leq \frac{\sum_{r=1}^{s} u_r^{k^*} y_{rj} + u_k{}^* - w_j^{k^*}}{\sum_{i=1}^{m} v_i^{k^*} x_{ij}} \leq 1$, $j = 1,..,n$. Also, we have $E_k^{SA} = \frac{\sum_{r=1}^{s} u_r^{k^*} y_{rj} + u_k{}^* - \sum_{j=1}^{n} w_j^{k^*}}{\sum_{i=1}^{m} v_i^{k^*} x_{ij}}$, and $\frac{1}{E_j^{SA}}\left(\sum_{r=1}^{s} u_r^{k^*} y_{rj} + u_k{}^* - w_j^{k^*}\right)$-$E_k^{SA}\left(\sum_{i=1}^{m} v_i^{k^*} x_{ij}\right) \geq \frac{1}{E_j^{SA}}\left(\sum_{r=1}^{s} u_r^{k^*} y_{rj} + u_k{}^* - w_j^{k^*}\right)$-$\sum_{i=1}^{m} v_i^{k^*} x_{ij} \geq 0$, for any $j = 1,..,n$. If we put $r_k^k = 0$, $r_j^k = \frac{R}{n-1}$, $u_r^k = u_k{}^*$, $v_i^k = E_k^{SA} v_i^{k^*}$, $u_k = u_k{}^*$, $w_j^k = w_j^{k^*}$, and v_{m+1}^k be any value between

$\max\left\{\frac{n-1}{R} \max_{1 \leq j \leq n} \left\{\left(\left(\sum_{r=1}^{s} u_k{}^* y_{rj} + u_k{}^* - w_j^{k^*}\right) - E_k^{SA}\left(\sum_{i=1}^{m} v_i^{k^*} x_{ij}\right)\right) - ,0\right\}\right\}$,

$\frac{n-1}{R} \min\left\{\frac{1}{E_j^{SA}}\left(\left(\sum_{r=1}^{s} u_k{}^* y_{rj} + u_k{}^* - w_j^{k^*}\right) - E_k^{SA}\left(\sum_{i=1}^{m} v_i^{k^*} x_{ij}\right)\right), 0\right\}$, then $\sum_{j=1}^{n} r_j^k = R$,

$\frac{\sum_{r=1}^{s} u_r^k y_{rk} + u_k - \sum_{j=1}^{n} w_j^{k^*}}{\sum_{i=1}^{m} v_i^k x_{ik} + v_{m+1}^k r_k^k} = 1$, $E_j^{SA} \leq \frac{\sum_{r=1}^{s} u_r^k y_{rj} + u_k - w_j^k}{\sum_{i=1}^{m} v_i^k x_{ij} + v_{m+1}^k r_j^k} \leq 1$, $j = 1,..,n$, it show that

$\{u_r^k, u_k, w_j^k, v_i^k, v_{m+1}^k, r_j^k\}$, is a feasible solution to model (9) that makes DMU_k has efficiency score of one. ■

When algorithm terminates, the cross-efficiency score corresponding to for any DMU_j, $j = 1,..,n$, is equal to one. Also, all efficient DMUs become efficient after the fixed cost allocation, in the event that we consider the amounts of fixed cost allocated as an additional input for all DMUs.

4 Numerical Example

In this section, we use the proposed approach in this paper for the data set in Table 2. This data includes 12 DMUs, each with three inputs and outputs. This data set was previously used in Cook and Kress [7], Beasley [4], and Cook and Zhu [8]. As in Beasley [4]. We allocate a fixed cost of 100 among the 12 DMUs. The last column of Table 2 shows the efficiency scores of each DMUs based on constant returns to scale, variable returns to scale, and semi-additive technologies in the input oriented. In this paper, considering that the proposed approach is based on a semi-additive technology, we examine the results of this model. As shown in the last column of Table 2, units 4, 5, 8, 9, 11, and 12 are efficient and the other units are inefficient. We now use the algorithm presented in section 3 to find an optimal fixed cost allocation. For this purpose, we set the value $\epsilon = 10^{-6}$. Table 3 shows the specific fixed cost values based on equation (11) in iteration 13. The last row of Table 3 shows the amount of fixed costs allocated to 12 DMUs. If we increase the number of iterations of the algorithm, we can obtain a fairer fixed cost allocation. Table 4 shows the specific fixed cost values based on equation (11) at the end of the algorithm after 55 iterations. As can be seen, the largest allocated cost is related to DMU 9 and the lowest fixed cost is related to DMUs 1 and 7, which is equal to zero. In Table 5, we compare the results of the proposed approach in this paper for the allocation of fixed costs with the results of other previous approaches that based on the constant returns to scale technology. But the proposed approach in this paper is presented in semi-additive technology. As previously stated, all 12 DMUs are efficient at the end of the algorithm, taking into account the specific fixed cost as a new input for all DMUs. As can be seen in semi-additive technology, we can obtain a different allocation scheme than the proposed previous approaches in this field. As can be seen in Table 5, units 9 and 11 have the same input, but based on the proposed approach in this paper, these units have different specific fixed cost amounts. This is a logical result because these two units have different output levels. But based on the approach provided by Cook and Kress [7], they have the same amount of specific fixed cost. We can have similar interpretations for DMUs 10 and 12. The proposed approach in this paper may not lead to a unique cost allocation scheme. However, if we want to obtain a unique fixed cost allocation scheme, we can impose additional restrictions on the fixed cost amount allocated to the DMUs in model (9). For example, we can put $r_1^k = 6.78$, $r_2^k = 7.21$, $r_3^k = 6.83$, $r_{10}^k = 10.08$. In this case, by solving model (9), the values of fixed costs specific to other DMUs are obtained as: $\tilde{r}_1 = 6.78$, $\tilde{r}_2 = 7.21$, $\tilde{r}_3 = 6.83$, $\tilde{r}_4 = 14.89$, $\tilde{r}_5 = 8.22$, $\tilde{r}_6 = 5.47$, $\tilde{r}_7 = 0$, $\tilde{r}_8 = 7.21$, $\tilde{r}_9 = 24.97$, $\tilde{r}_{10} = 10.08$, $\tilde{r}_{11} = 0.13$, $\tilde{r}_{12} = 8.21$, which are different from the results obtained from previous approaches as Beasley [4], and Cook and Zhu [8], Du et al. [10] based on models based on constant returns to scale technology.

Table 2: Numerical Example**.**

DMU	Input1	Input2	Input3	Output1	Output2	CCR efficiency	BCC efficiency	Semi efficiency
1	350	39	9	67	751	0.7567	0.8292	0.8292
2	298	26	8	73	611	0.923	0.9348	0.9348
3	422	31	7	75	584	0.747	0.7483	0.7483
4	281	16	9	70	665	1	1	1
5	301	16	6	75	445	1	1	1
6	360	29	17	83	1070	0.9612	1	0.9612
7	540	18	10	72	457	0.8604	0.8889	0.8889
8	276	33	5	78	590	1	1	1
9	323	25	5	75	1074	1	1	1
10	444	64	6	74	1072	0.8318	0.8333	0.8333
11	323	25	5	25	350	0.3333	1	1
12	444	64	6	104	1199	1	1	1

Table 3 shows the cross cost fixed allocation scores obtained on the step fourth of proposed algorithm in repetition 13. Also, last row of Table 3 shows the amount of fixed cost R=100 distributed to all DMUs obtained on the step fourth of proposed algorithm in repetition 13.

Table 3: The Cross Cost Fixed Allocation (In Repetition 13).

DMU	The cross cost fixed allocation											
1	0.000	7.293	3.616	15.567	12.824	3.903	0.000	8.365	37.119	1.482	1.104	8.728
2	0.000	6.794	2.746	16.225	11.693	6.624	0.000	6.676	38.990	1.305	0.973	7.975
3	0.000	6.800	2.730	16.254	11.690	6.701	0.000	6.656	39.008	1.269	0.946	7.946
4	0.000	6.781	2.780	16.166	11.698	6.471	0.000	6.714	38.955	1.378	1.027	8.032
5	0.000	6.833	2.642	16.408	11.678	7.100	0.000	6.557	39.099	1.080	0.805	7.798
6	0.000	6.686	3.026	15.732	11.733	5.345	0.000	6.994	38.697	1.912	1.425	8.452
7	0.000	6.712	2.888	16.245	11.583	7.276	0.816	6.254	38.665	1.063	0.792	7.707
8	0.000	6.798	2.734	16.248	11.691	6.685	0.000	6.660	39.004	1.277	0.952	7.953
9	0.000	6.685	3.029	15.728	11.734	5.334	0.000	6.996	38.694	1.917	1.429	8.455
10	0.000	6.685	3.028	15.727	11.732	5.337	0.000	6.994	38.695	1.918	1.427	8.457
11	0.000	6.345	2.586	15.870	10.882	7.485	0.000	6.050	39.522	2.163	0.000	9.096
12	6.473	12.522	6.373	14.289	16.242	0.000	0.000	15.245	14.444	0.000	14.413	0.000
Mean	0.539	7.244	3.181	15.871	12.098	5.688	0.068	7.513	36.741	1.397	2.107	7.550

Table 4 shows the final amount of fixed cost R=100 distributed to all DMUs obtained on the step fourth of proposed algorithm in final repetition. Also, last row of Table 3 shows the optimal amount of fixed cost R=100 distributed to all DMUs obtained on the step fourth of proposed algorithm when the algorithm stops.

Table 4: The Optimal Cross Cost Fixed Allocation (At the End of the Algorithm).

DMU	The optimal cross cost fixed allocation											
1	0.000	6.926	3.149	15.867	12.179	4.744	0.000	7.376	38.336	1.517	2.018	7.889
2	0.000	6.173	2.701	15.498	10.686	7.017	0.000	6.067	39.466	2.735	0.000	9.659
3	0.000	6.172	2.701	15.498	10.686	7.016	0.000	6.067	39.466	2.736	0.000	9.660
4	0.000	6.173	2.701	15.499	10.686	7.017	0.000	6.067	39.465	2.734	0.000	9.658
5	0.000	6.171	2.701	15.496	10.685	7.014	0.000	6.066	39.466	2.738	0.000	9.662
6	0.000	6.176	2.699	15.504	10.689	7.023	0.000	6.067	39.465	2.726	0.000	9.651
7	0.000	6.171	2.701	15.496	10.685	7.014	0.000	6.066	39.466	2.738	0.000	9.662
8	0.000	7.032	2.145	17.287	11.602	9.409	0.000	5.993	39.587	0.000	0.000	6.945
9	0.000	7.032	2.146	17.287	11.603	9.408	0.000	5.994	39.586	0.000	0.000	6.946
10	0.000	7.031	2.147	17.284	11.601	9.404	0.000	5.994	39.586	0.004	0.000	6.950
11	0.000	7.032	2.145	17.287	11.601	9.410	0.000	5.992	39.587	0.000	0.000	6.945
12	0.000	7.406	2.489	17.678	12.404	8.401	0.000	6.715	40.403	0.000	0.793	3.712
Mean	0.000	6.624	2.535	16.307	11.259	7.740	0.000	6.205	39.490	1.494	0.234	8.111

Table 5: Fixed Cost Allocation Results for the Different Approaches.

DMU	Our approach	Feng et al. [12]	cook and kress [7]	beasly [4]	cook and zhu [8]	Du et al. [10]	Li et al. [21]
1	0	8.09	14.52	6.78	11.22	5.79	5.54
2	6.624	7.36	6.74	7.21	0	7.95	7.53
3	2.535	8.91	9.32	6.83	16.95	6.54	7.35
4	16.307	6.94	5.6	8.47	0	11.1	7.87
5	11.259	7.13	5.79	7.08	0	8.69	6.38
6	7.74	9	8.15	10.06	15.43	13.49	11.5
7	0	9.77	8.86	5.09	0	7.1	5.9
8	6.205	7.41	6.26	7.74	0	6.83	7.77
9	39.49	8.2	7.31	15.11	17.62	16.68	11.9
10	1.494	10.21	10.08	10.08	21.15	5.42	11.38
11	0.234	5.46	7.31	1.58	17.62	0	2.74
12	8.111	11.52	10.08	13.97	0	10.41	14.14

5 Empirical Study

In this section, we apply the proposed approach to data set was previously used in Li, et al. [21]. This data is involving a city commercial bank in Chengdu, Sichuan Province of China, which has 18 branches. As stated in Li, et al. [21], the city commercial bank was charged 29 million CNY for information and technology maintenance. The headquarters of that city commercial bank intends to distribute the total maintenance charge among its 18 branches. We consider each branch is considered as a homogeneous and independent DMU, and the total fixed cost is R = 2900 units (1 unit = 10 thousand CNY). Li et al. [21] consider three inputs and three outputs.

The input and output variables have come in Table 6. Li et al. [21] consider the inputs as: staff (x1), that refers to human resource investment and manpower; Second input is fixed assets (x2), referring to the asset value of physical capital that can be used for business activities; Third input is operation costs (x3), that refers to the costs generated during the bank operations other than the labor costs. Outputs include deposit operating amount (y1), loan operating amount (y2) (i.e. total score of loans given by the bank), and revenue income (y3), that takes into account both interest income and non-interest income.

Table 6: Input and Output Variables Regarding Branches (Li et al. [21]).

Input/output	**Variable**	**Unit**
Input	Staffs (x1)	Person
	Fixed assets (x2)	10 thousand CNY
	Operation costs except for the labor costs (x3)	10 thousand CNY
Output	Deposits (y1)	10 thousand CNY
	Loans (y2)	10 thousand CNY
	Income (y3)	10 thousand CNY

The values of inputs and outputs of city commercial bank branches are given in Table 7.

Table 7: The Dataset of 18 Branches of the City Commercial Bank (Li et al. [21]).

DMU	**I1**	**I2**	**I3**	**O1**	**O2**	**O3**	**CCR efficiency**	**BCC efficiency**	**Semi efficiency**
1	62	1822	1361	140117	130288	5260	1	1	1
2	80	1833	1565	213774	145761	10773	1	1	1
3	129	3595	1378	194084	130556	8006	0.6245	0.6249	0.6249
4	62	1978	333	87876	49454	4479	0.8173	1	1
5	89	2138	549	107091	60872	5897	0.6766	0.697	0.697
6	84	1910	704	97472	94310	3849	0.6672	0.8112	0.8112
7	36	1234	840	114001	80019	5292	1	1	1
8	172	4348	959	366423	306926	12479	1	1	1
9	62	879	1253	107393	86485	5132	1	1	1
10	53	2566	483	69691	43907	3869	0.5483	0.7478	0.7478
11	92	1348	419	148458	87193	7234	1	1	1
12	39	1229	513	83752	40046	3984	0.8343	1	1
13	144	4640	1323	223539	211466	10655	0.7922	0.8177	0.7922
14	47	2248	670	70555	65110	2205	0.6931	0.7967	0.7967
15	39	1571	362	99143	66736	5271	1	1	1
16	56	1635	669	112513	79366	5202	0.814	0.873	0.873
17	34	939	867	87660	56157	3000	0.8962	1	1
18	58	1807	419	88334	67160	4171	0.6994	0.8668	0.8668

Based on the data in Table 7, we have a series of CCR efficiencies through CCR model, as shown in the eighth column of Table 7. We can learn from Table 7 that seven bank branches are identified as efficient (1, 2, 7, 8, 9, 11, and 15), while the other eleven bank branches are inefficient. Based on the BCC model, as shown in the ninth column of Table 7. We can learn from Table 7 that ten bank branches are identified as efficient (1,2, 4, 7, 8, 9, 11, 12, 15, and 17), while the other eight bank branches are inefficient. In this paper, given that the proposed approach is based on a semi-additive technology, we consider the results in this technology. The results are similar to the results of BCC model, except that the efficiency score for the thirteen units of the two models are different. The efficiency scores for unit thirteen in technologies variable returns to scale and semi-additive are equal to 0.8177, 0.7922, respectively. Now, we apply the algorithm presented in section 3 to find an optimal fixed cost allocation. For this purpose, we set the value $\epsilon = 10^{-6}$. For comparison purposes, the fixed cost allocation scheme from Li et al. [20] are also listed in in Table 8. We have given the corresponding rank for each unit based on the fixed cost allocated to each DMU based on both approaches.
From Table 8, the fixed cost allocation scheme obtained from the proposed approach is feasible as each DMU is allocated a positive fixed cost. DMUs 8, 13 and 2 have the best rankings based on the approach presented in this paper, respectively. DMUs 8, 13 and 2 have the best rankings based on the approach presented in this paper and the proposed approach provided by Li et al. [21], respectively. The results of Li et al. [21] based on the constant returns to scale technology, but the proposed approach in this paper is presented in semi-additive technology.

Table 8: Fixed Cost Allocation Results for the Different Approaches for Bank Branches.

DMU	**Our approach**	**Rank**	**Li et al. [21]**	**Rank**	**DMU**	**Our approach**	**Rank**	**Li et al. [21]**	**Rank**
1	132.297	8	157.41	6	10	0.000	16	80.18	17
2	297.140	3	287.63	3	11	248.044	4	197.79	5
3	49.738	13	202.67	4	12	74.160	11	96.96	15
4	91.821	10	107.2	14	13	304.743	2	293.7	2
5	44.387	14	137.98	10	14	0.000	16	57.24	18
6	58.190	12	110.81	12	15	227.663	5	140.81	9
7	175.690	6	144.92	7	16	151.588	7	140.21	8
8	896.739	1	416.6	1	17	28.476	15	81.78	16
9	0.000	16	135.6	11	18	119.322	9	110.52	13

Table 9 shows the final amount of fixed cost R=2900 distributed to all DMUs obtained on the step fourth of proposed algorithm in final repetition. Also, last row of Table 9 show the optimal amount of fixed cost R=2900 distributed to all DMUs obtained on the step fourth of proposed algorithm when the algorithm stops.

6 Conclusion

In this paper, we first introduce semi-additive technology by introducing the underlying assumptions for estimating the frontier of PPS. Also, we presented the performance evaluation model in semi-additive technology in the form of fractions and multiples. In the following, we present an interactive process for fixed cost allocation based on the concept of cross-efficiency in DEA. The proposed approach is always feasible. Cross-efficiency scores in semi-additive technology corresponding to all DMUs are improved at each stage of the interactive process, and at the end of the algorithm, if the fixed cost amount assigned to each DMUs is added as a new input to all DMU, these units are efficient in semi-additive technology. Although the proposed approach may not lead to a unique cost allocation plan, however, by imposing constraints on the corresponding cost amounts of some DMUs, we can obtain a unique cost allocation plan corresponding to each of the DMUs. The proposed algorithm is convergent in the interactive process. Due to the linearization of models, it can be easily solved with conventional optimization software. As future work, we can use the proposed new models in semi-additive technology to determine the returns to scale class of DMUs. We can also obtain a fixed cost allocation scheme for new aggregate DMUs and extend the proposed approach to other data structures in DEA, such as a two-stage network structure.

Table 9: The Optimal Cross Cost Fixed Allocation to Bank Branches (At the End of the Algorithm).

DMU	**The optimal cross cost fixed allocation**																	
1	149.389	350.307	0.000	79.546	0.000	0.000	231.281	867.168	0.000	0.000	229.771	103.156	265.833	0.000	273.084	172.090	62.102	116.276
2	114.921	317.141	0.000	110.324	26.684	0.000	235.252	831.001	0.000	0.000	282.202	147.639	142.309	0.000	278.826	183.071	102.417	128.213
3	98.580	287.634	0.000	122.801	44.654	17.735	222.985	826.587	0.000	0.000	305.960	152.572	122.348	0.000	275.083	183.261	102.581	137.222
4	149.388	350.307	0.000	79.546	0.000	0.000	231.281	867.169	0.000	0.000	229.771	103.156	265.833	0.000	273.084	172.090	62.102	116.276
5	149.389	350.307	0.000	79.545	0.000	0.000	231.281	867.169	0.000	0.000	229.770	103.155	265.835	0.000	273.084	172.090	62.101	116.275
6	149.388	350.307	0.000	79.546	0.000	0.000	231.281	867.168	0.000	0.000	229.772	103.157	265.830	0.000	273.084	172.090	62.103	116.276
7	166.812	345.133	0.000	65.275	0.000	35.067	133.148	928.886	0.000	0.000	212.838	56.328	397.675	0.000	265.054	162.266	10.340	121.180
8	168.124	376.474	0.000	80.433	0.000	13.535	240.767	698.785	0.000	0.000	241.709	93.375	338.092	0.000	291.940	182.134	47.116	127.517
9	122.053	244.134	113.481	93.987	101.446	130.447	116.531	837.253	0.000	0.000	280.860	44.191	377.743	0.000	172.814	136.686	1.712	126.662
10	116.011	310.044	63.089	100.726	67.180	48.874	170.980	810.774	0.000	0.000	240.448	69.738	379.503	0.000	241.481	154.270	0.000	126.882
11	167.111	342.529	26.400	71.871	16.724	55.259	196.391	988.052	0.000	0.000	0.000	49.309	435.637	0.000	259.841	163.390	0.000	127.487
12	165.073	335.341	12.952	62.486	5.049	43.796	194.138	928.903	0.000	0.000	208.925	0.000	416.916	0.000	252.255	155.653	0.000	118.512
13	156.248	328.616	61.118	90.818	47.481	81.614	183.226	1054.559	0.000	0.000	277.546	55.060	0.000	0.000	255.206	168.711	0.000	139.796
14	141.283	196.042	101.394	82.497	57.348	133.962	111.779	968.156	0.000	0.000	239.804	27.146	405.592	0.000	173.799	131.805	0.000	129.393
15	94.660	220.417	130.058	115.328	110.073	125.912	109.990	972.707	0.000	0.000	320.075	56.227	364.348	0.000	0.000	142.920	0.000	137.285
16	93.279	217.201	128.156	113.642	108.463	124.070	108.385	958.499	0.000	0.000	315.397	55.406	359.028	0.000	183.195	0.000	0.000	135.280
17	86.545	209.798	130.722	110.987	105.613	113.308	105.553	911.804	0.000	0.000	305.158	59.959	324.521	0.000	173.266	135.498	0.000	127.268
18	93.100	216.783	127.913	113.426	108.257	123.835	108.176	956.667	0.000	0.000	314.796	55.300	358.340	0.000	182.844	140.563	0.000	0.000
Mean	132.297	297.140	49.738	91.821	44.387	58.190	175.690	896.739	0.000	0.000	248.044	74.160	304.743	0.000	227.663	151.588	28.476	119.322

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