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Transversal spaces and common fixed point Theorem

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Abstract

In this paper we formulate and prove some fixed and common fixed point Theorems for self-mappings defined on complete lower Transversal functional probabilistic spaces.

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1 Introduction

Let T is self-mapping on X. The mapping T has a fixed point if there exist $x_0 \in X$ such that $Tx_0 = x_0$. Transversal spaces were introduced by M. R. Tascović in [5]. Some of the first results in fixed point Theory for mappings defined on Transversal functional probabilistic spaces are given in [2,5]. Lower Transversal functional probabilistic spaces as a natural extension of metric spaces and Fuzzy metric spaces were introduced by S. N. gesić, M. R. Tasković and N.A.Babacev in [3]. They also studied some fixed and common fixed point Theorems for compatible mappings defined on complete lower Transversal functional probabilistic spaces. In this paper, we investigate some fixed and common fixed point Theorem for semi-compatible mappings defined on complete lower Transversal functional probabilistic spaces. First, we recall some definitions and examples.

Definition 1 *[3] Let X be a nonempty set. The symmetric function $\rho : X \times X \times [0, \infty) \to [0, 1]$ is called a lower functional probabilistic Transverse on X, if there exists a function $d : [0, 1] \times [0, 1] \to [0, 1]$, called a lower probabilistic bisection function, such that*

$$\rho(p, q)(x) \geq \min\{\rho(p, s)(x), \rho(s, q)(x), d(\rho(p, s)(x), \rho(s, q)(x))\}, \quad (1.1)$$

for all $p, q, s \in X$, and for each $x \in [0, \infty)$. The triple (X, ρ, d) is called a lower Transversal functional probabilistic space.

Example 1.1 *[3] Every metric space (X, δ) can be considered as a lower Transversal functional probabilistic space (X, ρ, d) with the lower probabilistic bisection function $d(a, b) = \min\{a, b\}$, and the lower functional Transverse $\rho(p, q)(x) = \frac{\theta(x)}{\theta(x)+\delta(p,q)}$ where $\theta : [0, \infty) \to [0, \infty)$ and $\theta(0) = 0$ is a bijection function such that, $\lim_{x\to+\infty} \theta(x) = +\infty$. The triple (X, ρ, d) is said to be a lower Transversal functional probabilistic space induced by the metric δ.*

Before we give another example, first we introduce Fuzzy metric spaces.

Definition 2 *[4] A binary operation $* : [0, 1] \times [0, 1] \to [0, 1]$ is a con-*

tinuous t-norm if $*$ satisfies the following conditions

(a) $*$ *is commutative and associative,*
(b) $*$ *is continuous,*
(c) $a * 1 = a$ *for all* $a \in [0,1]$,
(d) $a * b \leq c * d$ *whenever* $a \leq c$ *and* $b \leq d$, *and* $a, b, c, d \in [0,1]$.

Examples of t-norm are $a * b = \min\{a, b\}$ and $a * b = ab$.

Definition 3 *[4] A 3-tuple $(X, M, *)$ is said to be a Fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm and M is a Fuzzy set on $X^2 \times (0, \infty)$ satisfying the following conditions*

(Fm1) $M(x, y, t) > 0$,
(Fm2) $M(x, y, t) = 1$ *if and only if* $x = y$,
(Fm3) $M(x, y, t) = M(y, x, t)$
(Fm4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,
(Fm5) $M(x, y, t) : (0, \infty) \to (0, 1]$ *is continuous,*

for all $x, y, z \in X$ and $s, t > 0$. Then M is called a Fuzzy metric on X.

Examples of Fuzzy metric spaces introduced by A. George and P. Veeramani [1]. Note that by [3] every Fuzzy metric space can be considered as a lower Transversal functional probabilistic space. In this case the lower functional probabilistic Transverse is defined as $\rho(p, q)(x) = M(p, q, x)$, and the lower probabilistic bisection function is defined with

$$d(\rho(p, s)(x), \rho(s, q)(x)) = \rho(p, s)(\frac{x}{2}) * (\rho(s, q)(\frac{x}{2}).$$

The inequality that defines the lower Transversal functional probabilistic space follows from the next inequalities,

$$\begin{aligned} \rho(p, q)(x) = M(p, q, x) &\geq M(p, s, \frac{x}{2}) * M(s, q, \frac{x}{2}) \\ &= d(\rho(p, s)(x), \rho(s, q)(x)) \\ &\geq \min\{\rho(p, s)(x), \rho(s, q)(x), d(\rho(p, s)(x), \rho(s, q)(x))\}, \end{aligned}$$

for all $p, q, s \in X$ and all $x > 0$.

2 Preliminaries

The following definitions and lemmas play an important role in the proof of main results.

Definition 4 *[3] Let (X, ρ, d) be a lower Transversal functional probabilistic space.*

(a) A sequence (p_n) in (X, ρ, d) converges to a point $p \in X$, if for each $x > 0$ and each $\lambda \in (0, 1)$ there exists an integer n_0 such that

$$\rho(p, p_n)(x) > 1 - \lambda,$$

for all $n \geq n_0$.

(b) A sequence (p_n) is said to be a Cauchy sequence if for each $x > 0$ and each $\lambda \in (0, 1)$ there exists an integer n_0 such that,

$$\rho(p_m, p_n)(x) > 1 - \lambda,$$

for all $n, m \geq n_0$.

(c) A lower Transversal functional probabilistic space in which every Cauchy sequence is convergent is said to be complete.

Throughout this paper, we consider lower Transversal functional probabilistic spaces with the lower functional probabilistic Transverse ρ which satisfies the following conditions

(T1) $\rho(p, q)(x)$ is a left-continuous function for $x \in (0, \infty)$ and right-continuous at the point $x = 0$,
(T2) $\rho(p, q)(x) = 1$ for all $x > 0$ if and only if $p = q$,
(T3) $\rho(p, q)(x)$ is a non-decreasing function,
(T4) $\lim_{x \to +\infty} \rho(p, q)(x) = 1$, for all $p, q \in X$,
(T5) $\rho(p, q)(x) = \rho(q, p)(x)$.

Also, we assume that the lower probabilistic bisection function $d(x, y)$ satisfies the following conditions

(B1) $d(x, y)$ is a non-decreasing and continuous function,

(B2) $d(x,x) \geq x$,
(B3) $\lim_{x\to 1} d(a,x) = a$.

Lemma 5 *[3] Let (X,ρ,d) be a lower Transversal functional probabilistic space, with the lower functional probabilistic Transverse satisfying $(T1)-(T4)$ and lower bisection function satisfying $(B1)-(B3)$. If*

$$\liminf_{n\to+\infty} p_n = p, \quad \liminf_{n\to+\infty} q_n = q,$$

then

$$\liminf_{n\to+\infty} \rho(p_n,q_n)(x) = \rho(p,q)(x).$$

Definition 6 *Two self-mappings S and T defined on a lower Transversal functional probabilistic space (X,ρ,d) are said to be semi-compatible if*

$$\lim_{n\to+\infty} \rho(ASy_n, Sy)(x) = 1,$$

for all $x > 0$, whenever (y_n) is a sequence in X such that

$$\lim_{n\to+\infty} Ay_n = \lim_{n\to+\infty} Sy_n = y.$$

Lemma 7 *[3] Let (X,ρ,d) be a lower Transversal functional probabilistic space with the lower functional probabilistic Transverse satisfying $(T1)-(T4)$. Let $\varphi : (0,+\infty) \to (0,+\infty)$ be a continuous, non-decreasing function which satisfies $\varphi(x) < x$, for all $x > 0$. If for $p,q \in X$ it holds that $\rho(p,q)(\varphi(x)) \geq \rho(p,q)(x)$ for all $x > 0$, then $p = q$.*

Definition 8 *Two self-mappings S and T defined on a lower Transversal functional probabilistic space (X,ρ,d) are said to be weak compatible if they commute at their coincidence points, that is, $Tx = Sx$ implies that $TSx = STx$.*

3 Main results

Theorem 9 *Let A, B, S and T be self-mappings on complete lower Transversal functional probabilistic space (X,ρ,d), Satisfying the condition*

(a) $A(X) \subseteq T(X), B(X) \subset S(X)$,
(b) *the pair* (A,S) *is semi-compatible and* (B,T) *is weak compatible,*
(c) *one of* A *or* S *is continuous,*
(d) *there exists* $k \in (0,1)$ *such that*

$$\rho(Ap, Bq)(kx) \geq \rho(Sp, Tq)(x), \tag{3.1}$$

for all $x > 0$ *and* $p, q \in X$.

Then A, B, S *and* T *have a unique common fixed point.*

Proof. Let $x_0 \in X$ be any arbitrary point as $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$, there exist $x_1, x_2 \in X$ such that $Ax_0 = Tx_1$, $Bx_1 = Sx_2$. Inductively, construct sequences (x_n) and (y_n) in X such that

$$y_{2n+1} = Ax_{2n} = Tx_{2n+1}, \quad y_{2n+2} = Bx_{2n+1} = Sx_{2n+2},$$

for $n = 0, 1, 2, ...$ Now using (3.1) with $p = x_{2n}, q = x_{2n+1}$, we get

$$\rho(Ax_{2n}, Bx_{2n+1})(kx) \geq \rho(Sx_{2n}, Tx_{2n+1})(x),$$

that is,

$$\rho(y_{2n+1}, y_{2n+2})(kx) \geq \rho(y_{2n}, y_{2n+1})(x).$$

Similarly, by putting $p = x_{2n+2}$ and $q = x_{2n+1}$ in (3.1), we have

$$\rho(y_{2n+3}, y_{2n+2})(kx) \geq \rho(y_{2n+2}, y_{2n+1})(x).$$

Thus, for any n and x, we have

$$\rho(y_{n+1}, y_n)(kx) \geq \rho(y_{n-1}, y_n)(x). \tag{3.2}$$

We show that (y_n) is a Cauchy sequence in X. Since

$$\rho(y_n, y_{n+p})(x) \geq \min\{\rho(y_n, y_{n+1})(x), \rho(y_{n+1}, y_{n+p})(x), d(\rho(y_n, y_{n+1})(x), \rho(y_{n+1}, y_{n+p})(x))$$

. If $\rho(y_n, y_{n+p})(x) \geqslant \rho(y_n, y_{n+1})(x)$ by (3.2) and $(T4)$, we get

$$\begin{aligned} \rho(y_n, y_{n+p})(x) &\geq \rho(y_n, y_{n+1})(x) \\ &\geq \rho(y_{n-1}, y_n)(\frac{x}{k}) \geq ... \geq \\ &\geq \rho(y_0, y_1)(\frac{x}{k^n}) \to 1 \end{aligned}$$

as $n \to \infty$. If
$$\rho(y_n, y_{n+p})(x) \geq \rho(y_{n+1}, y_{n+p})(x),$$
as mentioned in previous case we have $\rho(y_n, y_{n+p})(x) \to 1$ as $n \to \infty$. If $\rho(y_n, y_{n+p})(x) \geqslant d(\rho(y_n, y_{n+1})(x), \rho(y_{n+1}, y_{n+p})(x))$, using (3.2) and $(B1)$, we get

$$\begin{aligned}\rho(y_n, y_{n+p})(x) &\geq d(\rho(y_n, y_{n+1})(x), \rho(y_{n+1}, y_{n+p})(x)) \\ &\geq d(\rho(y_{n-1}, y_n)(\frac{x}{k}), \rho(y_n, y_{n+p-1})(\frac{x}{k})) \geq ... \geq \\ &\geq d(\rho(y_0, y_1)(\frac{x}{k^n}), \rho(y_1, y_p)(\frac{x}{k^n})).\end{aligned}$$

Using $(B2)$ and $(B3)$, we get $\rho(y_n, y_{n+1})(x) \to 1$ as $n \to \infty$. Hence, (y_n) is a Cauchy sequence in X, which is complete. Therefore (y_n) converges to $u \in X$. Its subsequences $(Ax_{2n}), (Bx_{2n+1}), (Sx_{2n}), (Tx_{2n+1})$ also converges to u, that is,

$$Ax_{2n} \to u, \quad Bx_{2n+1} \to u, \quad Sx_{2n} \to u, \quad Tx_{2n+1} \to u. \tag{3.3}$$

Case I (S is continuous). In this case, we have

$$SAx_{2n} \to Su, \quad S^2x_{2n} \to Su.$$

Also semi-compatibility of the pair (A, S) gives

$$\lim_{n\to\infty} ASx_{2n} = Su.$$

Step1. By putting $p = Sx_{2n}, q = x_{2n+1}$ in (3.1), we obtain that

$$\rho(ASx_{2n}, Bx_{2n+1})(Kx) \geq \rho(SSx_{2n}, Tx_{2n+1})(x),$$

for all $x > 0$. By taking $\liminf$ from two side of previous inequality as $n \to \infty$, also using Lemma 2.2, we have

$$\rho(Su, u)(kx) \geq \rho(Su, u)(x),$$

for all $x > 0$. By Lemma 2.4, we have

$$Su = u.$$

Step2. By putting $p = u$ and $q = x_{2n+1}$ in (3.1), we have

$$\rho(Au, Bx_{2n+1})(kx) \geq \rho(Su, Tx_{2n+1})(x)$$

for all $x > 0$. By taking $\liminf$ from two side of previous inequality as $n \to \infty$ also using lemma 2.2 and $(T2)$, we get

$$\rho(Au, u)(kx) \geq \rho(Su, u)(x) = \rho(u, u)(x) = 1$$

for all $x > 0$, which gives $u = Au$. Hence

$$Au = u = Su.$$

Step3. As $A(X) \subseteq T(X)$, there exists $w \in X$ such that

$$Au = Su = u = Tw.$$

By putting $p = x_{2n}, q = w$ in (3.1), we obtain that

$$\rho(Ax_{2n}, Bw)(kx) \geq \rho(Sx_{2n}, Tw)(x),$$

for all $x > 0$. By taking $\liminf$ from two side of previous inequality, using lemma 2.2 and $(T2)$, we get

$$\rho(u, Bw)(kx) \geq \rho(u, Tw)(x) = \rho(u, u)(x) = 1,$$

hence we have $u = Bw$. Therefore $Bw = Tw = u$. Since (B, T) is weak compatible, we get that $TBw = BTw$. that is,

$$Bu = Tu.$$

By putting $x = u, y = u$ in (3.1) and using lemma 2.4, we obtain $u = Au = Su = Bu = Tu$, that is, u is a common fixed point of A, B, S and T.

Case II (A is continuous). In this case, using (3.2) we have

$$ASx_{2n} \to Au.$$

The semi-compatibility of pair (A, S) gives

$$ASx_{2n} \to Su.$$

By lemma 2.2 we have

$$1 = \liminf_{n\to\infty} \rho(ASx_{2n}, ASx_{2n})(x) = \rho(Au, Su)(x),$$

for all $x > 0$. Using T2 we obtain that $Au = Su$.

Step(4). By putting $p = u$ and $q = x_{2n+1}$ in (3.1), we obtain that

$$\rho(Au, Bx_{2n+1})(kx) \geq \rho(Su, Tx_{2n+1})(x),$$

for all $x > 0$. By taking lim inf from two side of previous inequality and using lemma 2.2, we get

$$\rho(Au, u)(kx) \geq \rho(Su, u)(x) = \rho(Au, u)(x),$$

for all $x > 0$. Using lemma 2.4, which gives $u = Au$ and the rest of the proof follows from step 3 onwards of the previous case.

Uniqueness. Let z be another common fixed point of A, B, S, and T. Then $z = Az = Bz = Sz = Tz$. Putting $p = u$ and $q = z$ in (3.1), we get

$$\rho(u, z)(kx) = \rho(Au, Bz)(kx) \geq \rho(Su, Tz)(x) = \rho(u, z)(x).$$

Using lemma 2.4 we have $u = z$. Therefore, u is the unique common fixed point of the self-maps A, B, S, and T. □

Corollary 10 *Let A be a self-mapping defined on a complete lower Transversal functional probabilistic space (X, ρ, d), and there exists $k \in (0, 1)$ such that*

$$\rho(Ap, Aq)(kx) \geqslant \rho(p, q)(x),$$

for all $x > 0$ and $p, q \in X$. Then A has a unique fixed point.

Proof. Taking that $A = B$ and $S = T = I$ identical mapping, all the conditions of theorem 3.1 are satisfied, i.e. the statement follows from theorem 3.1. □

Also, since Fuzzy metric spaces and metric spaces are lower Transversal functional probabilistic space, from Theorem 3.1 we get similar results for mappings defined on these spaces.

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