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Numerical Solution of a New Type Fuzzy Nonlinear Volterra Integral Equations

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Abstract

Fuzzy integral equations play a fundamental role in the many fields of engineering and applied mathematics. The paper presented, a new type of fuzzy Volterra integral equations of the second kind with nonlinear fuzzy kernels. Numerical solutions of a new type of nonlinear fuzzy Volterra integral equations with nonlinear fuzzy kernels through Variational Homotopy perturbation (VHP) method based on the parametric form of a fuzzy number,is investigated. To find the approximate solution and to get an approximation for fuzzy solution of the new type of nonlinear fuzzy Volterra integral equations the VHPM is applied and it is shown that VHPM is an effective and reliable approach to solve these equations. Finally, a few numerical examples are given and results unfold that VHPM is very close to exact solutions. The obtained approximate solutions are contrasted with the exact solution and absolute error between obtaining numerical results and an exact solution are found. One of the examples shows a comparison between VHPM and HPM.

1 Introduction

Studying of fuzzy integral equations is important in great topics for many problems in applied mathematics, such as physics, mechanics, geography, medical and fuzzy control. A large class of initial value and boundary differential equations can be converted to partial differential equations or Volterra integral equations. The numerical methods are usually applied for solving nonlinear equations. Some of these numerical methods have high accuracies such as Adomian decomposition method, Variational iteration method and Homotopy perturbation method. Homotopy perturbation method (HPM) is a technique that introduced and improved by He and used for nonlinear problems [11]. This method has a very high convergency rate in most cases, few iterations lead to approximate solutions with low errors [13–15]. Variational iteration method (VIM) is an analytical scheme to use it for nonlinear problems and give approximate solutions closed to the exact solutions [12, 16]. The Adomian decomposition method (ADM) was presented by Adomian that is a semi-analytical technique for solving nonlinear equations by using of series [1, 2]. Variational Homotopy Perturbation method (VHPM) which proffered bases on HPM and VIM by applying ADM in nonlinear terms [13, 14]. Some numerical methods used for solving fuzzy Volterra integral equations of the second kind studied by many researchers will be expressed in following:

Jafarian and et. al in [19], solved fuzzy Volterra integral equations by the Taylor expansion method in system terms. Variational iteration method and Homotopy perturbation method utilized to solve Volterra integral equations by Mirzaei, [25]. Nonlinear fuzzy Volterra integral equations of the second kind were solved with applying Nystorm techniques by Salehi and Nejatian in [30]. Fuzzy Bernstein polynomials are used to solve fuzzy Volterra integral equations of the second kind by Mosleh and Otadi in [23]. Fadravi et al. studied the solutions of fuzzy Fredholm integral equations using neural networks, [18] and Bica and Popescu [5, 6], applied successive

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approximations method to find the solution of fuzzy Hammerstein-Volterra integral equation. Hassan studied the numerical and analytical solution of fuzzy nonlinear integrals equations by the Homotopy analysis method, [10]. Narayanamoorthy and Sathiyapriya applied the Homotopy Perturbation method to approximate the solution of linear and nonlinear fuzzy Volterra integral equations of second kind, [26].

In all of the papers mentioned in above and in many others, the authors considered fuzzy integral equations with one integral that is special manner of fuzzy integral equations that is presented in this article. This new type of nonlinear fuzzy integral equations with nonlinear fuzzy kernels is appeared in transforming of differential equations of n-th order with fuzzy initial values [29]. The paper is organized in below: In section 2, some basic concepts reviewed briefly. In section 3, the new type of fuzzy integral equation with nonlinear kernels is introduced at the first time and the solution of it is found by used Variational Homotopy perturbation method. In 4, the examples are presented to illustrate the method more and the results are presented in figures and tables. One of the examples that is a special case of this new type of fuzzy integral equations is compared with HPM. Finally, in section 5 a conclusion is drawn.

2 Basic concepts

The basic definitions of a fuzzy number are given in following:

Definition 2.1. *[7] A fuzzy number is a fuzzy set like $u : \mathbb{R} \to [0,1]$ which satisfies:*

1. *u is an upper semi-continuous function,*
2. *$u(x) = 0$ outside some interval [a,d],*
3. *There are two real numbers b, c such as $a \le b \le c \le d$ and*

   3.a $u(x)$ *is a monotonic increasing function on [a, b],*

   3.b $u(x)$ *is a monotonic decreasing function on [c, d],*

   3.c $u(x) = 1$ *for all* $x \in [b, c]$.

Theorem 2.1. *[31] The metric space $(P_K(R^n), d)$ is separable and complete. Let $I = [c, d] \subset R$ is a compact interval and denote*

$$E^n = \{u : R^n \to [0,1] \mid u \ \ satisfies \ \ (i) - (iv) \ \ below\}$$

where

(1) $u\, is\, normal,\ i.e.\, there\, exists\, an\, x_0 \in R^n\, such\, that\, u(x_0) = 1,$

(2) $u\, is\, fuzzy\, convex$

(3) $u\, is\, upper\, semicontinuous,$

(4) $[u]^0 = cl\{x \in R^n | u(x) > 0\}\, is\, compact.$

For $0 < \alpha \le 1$ denote $[u]^\alpha = \{x \in R^n | u(x) \ge \alpha\}$, then from (1)-(4) it follows that the α-level set $[u]^\alpha \in P_k(R^n)$ for all $0 \le \alpha \le 1$.

Definition 2.2. *[32] If $\tilde{A} = \{(x, \mu_{\tilde{A}}(x))|x \in X\}$ is a fuzzy set on the $X \in \mathbb{R}$, the $\alpha - cut$ of subsets of $\tilde{A}$ is:*

$$\tilde{A}_\alpha = \{x \in X|\mu_{\tilde{A}}(x) \geq \alpha\}$$

that $\mu_{\tilde{A}} : X \to [0, 1]$ is named membership functions of $\tilde{A}$. For $\alpha = 1$, the 1-cut of $\tilde{A}$ is named core of $\tilde{A}$.

Definition 2.3. *[21] A mapping $F : [a, b] \to E^n$ is strongly measurable if for all $\alpha \in [0, 1]$ the set-valued mapping $F_\alpha : I \to P_K(R^n)$ defined by*

$$F_\alpha(t) = [F(t)]^\alpha$$

is Lebesque measurable, when $P_K(R^n)$ is dedicated with the topology generated by the Hausdorff metric d.

Definition 2.4. *[21] Let $F : I \to E^n$. The integral of F over I, denoted by $\int_I F(t)dt$, is defined level-wise by the equation*

$$(\int_I F(t)dt)^\alpha = \int_I F_\alpha(t)dt = \{f(t)dt \mid f : I \to R^n \; is \; a \; measurable \; selection \; for \; F_\alpha\}$$

for all $0 < \alpha \leq 1$. A strongly measurable and integrable bounded mapping $F : I \to E^n$ is integrable over I if $\int_I F(t)dt \in E^n$.

Theorem 2.2. *[31] If $f : [a, b] \to E^n$ is integrable and $c \in [a, b]$, $\lambda \in \mathbb{R}$. Thus:*
(i) $\int_{t_0}^{t_0+a} F(t)dt = \int_{t_0}^{c} F(t)dt + \int_{c}^{t_0+a} F(t)dt$,
(ii) $\int_I (F(t) + G(t))dt = \int_I F(t)dt + \int_I G(t)dt$,
(iii) $\int_I \lambda F(t)dt = \lambda \int_I F(t)dt$,
(iv) $D(F, G)$ is integrable,
(V) $D(\int_I F(t)dt, \int_I G(t)dt) \leq \int_I D(F, G)$

3 A New Type Nonlinear Fuzzy Volterra Integral Equations and VHPM Solution's

In this section, a new type of fuzzy Volterra integral equation with nonlinear fuzzy kernels are introduced and Variational Homotopy Perturbation Method is used to solve them. These nonlinear fuzzy integral equations can be appeared in transforming Nth-order fuzzy differential equations with fuzzy boundary values or fuzzy initial values or fuzzy Nth-order integro-differential equations with fuzzy nonlinear kernels and fuzzy boundary values or fuzzy initial values.
A new type of nonlinear fuzzy Volterra integral equations with nonlinear fuzzy kernels is defined in following:

$$u(t) = f(t) + \int_{a_0}^{t} g_1(t, s, u(s))ds + \int_{a_0}^{t}\int_{a_1}^{t_1} g_2(t, t_1, s, u(s))dsdt_1 ... + \underbrace{\int_{a_0}^{t} ... \int_{a_{n-1}}^{t_{n-1}}}_{n} g_n(t, t_1, ..., t_{n-1}, s, u(s))dsdt_{n-1}...dt_2dt_1 \tag{3.1}$$

Where $t, t_1, ..., t_{n-1} \geq 0$ and $u(t)$ is a function of t, $f(t)$ is a set-valued function and $g_1(t, s, u(s))$ and $g_2(t, t_1, s, u(s))$,..., $g_n(t, t_1, ..., t_{n-1}, s, u(s))$ are nonlinear fuzzy functions and all continuous and $a_0, ..., a_{n-1}$ are positive real numbers. Eq. (1) is named NC-NFVIE.
Now Variation Homotopy Perturbation method is considered for solving NC-NFVIE. For this, consider the following equation:

$$Lu + Nu = k(x) \tag{3.2}$$

where L and N are linear and nonlinear operators, respectively and $k(x)$ is the inhomogeneous term. The Variational iteration method, for Eq. (2) is got in following equation:

$$u_{i+1}(x) = u_i(x) + \int_0^x \lambda\{Lu_i(t) + N\hat{u}_i(t) - k(t)\}dt \tag{3.3}$$

which λ is the general Lagrangian multiplier that is identified by variational iteration method optimally. $\hat{u}_i$ is a restricted variation which means that $\delta\hat{u}_i = 0$. The successive approximations u_{i+1}, $i = 0, 1, 2, ...$, of the solution $u(x)$ will be readily obtained upon using Lagrangian multiplier determined and any selective function u_0. Consequently, the solution is given by following term:

$$u(x) = lim_{n\longrightarrow\infty}u_i(x)$$

The solution of Eq. (3) is considered as a fixed point of the following function under a suitable choice of first term u_0:

$$u_{i+1}(x) = u_i(x) + \int_0^x \lambda\{Lu_i(t) + Nu_i(t) - k(t)\}dx \tag{3.4}$$

Now, the variational iteration method is considered for NC-NFVIE then from Eqs. (1) and (4) the following term is got in below:

$$\widetilde{u}_{i+1}(x) = \widetilde{u}_i(x) + \int_0^x \lambda(t)\{\widetilde{u}_i(t) - f(t) - \int_{a_0}^t g_1(t, s, \widetilde{u}_i(s))ds - ... - \underbrace{\int_{a_0}^t ... \int_{a_{n-1}}^{t_{n-1}}}_{n} g_n(t, t_1, ..., t_{n-1}, s, \widetilde{u}_i(s))ds...dt_1\}dx \tag{3.5}$$

that $\delta(\widetilde{u})$ is a restricted variation.
Now the Homotpy perturbation method is applied for Eq. (3) as follows:

$$\sum_{n=0}^{\infty} P^i u_i(x) = u_0(x) + P\int_0^x \lambda(t)(L(\sum_{n=0}^{\infty} P^i u_i)dx) + N(\sum_{n=0}^{\infty} P^i u_i)dx) - \int_0^x \lambda(t)k(t)dx)$$

The Variational Homotopy perturbation Method for NC-NFVIE is in the following equation:

$$\sum_{n=0}^{\infty} P^i u_i(x) = u_0(x)+P\int_0^x \lambda(t)(\sum_{n=0}^{\infty} P^i u_i(x)-f(t)-\int_{a_0}^t g_1(t, s, \sum_{i=0}^{\infty} P^i u_i(x))ds-...-\underbrace{\int_{a_0}^t ... \int_{a_{n-1}}^{t_{n-1}}}_{n} g_n(t, s, \sum_{i=0}^{\infty} P^i u_i(x))ds...dt_1) \tag{3.6}$$

Indeed the variational homotopy perturbation method is formulated by variational iteration method and Homotopy perturbation method by Adomian's polynomials. The parameter $p \in (0, 1]$ is considered as an expanding factor that is obtained as follows:

$$f = \sum_{i=0}^{\infty} p^i u_i = u_0 + pu_1 + p^2 u_2 + ... \tag{3.7}$$

If $p \to 1$, then Eq. (7) becomes the approximate solution of the form

$$u = lim_{p\to 1} f = u_0 + u_1 + u_2 + ...$$

A comparison of like powers of p gives solutions of various orders.

4 Examples

In this section three examples are presented and solved by the Variational Homotopy Perturbation method, tables and figures are presented that approximation and exact solutions are compared with those.
Example (4.1) is solved by Homotopy perturbation method in [26]. This integral equation is a special case NC-NFVIE.

Example 4.1. *Consider the following fuzzy integral equation:*

$$u(t) = [(2-r)^2(\frac{t^6}{2}+t^5-t^3+\frac{11}{32}t^2)-\frac{11}{32}t^2+rt+r, r^2(\frac{t^6}{2}+t^5-t^3+\frac{11}{32}t^2)+(2-r)(\frac{-11}{32}(2-r)(t)^2+t+1)]+\int_0^t s^2(1-2t)u^2(s)ds$$

The optimal λ by solving is founded $\lambda=-1$. Now, by exerting the VHPM, it is possible to obtain an equation as follows:
$u_0+pu_1+p^2u_2+\ldots=[(2-r)^2(\frac{t^6}{2}+t^5-t^3+\frac{11}{32}t^2)-\frac{11}{32}t^2+rt+r, r^2(\frac{t^6}{2}+t^5-t^3+\frac{11}{32}t^2)+(2-r)(\frac{-11}{32}(2-r)(t)^2+t+1)]+\int_0^t s^2(1-2t)(u_0+pu_1+p^2u_2+\ldots)^2(s)ds$
The exact solution is $u(t)=[r(t+1),(2-r)(t+1)]$. Results for $x=0.01$ and $t=0.5$ by VHPM and HPM and compare between and absolute errors for $\underline{u}(t)$ and $\overline{u}(t)$ are shown in Table (1) and figure (1).

Example 4.2. *Consider the following fuzzy integral equation:*

$$u(t) = [2rt-\frac{t^6}{5}(2r)^2-\frac{t^6}{24}(2r)^3,(3-r)t-\frac{t^6}{5}(3-r)^2-\frac{t^6}{24}(3-r)^3]+\int_0^t s^2tu^2(s)ds+\int_0^t\int_0^y yu^3(s)dsdy$$

The optimal λ by solving is founded $\lambda=-1$. Now, by exerting the VHPM, it is possible to get an equation as follows:
$u_0+pu_1+p^2u_2+\ldots=[2rt-\frac{t^6}{5}(2r)^2-\frac{t^6}{24}(2r)^3,(3-r)t-\frac{t^6}{5}(3-r)^2-\frac{t^6}{24}(3-r)^3]-\frac{t^6}{24}(2r)^3+p(\int_0^x\int_0^t s^2t(u_0+pu_1+p^2u_2+\ldots)^2(s)ds+\int_0^x\int_0^t\int_0^y y(u_0+pu_1+p^2u_2+\ldots)^3)dsdydx.$
By comparing the terms with identical powers of p, we have the following results:
$p^0: u_0(t)=[2rt-\frac{t^6}{5}(2r)^2-\frac{t^6}{24}(2r)^3,(3-r)t-\frac{t^6}{5}(3-r)^2-\frac{t^6}{24}(3-r)^3]$
$p^1: u_1(t)=\int_0^x\int_0^t s^2t(u_0)^2(s)dsdx+\int_0^x\int_0^t\int_0^y y(u_0)^3(s)dsdydx$
$p^2: u_2(t)=\int_0^x\int_0^t s^2t(2u_0u_1)(s)dsdx+\int_0^x\int_0^t\int_0^y y(3u_0^2u_1)(s)dsdydx$
$p^2: u_3(t)=\int_0^x\int_0^t s^2t(2u_0u_2+u_1^2)(s)dsdx+\int_0^x\int_0^t\int_0^y y(3u_0u_2+3u_0u_2^2)(s)dsdydx$
$\vdots$
that is solve for two $u_0(t)$:
$\underline{u}_0(t)=2rt-\frac{t^6}{5}(2r)^2-\frac{t^6}{24}(2r)^3$ *and* $\overline{u}_0(t)=(3-r)t-\frac{t^6}{5}(3-r)^2-\frac{t^6}{24}(3-r)^3$
The exact solution is $u(t)=[2rt,(3-r)t]$. Results for $x=0.01$ and $t=0.01$ are shown in Table (2) and figure (2). Also in table (3) and figure (3), results for $x=0.01$ and $t=0.001$ are shown and absolute errors are presented in tables (2) and (3) for $\underline{u}(t)$ and $\overline{u}(t)$

Example 4.3. *Consider the following fuzzy third order integral equation:*

$$u(t) = [(1+3r)t^2-\frac{t^{12}}{40}(1+3r)^3-\frac{t^9}{40}(1+3r)^2-\frac{t^{18}}{1155}(1+3r)^2,(6-2r)t^2-\frac{t^{12}}{40}(6-2r)^3-\frac{t^9}{40}(6-2r)^2-\frac{t^{18}}{1155}(6-2r)^2]$$

r	$\underline{u}(VHPM)$	$\underline{u}(Exact)$	$\lvert Error(\underline{u}(VHPM))\rvert$	$\underline{u}(HPM)$	$\lvert Error(\underline{u}(HPM))\rvert$	$\overline{u}_{VHPM}$	$\overline{u}_{Exact}$	$\lvert Error(\overline{u}(VHPM))\rvert$	$\overline{u}(HPM)$	$\lvert Error(\overline{u}(HPM))\rvert$
0	2.57E-11	0	2.57E(-11)	0	0	2.999999997	3	3E(-9)	2.98221	0.01779
0.1	0.150000005	0.15	5E(-8)	0.148744	0.001256	2.849999996	2.85	4E(-9)	2.83473	0.01527
0.2	0.300000015	0.3	1.5E(-8)	0.297259	0.002741	2.699999995	2.7	5E(-9)	2.68701	0.01299
0.3	0.450000002	0.45	2E(-8)	0.445361	0.004639	2.549999994	2.55	6E(-9)	2.53908	0.01092
0.4	0.600000025	0.6	2.5E(-8)	0.599574	0.000426	2.399999993	2.4	7E(-9)	2.39095	0.00905
0.5	0.750000035	0.75	3.5E(-8)	0.749317	0.000683	2.499999992	2.25	8E(-9)	2.24261	0.00739
0.6	0.900000045	0.9	4.5E(-8)	0.899174	0.000826	2.099999991	2.1	9E(-9)	2.09409	0.00591
0.7	1.050000005	1.05	5E(-8)	1.049622	0.000378	1.949999999	1.95	1E(-9)	1.94538	0.00462
0.8	1.20000055	1.2	5.5E(-8)	1.199686	0.000314	1.799999998	1.8	2E(-9)	1.79650	0.00350
0.9	1.3500006	1.35	6E(-8)	1.349564	0.000436	1.649999997	1.65	3E(-9)	1.64746	0.00254
1	1.50000065	1.5	6.5E(-8)	1.499300	0.000700	1.499999996	1.5	4E(-9)	1.49827	0.00173

Table 1:Numerical results for $x = 0.01$ and $t = 0.5$ and compare with HPM in Example (4.1)

r	$\underline{u}_{VHPM}$	$\underline{u}_{Exact}$	$\lvert Error(\underline{u}(VHPM))\rvert$	$\overline{u}_{VHPM}$	$\overline{u}_{Exact}$	$\lvert Error(\overline{u}(VHPM))\rvert$
0	0	0	0	0.03000000000	0.030	0
0.1	0.0020000000	0.002	0	0.02900000000	0.029	0
0.2	0.0040000000	0.004	0	0.02800000000	0.028	0
0.3	0.0060000000	0.006	0	0.02700000000	0.027	0
0.4	0.0080000000	0.008	0	0.02600000000	0.026	0
0.5	0.0100000000	0.010	0	0.02500000000	0.025	0
0.6	0.0120000000	0.012	0	0.02400000000	0.024	0
0.7	0.0140000000	0.014	0	0.02300000000	0.023	0
0.8	0.0160000000	0.016	0	0.02200000000	0.022	0
0.9	0.0180000000	0.018	0	0.02100000000	0.021	0
1	0.0200000000	0.020	0	0.02000000000	0.02	0

Table 2: Numerical results for $x = 0.01$ and $t = 0.01$ in Example (4.2)

$$+\int_0^t s^3t^2u^3(s)ds + \int_0^t\int_0^y ty^2u^2(s)dsdy + \int_0^t\int_0^y\int_0^z s^2t^3z^3y^3u^2(s)dsdzdy$$

Now, by exerting the VHPM, it is then possible to obtain an equation as follows:

$u_0 + pu_1 + p^2u_2 + \ldots = [(1+3r)t^2 - \frac{t^{12}}{40}(1+3r)^3 - \frac{t^9}{40}(1+3r)^2 - \frac{t^{18}}{1155}(1+3r)^2, (6-2r)t^2 - \frac{t^{12}}{40}(6-2r)^3 - \frac{t^9}{40}(6-2r)^2 - \frac{t^{18}}{1155}(6-2r)^2] + p\int_0^x(+\int_0^t s^3t^2(u_0+pu_1+p^2u_2+\ldots)^3(s)ds + \int_0^t\int_0^y ty^2(u_0+pu_1+p^2u_2+\ldots)^2(s)dsdydx + \int_0^t\int_0^y\int_0^z s^2t^3z^3y^3(u_0+pu_1+p^2u_2+\ldots)^2(s)dsdzdy)$

By comparing the terms with identical powers of p*, we have the results for* $x = 0.01$ *and* $t = 0.01$ *are shown in Table (4) and figure (4). In table (5) and figure (5)results for* $x = 0.05$ *and* $t = 0.001$ *are presented. The absolute errors are presented in tables (4) and (5). The exact solution is* $u(t) = [(1+3r)t^2, (6-2r)t^2]$.

r	$\underline{u}_{VHPM}$	$\underline{u}_{Exact}$	$\|Error(\underline{u}(VHPM))\|$	$\overline{u}_{VHPM}$	$\overline{u}_{Exact}$	$\|Error(\overline{u}(VHPM))\|$
0	0	0	0	0.0020000000053	0.003	5.3E(-11)
0.1	0.0002000000000	0.0002	0	0.0029000000048	0.0029	4.8E(-11)
0.2	0.0004000000001	0.0004	1E(-13)	0.0028000000043	0.0028	4.3E(-11)
0.3	0.0006000000004	0.0006	4E(-13)	0.0027000000038	0.0027	3.8E(-11)
0.4	0.0008000000001	0.0008	1E(-12)	0.0026000000034	0.0026	3.4E(-11)
0.5	0.0010000000002	0.001	2E(-12)	0.0025000000031	0.0025	3.1E(-11)
0.6	0.0012000000003	0.0012	3E(-12)	0.0024000000027	0.0024	2.7E(-11)
0.7	0.0014000000005	0.0014	5E(-12)	0.0023000000024	0.0023	2.3E(-11)
0.8	0.0016000000008	0.0016	8E(-12)	0.0022000000021	0.0022	2.1E(-11)
0.9	0.0018000000011	0.0018	1.1E(-11)	0.0021000000018	0.0021	1.8E(-11)
1	0.0020000000016	0.002	1.6E(-11)	0.0020000000016	0.002	1.6E(-11)

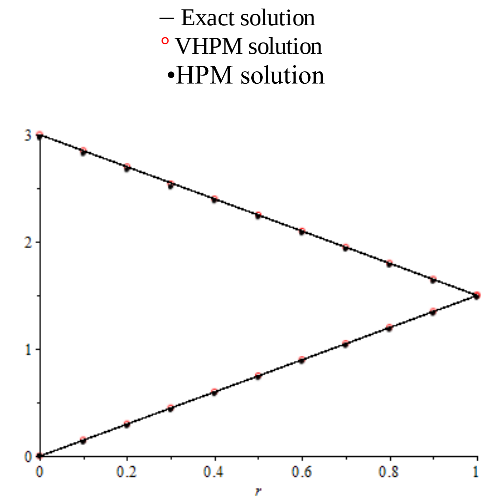
Table 3: Numerical results for $x = 0.05$ and $t = 0.001$ in Example (4.2)

r	$\underline{u}_{VHPM}$	$\underline{u}_{Exact}$	$\|Error(\underline{u}(VHPM))\|$	$\overline{u}_{VHPM}$	$\overline{u}_{Exact}$	$\|Error(\overline{u}(VHPM))\|$
0	0.0001000000000	0.00010	0	0.000600000000	0.0006	0
0.1	0.0001300000000	0.00013	0	0.000590000000	0.00058	0
0.2	0.0001600000000	0.00016	0	0.000560000000	0.00056	0
0.3	0.0001900000000	0.00019	0	0.000540000000	0.00054	0
0.4	0.0002200000000	0.00022	0	0.000520000000	0.00052	0
0.5	0.0002500000000	0.00025	0	0.000500000000	0.00050	0
0.6	0.0002800000000	0.00028	0	0.000480000000	0.00048	0
0.7	0.0003100000000	0.00031	0	0.000460000000	0.00046	0
0.8	0.0003400000000	0.00034	0	0.000440000000	0.000440	0
0.9	0.0003700000000	0.00037	0	0.000420000000	0.00042	0
1	0.0004000000000	0.00040	0	0.000420000000	0.00042	0

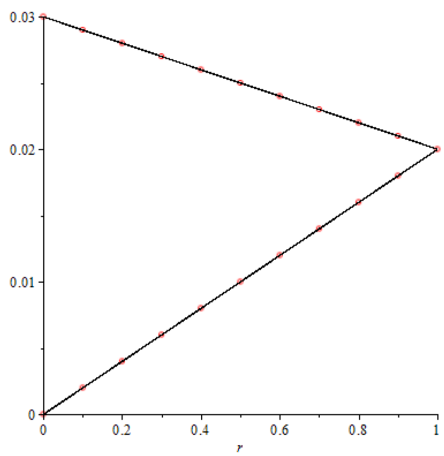
Table 4: Numerical results for $x = 0.01$ and $t = 0.01$ in Example (4.3)

r	$\underline{u}_{VHPM}$	$\underline{u}_{Exact}$	$\|Error(\underline{u}(VHPM))\|$	$\overline{u}_{VHPM}$	$\overline{u}_{Exact}$	$\|Error(\overline{u}(VHPM))\|$
0	0.0000010000000000	0.000001	0	0.000006000000004	0.000006	4E(-15)
0.1	0.0000013000000000	0.0000013	0	0.000005800000004	0.0000058	4E(-15)
0.2	0.0000016000000000	0.0000016	0	0.000005600000003	0.0000056	3E(-15)
0.3	0.0000019000000000	0.0000019	0	0.000005400000003	0.0000054	3E(-15)
0.4	0.000002200000001	0.0000022	1E(-15)	0.000005200000003	0.0000052	3E(-15)
0.5	0.000002500000001	0.0000025	1E(-15)	0.000005000000003	0.000005	3E(-15)
0.6	0.000002800000001	0.0000028	1E(-15)	0.000004800000003	0.0000048	3E(-15)
0.7	0.000003100000001	0.0000031	1E(-15)	0.000004600000002	0.0000046	2E(-15)
0.8	0.000003400000001	0.0000034	1E(-15)	0.000004400000002	0.0000044	2E(-15)
0.9	0.000003700000001	0.0000037	1E(-15)	0.000004200000002	0.0000042	2E(-15)
1	0.000004000000002	0.000004	2E(-15)	0.000004000000002	0.000004	2E(-15)

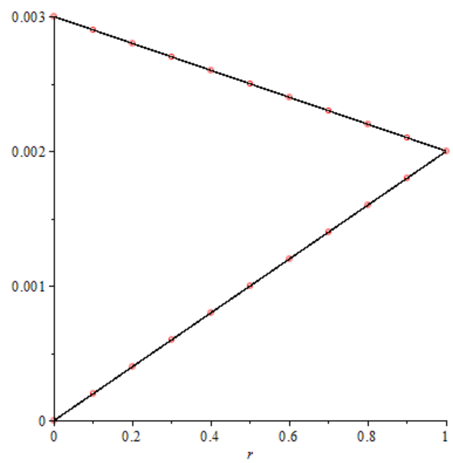
Table 5: Numerical results for $x = 0.05$ and $t = 0.001$ in Example (4.3)



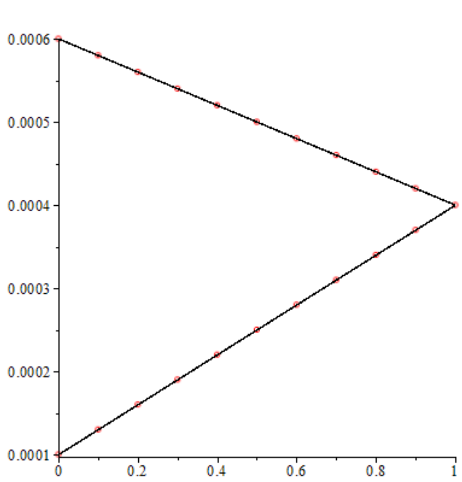
Compare between VHPM, HPM and Exact solution in Example (4.1) with x=0.01 & t=0.5



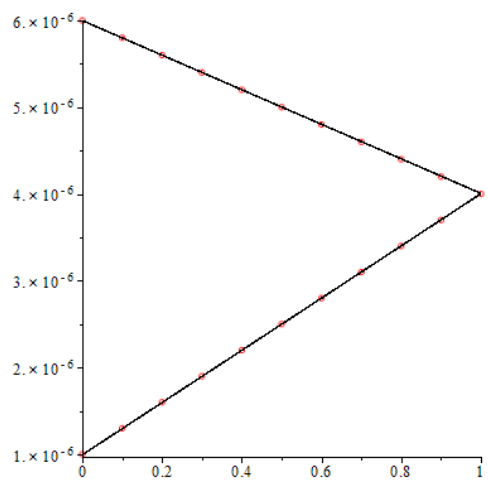
Comparison between VHPM and Exact solution in Example (4.2) x=0.01 & t=0.01



Comparison between VHPM and Exact solution in Example (4.2) with x=0.05 & t=0.001



Comparison between VHPM and Exact solution in Example (4.3) x=0.01 & t=0.01



Comparison between VHPM and Exact solution in Example (4.3) with x=0.05 & t=0.001

5 Conclusion

In this work, a new type of nonlinear fuzzy Volterra integral equations was introduced. These equations were included N multifold integrals, which $N = 1, 2, 3, ...$, with nonlinear fuzzy kernels are presented for the first time. The variational homotopy perturbation method was successfully employed for solving them. This method was based on the variational iteration method and homotopy perturbation method by using the Adomian decomposition method. The obtained results by this method was illustrated without absolute error or very near to exact solutions. Obtained data evince that the convergence rate is very fast, and lower approximations can accede high accuracy. In fact the numerical method was developed by variational Homotopy perturbation method for mentioning nonlinear fuzzy integral equations and special case, for all fuzzy integral equations. The computations in this paper were performed by using Maple 18.

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