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A numerical solution of Nagumo telegraph equation by Adomian decomposition method

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Abstract

In this work, the solution of a boundary value problem is discussed via a semi analytical method. The purpose of the present paper is to inspect the application of the Adomian decomposition method for solving the Nagumo telegraph equation. The numerical solution is obtained for some special cases so that demonstrate the validity of method.

Keywords: Nagumo telegraph equation, Adomian decomposition method

1 Introduction

Nonlinear phenomena, that appear in many areas of scientific fields such as plasma physics, mathematical biology, fluid dynamics and chemical kinetics can be modeled by partial differential equation. As a sample, telegraph equations are used in signal analysis for transmission and propagation of electrical signals [16] and also has applications in other filed [19, 6, 12].

In the present paper we are dealing with the numerical approximation of the following Nagumo telegraph equation [20, 7]

$$\tau u_{tt} + (1 - \tau[a - 2(1 + a)u + 3u^2])u_t = u_{xx} + u(a - u)(1 - u), \tag{1.1}$$

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subject to the boundary conditions

$$u_x(0,t) = 0 \quad and \quad u_x(L,t) = 0, \quad for\, some\, L > 0, \tag{1.2}$$

and the initial condition

$$u(x,0) = \rho \in [0,1], \tag{1.3}$$

for $a \in [0,1]$ and $\tau \in \mathbb{R}$. Physically, the initial data provides information on a distribution or concentration, at the time $t = 0$ [7]. For suitable initial data, the concentration should be expected to approach the fixed point $a \in (0,1)$ for large time. The parameter a acts as the ambient concentration and the parameter τ acts as a measure of the memory delay effect in the equation (1.1). We see that, as $\tau \to 0$, the Nagumo telegraph equation reduce to the Nagumo reaction-diffusion equation [20, 7].

In this paper, we do not discuss solutions to ordinary differential equations, such as those obtained in the traveling wave case [2, 18]. In fact, we directly obtain the numerical solution of partial differential equation for special initial data.

2 Adomian decomposition method (ADM)

The decomposition method has been denoted to solve [8, 9, 13, 14, 3, 5, 4, 15] effectively, easily and accurately a large class of linear or nonlinear, ordinary or partial, deterministic or stochastic, differential or integral equations. The method is well-suited to physical problems since it dose not require unnecessary linearization, perturbation and other restrictive methods and assumptions which may change the problem being solved sometimes seriously.

For the aim of illustration of the methodology to the proposed method, using ADM, we begin by considering the differential equation

$$Lu(x_1,\ldots,x_n) + Ru(x_1,\ldots,x_n) + Nu(x_1,\ldots,x_n) = g(x_1,\ldots,x_n), \tag{2.1}$$

where L is the invertible operator of highest-order derivative with respect to $x_j (1 \leq j \leq n)$, let the order of this operator is k, N is the nonlinear operator, R is a the remainder linear operator and $g(x_1,\ldots,x_n)$ is a given function. Let the inverse operator of L is defined as

$$L^{-1}(.) = \underbrace{\int\int\ldots\int(.)dx_j\ldots dx_j}_{k\, times}, \tag{2.2}$$

where k is defined as the integration level, therefore we can get the solution of (2.1) as the following

$$u(x_1,\ldots,x_n) = f(x_1,\ldots,x_n) - L^{-1}[Nu(x_1,\ldots,x_n) + Ru(x_1,\ldots,x_n)], \tag{2.3}$$

where $f(x_1, \ldots, x_n)$ is appearing using integrating the source function $g(x_1, \ldots, x_n)$ and the given initial conditions, namely

$$f(x_1, \ldots, x_n) = \psi(x_1, \ldots, x_n) + L^{-1}[g(x_1, \ldots, x_n)], \tag{2.4}$$

and ψ is the solution of the following homogeneous differential equation

$$L[\psi(x_1, \ldots, x_n)] = 0. \tag{2.5}$$

According to the standard ADM, the unknown function $u(x_1, \ldots, x_n)$ can be expressed by an infinite series

$$u(x_1, \ldots, x_n) = \sum_{m=0}^{\infty} u_m(x_1, \ldots, x_n), \tag{2.6}$$

and the nonlinear operator $Nu(x_1, \ldots, x_n)$ can be decomposed of the form

$$Nu(x_1, \ldots, x_n) = \sum_{m=0}^{\infty} A_m(x_1, \ldots, x_n). \tag{2.7}$$

Each term A_m is called Adomian's polynomial and it is given by

$$A_m(x_1, \ldots, x_n) = \frac{1}{m!}[\frac{d^m}{d\lambda^m} N(\sum_{i=0}^{\infty} \lambda^i u_i(x_1, \ldots, x_n)]|_{\lambda=0}, \ m \geq 0, \tag{2.8}$$

where λ is a parameter inserted for convenience. At last, the approximated solution of (2.3) is deduced by the following recurrence relation

$$\begin{aligned} &u_0(x_1, \ldots, x_n) = f(x_1, \ldots, x_n), \\ &u_{m+1}(x_1, \ldots, x_n) = -L^{-1}[Ru_m(x_1, \ldots, x_n) + A_m(x_1, \ldots, x_n)], \ \ m \geq 0. \end{aligned} \tag{2.9}$$

The efficiency of method can be improved by taking further components of the solution series. However, for real problem, the $(m+1)$-terms approximate ϕ_m defined by $\phi_m = \sum_{i=0}^{m} u_i(x_1, \ldots, x_n)$, $m \geq 0$, where

$$\lim_{m \to \infty} \phi_m = u(x_1, \ldots, x_n). \tag{2.10}$$

The convergence of the decomposition series have investigated by several authors, for more details see [11, 17].

3 Application of ADM to Nagumo telegraph equation

Now the solution series given by ADM is applied for solving the Nagumo telegraph equation. Consider the Nagumo telegraph equation (1.1) with initial condition (1.3).

(1.1) can be written in an operator form

$$\begin{aligned} &Lu = -\tfrac{1-a\tau}{\tau}u_t + \tfrac{a}{\tau}u + \tfrac{1}{\tau}u_{xx} - 2(1+a)uu_t + 3u^2u_t - \tfrac{a+1}{\tau}u^2 + \tfrac{1}{\tau}u^3, \\ &u(x,0) = \rho \in [0,1], \end{aligned} \tag{3.1}$$

where the differential operator L is

$$L \equiv \frac{\partial^2}{\partial t^2}. \tag{3.2}$$

It is assumed that the inverse operator L^{-1} is an integral operator given by

$$L^{-1} = \int_0^t \int_0^s drds. \tag{3.3}$$

According to the previous section, the ADM [10, 21] assumes that the unknown function $u(x,t)$ can be represented by an infinite series of the form

$$u(x,t) = \sum_{m=0}^{\infty} u_m(x,t), \tag{3.4}$$

and the nonlinear operators

$$F_1(u) = uu_t, \qquad F_2(u) = u^2u_t, \qquad F_3(u) = u^2, \qquad F_4(u) = u^3, \tag{3.5}$$

can be decomposed by an infinite series of polynomials given by

$$F_1(u) = \sum_{m=0}^{\infty} A_m, \qquad F_2(u) = \sum_{m=0}^{\infty} B_m, \qquad F_3(u) = \sum_{m=0}^{\infty} C_m, \qquad F_4(u) = \sum_{m=0}^{\infty} D_m, \tag{3.6}$$

where the components $u_m(x,t)$ will be determined recurrently, and A_m, B_m, C_m, D_m are the so-called Adomian polynomials of $u_0, u_1, \ldots, u_n$ defined by

$$\begin{aligned} A_m &= \tfrac{1}{m!}[\tfrac{d^m}{d\lambda^m}F_1(\textstyle\sum_{i=0}^{\infty}\lambda^i u_i)]|_{\lambda=0}, \ m \geq 0, \\ B_m &= \tfrac{1}{m!}[\tfrac{d^m}{d\lambda^m}F_2(\textstyle\sum_{i=0}^{\infty}\lambda^i u_i)]|_{\lambda=0}, \ m \geq 0, \\ C_m &= \tfrac{1}{m!}[\tfrac{d^m}{d\lambda^m}F_3(\textstyle\sum_{i=0}^{\infty}\lambda^i u_i)]|_{\lambda=0}, \ m \geq 0, \\ D_m &= \tfrac{1}{m!}[\tfrac{d^m}{d\lambda^m}F_4(\textstyle\sum_{i=0}^{\infty}\lambda^i u_i)]|_{\lambda=0}, \ m \geq 0. \end{aligned} \tag{3.7}$$

It is now well known that these polynomials can be constructed for all classes of nonlinearity according to algorithms set by Adomian [8] and recently developed in [1, 22]. Operating with the integral operator L^{-1} on both sides of (3.1) and using the initial condition, we have

$$u(x,t) = \rho + L^{-1}(-\frac{1-a\tau}{\tau}u_t + \frac{a}{\tau}u + \frac{1}{\tau}u_{xx} - 2(1+a)uu_t + 3u^2u_t - \frac{a+1}{\tau}u^2 + \frac{1}{\tau}u^3). \tag{3.8}$$

Substituting (3.4) and (3.6) into the functional (3.8) yields

$$\sum_{m=0}^{\infty} u_m(x,t) = \rho + L^{-1}\big(-\tfrac{1-a\tau}{\tau}\sum_{m=0}^{\infty} u_{mt} + \tfrac{a}{\tau}\sum_{m=0}^{\infty} u_m + \tfrac{1}{\tau}\sum_{m=0}^{\infty} u_{mxx} - 2(1+a)\sum_{m=0}^{\infty} A_m + 3\sum_{m=0}^{\infty} B_m - \tfrac{a+1}{\tau}\sum_{m=0}^{\infty} C_m + \tfrac{1}{\tau}\sum_{m=0}^{\infty} D_m\big). \tag{3.9}$$

Identify the zeroth component $u_0(x,t)$ by all terms that arise from the initial condition, and as a result, the remaining components $u_m(x,t)$, $m \geq 1$ can be determined by using the recurrence relation

$$\begin{aligned} u_0(x,t) &= \rho, \\ u_{m+1}(x,t) &= \\ &L^{-1}\big(-\tfrac{1-a\tau}{\tau}u_{mt} + \tfrac{a}{\tau}u_m + \tfrac{1}{\tau}u_{mxx} - 2(1+a)A_m + 3B_m - \tfrac{a+1}{\tau}C_m + \tfrac{1}{\tau}D_m\big), \end{aligned} \tag{3.10}$$

where $m \geq 0$ and A_m, B_m, C_m, D_m are Adomian polynomials that represent the nonlinear terms

$$uu_t, u^2u_t, u^2, u^3, \tag{3.11}$$

given by

$$\begin{aligned} A_0 &= u_0u_{0t}, \\ B_0 &= u_0^2u_{0t}, \\ C_0 &= u_0^2, \\ D_0 &= u_0^3, \\ A_1 &= u_1u_{0t} + u_0u_{1t}, \\ B_1 &= 2u_0u_{0t}u_1 + u_0^2u_{1t}, \\ C_1 &= 2u_0u_1, \\ D_1 &= 3u_0^2u_1, \\ A_2 &= u_2u_{0t} + u_0u_{2t} + u_1u_{1t}, \\ B_2 &= u_1^2u_{0t} + 2u_0u_{0t}u_2 + 2u_0u_1u_{1t} + u_0^2u_{2t}, \\ C_2 &= u_1^2 + 2u_0u_2, \\ D_2 &= 3u_0^2u_2 + 3u_0u_1^2, \\ A_3 &= u_3u_{0t} + u_0u_{3t} + u_2u_{1t} + u_1u_{2t}, \\ B_3 &= u_1^2u_{1t} + 2u_{0t}u_1u_2 + 2u_0u_2u_{1t} + 2u_0u_1u_{2t} + 2u_0u_{0t}u_3 + u_0^2u_{3t}, \\ C_3 &= 2u_1u_2 + 2u_0u_3, \\ D_3 &= 3u_0u_3 + 6u_0u_1u_2 + u_1^3. \end{aligned} \tag{3.12}$$

Other polynomials can be generated in a like manner.

The first few components of $u_m(x,t)$ follow immediately upon setting

$$\begin{aligned}
&u_0(x,t) = \rho,\\
&u_1(x,t) = L^{-1}(-\tfrac{1-a\tau}{\tau}u_{0t} + \tfrac{a}{\tau}u_0 + \tfrac{1}{\tau}u_{0xx} - 2(1+a)A_0 + 3B_0 - \tfrac{a+1}{\tau}C_0 + \tfrac{1}{\tau}D_0),\\
&u_2(x,t) = L^{-1}(-\tfrac{1-a\tau}{\tau}u_{1t} + \tfrac{a}{\tau}u_1 + \tfrac{1}{\tau}u_{1xx} - 2(1+a)A_1 + 3B_1 - \tfrac{a+1}{\tau}C_1 + \tfrac{1}{\tau}D_1),\\
&u_3(x,t) = L^{-1}(-\tfrac{1-a\tau}{\tau}u_{2t} + \tfrac{a}{\tau}u_2 + \tfrac{1}{\tau}u_{2xx} - 2(1+a)A_2 + 3B_2 - \tfrac{a+1}{\tau}C_2 + \tfrac{1}{\tau}D_2).
\end{aligned} \tag{3.13}$$

The scheme in (3.13) can easily determine the components $u_m(x,t)$, $m \geq 0$. It is possible to calculate more components in the decomposition series to enhance the approximation. Consequently, one can recursively determine every term of the series $\sum_{m=0}^{\infty} u_m(x,t)$, and therefore the solution $u(x,t)$ is readily obtained in a series form.

It is interesting to note that we obtained the solution series by using the initial condition only.

4 Numerical solutions

In this section, we consider some special cases to demonstrate the validity of ADM. We apply the ADM for solving the Nagumo telegraph equation with constant values of initial data $u(x,0) = \rho$, where $\rho \in [0,1]$. This permits us to provide a physically meaningful situation and analyze the influence of the physical parameters on the solutions.

To consider the upon constant initial data and $a = \rho$, a solution is given by $u(x,t) = \rho$. Therefore, we will consider cases in which a and ρ differ. The parameter a may be viewed as an ambient density of concentration, while ρ serves as an initial density or concentration over the finite space considered. So, we expect for the solution via ADM to tend from ρ to a, as we move forward in time [20, 2].

Let us fix $a = 0.5$, $L = 1$, $t^* = 0.5$ and $\tau = 0.1$ in the equation (1.1) and (1.2). Considering cases for initial value, we have the equation

$$0.1u_{tt} + 0.95u_t - 0.5u - u_{xx} + 0.3uu_t - 0.3u^2u_t + 1.5u^2 - u^3 = 0, \tag{4.1}$$

subject to the following different conditions

$$\begin{cases} u_x(0,t) = u_x(1,t) = 0,\\ u(x,0) = 0.55, \end{cases} \tag{4.2}$$

$$\begin{cases} u_x(0,t) = u_x(1,t) = 0,\\ u(x,0) = 0.5, \end{cases} \tag{4.3}$$

and

$$\begin{cases} u_x(0,t) = u_x(1,t) = 0,\\ u(x,0) = 0.45, \end{cases} \tag{4.4}$$

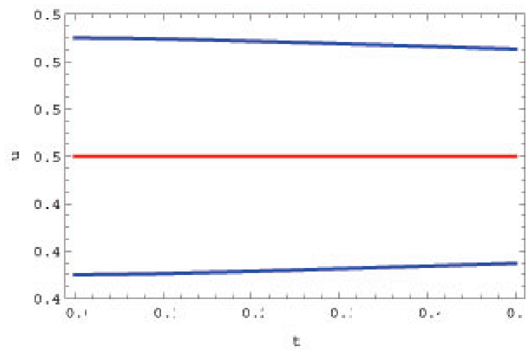

Fig. 1. Solutions to the Nagumo telegraph equation for variable $\rho = 0.55, m = 21$ (the above blue curve), $\rho = 0.5, m = 1$ (red line) and $\rho = 0.45, m = 20$ (the below blue curve).

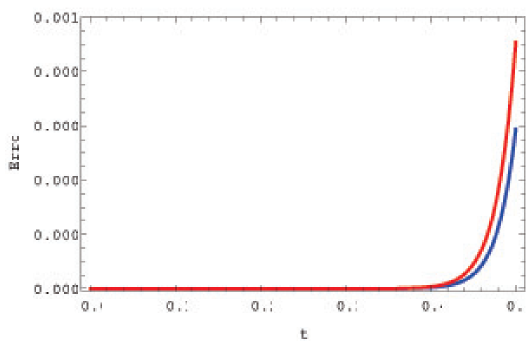

Fig. 2. The error functional for $u(x,0) = 0.55$ (blue curve) and $u(x,0) = 0.45$ (red curve).

that is, the density or concentration is uniform over the space at time $t = 0$.

Following the procedure in section 3, we solve the equation (4.1) with the conditions (4.2), (4.3) and (4.4) using iteration formula (3.10) by initial approximations $u_0(x,t) = 0.55$, $u_0(x,t) = 0.5$ and $u_0(x,t) = 0.45$, respectively.

Therefore, we find that the solutions are generally symmetric about the ambient density or concentration $a = 0.5$. We denote a plot of these functions over $t \in [0, 0.5]$ (for fixed $x = 0.5$) in Fig. 1.

The value of error functional for approximations are showed in Fig. 2. Note that for $u(x,0) = 0.5$, with one iteration of (3.10), we have $u_1(x,t) = u(x,t) = 0.5$ namely the error functional is zero.

5 Conclusion

In this paper, the Nagumo telegraph equation has been demonstrated over a closed interval to explain the validity of ADM. We find that in all cases the obtained error is rather good, and one can consider adding more iterations to get a better approximation.

The ADM can be applied to situations in which the initial data is more complicated, that is

$$\begin{aligned} u(x,0) &= \rho_1(x), \\ u_t(x,0) &= \rho_2(x). \end{aligned} \tag{5.1}$$

However, to account for the variability in the initial data, one must increase the number of iterations used in an approximate solution, in order to keep errors low. For this reason, we have considered constant initial data, which is still physically significant, and allows for rapid convergence to the solution with a minimal number of iterations.

In the case of constant initial data considered, when an initial concentration differs from the ambient concentration, we expect that the model will have a solution which tends toward the ambient concentration.

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