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Numerical solution of seven-order Sawada-Kotara equations by homotopy perturbation method

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Abstract

In this paper, an application of homotopy perturbation method is applied to finding the solutions of the seven-order Sawada-Kotera (sSK) and a Lax's seven-order KdV (LsKdV) equations. Then obtain the exact solitary-wave solutions and numerical solutions of the sSK and LsKdV equations for the initial conditions. The numerical solutions are compared with the known analytical solutions. Their remarkable accuracy are finally demonstrated for the both seven-order equations.

Keywords: Homotopy perturbation method, The seventh-order Sawada-Kotera equation, seventh-order KdV equation, Solitary-wave solution.

1 Introduction

In recent years, the application of the homotopy perturbation method (**HPM**) [10, 12] in nonlinear problems has been developed by scientists and engineers, because this method continuously deforms the difficult problem under study into a simple problem which is easy to solve. The homotopy perturbation method [11], proposed first by He in 1998 and was further developed and improved by He [12, 13, 16]. The method

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yields a very rapid convergence of the solution series in the most cases. Usually, one iteration leads to high accuracy of the solution. Although goal of He's homotopy perturbation method was to find a technique to unify linear and nonlinear, ordinary or partial differential equations for solving initial and boundary value problems. Most perturbation methods assume a small parameter exists, but most nonlinear problems have no small parameter at all. A review of recently developed nonlinear analysis methods can be found in [14]. Recently, the applications of homotopy perturbation theory among scientists were appeared [1-9], which has become a powerful mathematical tool, when it is successfully coupled with the perturbation theory [12, 15, 16].
In this work we would like to implement the **HPM** to the sSK and equations which can be shown in the form:

$$u_t + (63u^4 + 63(2u^2u_{xx} + uu_x^2) + 21(uu_{xxxx} + u_{xx}^2 + u_xu_{xxx}) + u_{xxxxxx})_x = 0, \quad (1.1)$$

$$u_t + (35u^4 + 70(u^2u_{xx} + uu_x^2) + 7(2uu_{xxxx} + 3u_{xx}^2 + 4u_xu_{xxx}) + u_{xxxxxx})_x = 0, \quad (1.2)$$

respectively. Eq. (1.1) equation is known as the seventh-order Sawada-Kotera equation [20] and Eq. (1.2) is known as Laxs seventh-order [19].

2 Basic idea of homotopy perturbation method

To illustrate **HPM** consider the following nonlinear differential equation:

$$A(u) - f(r) = 0, \qquad r \in \Omega, \tag{2.1}$$

with boundary conditions:

$$B(u, \partial u/\partial n) = 0, \qquad r \in \Gamma, \tag{2.2}$$

where A is a general differential operator, B is a boundary operator, $f(r)$ is a known analytic function and Γ is the boundary of the domain Ω.
The operator A can be generally divided into two parts F and N, where F is linear, whereas N is nonlinear. Therefore, Eq. (2.1) can be rewritten as follows:

$$F(u) + N(u) - f(r) = 0. \tag{2.3}$$

He [17] constructed a homotopy $v : \Omega \times [0, 1] \longrightarrow \mathrm{R}$ which satisfies:

$$H(v, p) = (1 - p)[F(v) - F(v_0)] + p[A(v) - f(r)] = 0, \tag{2.4}$$

or

$$H(v, p) = F(v) - F(v_0) + pF(v_0) + p[N(v) - f(r)] = 0, \tag{2.5}$$

where $r \in \Omega$, $p \in [0, 1]$ that is called homotopy parameter, and v_0 is an initial approximation of (2.1). Hence, it is obvious that:

$$H(v, 0) = F(v) - F(v_0) = 0, \qquad H(v, 1) = A(v) - f(r) = 0, \tag{2.6}$$

and the changing process of p from 0 to 1, is just that of $H(v,p)$ from $F(v) - F(v_0)$ to $A(v) - f(r)$. In topology, this is called deformation, $F(v) - F(v_0)$ and $A(v) - f(r)$ are called homotopic. Applying the perturbation technique [18], due to the fact that $0 \leq p \leq 1$ can be considered as a small parameter, we can assume that the solution of (2.4) or (2.5) can be expressed as a series in p, as follows:

$$v = v_0 + pv_1 + p^2 v_2 + p^3 v_3 + \dots, \tag{2.7}$$

when $p \to 1$, (2.4) or (2.5) corresponds to (2.3) and becomes the approximate solution of (2.3), i.e.,

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + v_3 + \dots. \tag{2.8}$$

The series (2.8) is convergent for most cases, and the rate of convergence depends on $A(v)$, [11].

3 The method of solution

In this section, we will describe **HPM** for solving Eqs. (1.1) and (1.2). Consider the standard form of the Equation (1.1) in an operator form:

$$\begin{aligned} L_t(u) + (63(K_1 u) + 63(2(K_2 u) + (M_1 u))+ \\ 21((M_2 u) + (N_1 u) + (N_2 u)) + L_x u)_x = 0. \end{aligned} \tag{3.1}$$

where the notations $K_1 u = u^4$, $K_2 u = u^2 u_{xx}$, $M_1 u = u u_x^2$, $M_2 u = u u_{xxxx}$, $N_1 u = u_{xx}^2$ and $N_2 u = u_x u_{xxx}$ symbolize the nonlinear term, respectively. The notation $L_t = \frac{\partial}{\partial t}$ and $L_x = \frac{\partial^6}{\partial x^6}$ symbolize the linear differential operators. Assuming the inverse of the operator L_t^{-1} exists and it can conveniently be taken as the definite integral with respect to t from 0 to t, i.e., $L_t^{-1} = \int_0^t (.) dt$. Thus, applying the inverse operator L_t^{-1} to (3.1) yields:

$$\begin{aligned} L_t^{-1} L_t(u) = -L_t^{-1}((63(K_1 u) + 63(2(K_2 u) + (M_1 u))+ \\ 21((M_2 u) + (N_1 u) + (N_2 u)) + L_x u)_x). \end{aligned} \tag{3.2}$$

Therefore, it follows that:

$$\begin{aligned} u(x,t) - u(x,0) = -L_t^{-1}((63(K_1 u) + 63(2(K_2 u) + (M_1 u))+ \\ 21((M_2 u) + (N_1 u) + (N_2 u)) + L_x u)_x). \end{aligned} \tag{3.3}$$

Since initial value is known and decompose the unknown function $u(x,t)$ as a sum of components defined by the decomposition series $u(x,t) = \sum_0^\infty v_n(x,t)$ with v_0 identified as $u(x,0)$.

For solving this equation by **HPM**, let $F(u) = u(x,t) - h(x,t) = 0$, where $h(x,t) = u(x,0)$. Hence, we may choose a convex homotopy such that:

$$H(v,p) = v(x,t) - h(x,t) + p \int_0^t ((63(K_1 u) + 63(2(K_2 u) + (M_1 u))+ \tag{3.4}$$

$$21((M_2u) + (N_1u) + (N_2u)) + L_xu)_x)dt = 0.$$

Substituting (2.7) into (3.4) and equating the terms with identical powers of p, we have:

$p^0 : v_0(x,t) = h(x,t),$

$$p^1 : v_1(x,t) = -\int_0^t (63v_0^4 + 63(2v_0^2(v_0)_{xx} + v_0(v_0^2)_x) + 21(v_0(v_0)_{xxxx} + (v_0^2)_{xx} + (v_0)_x(v_0)_{xxx}) + L_xv_0)_x dt,$$

$$p^2 : v_2(x,t) = -\int_0^t (63(4v_0^3v_1) + 63(v_0^2(v_1)_{xx} + 2v_0v_1(v_0)_{xx} + v_1(v_0^2)_x + 2v_0(v_1)_x(v_0)_x) + 21(v_0(v_1)_{xxxx} + v_1(v_0)_{xxxx} + (2v_0v_1)_{xx} + (v_0)_x(v_1)_{xxx} + (v_1)_x(v_0)_{xxx}) + L_xv_1)_x dt,$$

$$p^3 : v_3(x,t) = -\int_0^t (63(4v_0^3v_2 + 2v_0^2v_1^2 + 4v_0^2v_1^2) + 63(4v_0v_2(v_0)_{xx} + 2v_1^2(v_0)_{xx} + 2v_0^2(v_2)_{xx} + 2v_0(v_0)_x(v_1)_x + (v_1^2)_xv_0) + v_0^2(v_2)_x) + 21(v_0(v_1)_{xxxx} + v_1(v_0)_{xxxx} + (2v_0v_1)_{xx} + ((v_0)_x(v_1)_{xxx} + (v_1)_x(v_0)_{xxx}) + L_xv_1)_x dt,$$

$\vdots$

So we can calculate the terms of $u = \sum_{n=0}^{\infty} v_n$, term by term, otherwise by computing some terms say k, $u \approx \varphi_k = \sum_{n=0}^{k-1} v_n$, where $u = \lim_{k\to\infty} \varphi_k$ an approximation to the solution would be achieved.

4 Test examples

Example 1. We first consider sSk equation (1.1) with the initial condition by:

$$u(x,0) = \frac{4k^2}{3}(2 - 3\tanh^2(kx)). \tag{4.1}$$

A homotopy can be readily constructed as follows:

$$u(x,t) - h(x,t) + p\int_0^t ((63u^4 + 63(2u^2u_{xx} + uu_x^2) + 21(uu_{xxxx} + u_{xx}^2 + u_xu_{xxx}) + u_{xxxxxx})_x)dt = 0. \tag{4.2}$$

Substituting (2.7) into (4.2), and equating the terms with identical powers of p, gives:

$p^0 : v_0(x,t) = \frac{4k^2}{3}(2 - 3\tanh^2(kx)),$

$p^1 : v_1(x,t) = \frac{14336}{3}k^9(2 - 3\tanh^2(kx))^3 \tanh(kx)(1 - \tanh^2(kx))t + 2688k^5(2 - 3\tanh^2(kx))$

$(16k^4\tanh^2(kx)(1 - \tanh^2(kx)) - 8k^4(1 - \tanh^2(kx))^2)\tanh(kx)(1 - \tanh^2(kx))t -$

$244k^4(2 - 3\tanh^2(kx))^2(64k^5(1 - \tanh^2(kx))^2\tanh(kx) - 32k^5\tanh^3(kx)(1 -$

$\tanh^2(kx)))t - 2150k^8(2 - 3\tanh^2(kx))\tanh(kx)(1 - \tanh^2(kx))^2 t + 1792k^8$

$(2 - 3\tanh^2(kx))^2(1 - \tanh^2(kx))^2 t - 3584k^8(2 - 3\tanh^2(kx))^2\tanh^2(kx)(1 -$

$\tanh^2(kx))t + 336k^3\tanh(kx)(1 - \tanh^2(kx))(64k^6(1 - \tanh^2(kx))^3 - 352k^6(1 -$

$\tanh^2(kx))^2\tanh^2(kx) + 64k^6\tanh^4(kx)(1 - \tanh^2(kx)))t - 28k^2(2 - 3\tanh^2(kx))$

$(1664k^7(1 - \tanh^2(kx))^2\tanh^3(kx) - 1088k^7(1 - \tanh^2(kx))^3\tanh(kx) -$

$128k^7\tanh^5(kx)(1 - \tanh^2(kx)))t - 2688k^6(1 - \tanh^2(kx))^2\tanh^2(kx))t +$

$448k^6(2 - 3\tanh^2(kx))(1 - \tanh^2(kx))^2 t - 896k^6(2 - 3\tanh^2(kx))\tanh^2(kx)(1 -$

$\tanh^2(kx))t + 168k^4(1 - \tanh^2(kx))^2(64k^5(1 - \tanh^2(kx))^2\tanh(kx) - 32k^5\tanh(kx)$

$(1 - \tanh^2(kx)))t - 336k^4\tanh^2(kx)(1 - \tanh^2(kx))(64k^5(1 - \tanh^2(kx))^2\tanh(kx) -$

$32k^5\tanh^3(kx)(1 - \tanh^2(kx)))t - 1664k^7(1 - \tanh^2(kx))^2\tanh^3(kx)t +$

$1088k^7(1 - \tanh^2(kx))^3\tanh(kx)t + 128k^7\tanh^5(kx)(1 - \tanh^2(kx))t,$

$\vdots$

Continuing this process the complete solution $u(x,t) = \lim_{k\to\infty}\varphi_k$ found by means of n-term approximation $\varphi_k = \sum_{n=0}^{k-1} v_n$. The solution $u(x,t)$ in a series form and in a close form by [19] $u(x,t) = \frac{4k^2}{3}(2 - 3\tanh^2(k(x - \frac{256k^6}{3}t)))$. This result can be verified through substitution.

Example 2. Consider the LsKdV equation [19] (1.2) with the initial condition is given by:

$$u(x,0) = 2k^2\mathrm{sech}^2(kx). \tag{4.3}$$

The solution of this equation, we simply taken the equation in the form of Eq.(3.1) and using the initial value $u(x,0)$ (4.3) to obtain the components of $v_0(x,t),\ v_1(x,t),\ v_2(x,t)$ etc. A homotopy can be readily constructed as follows:

$$u(x,t)-h(x,t)+p\int_0^t((35u^4+70(u^2u_{xx}+uu_x^2)+7(2uu_{xxxx}+3u_{xx}^2+4u_xu_{xxx})+u_{xxxxxx})_x)dt=0. \tag{4.4}$$

Substituting (2.7) into (4.4), and equating the terms with identical powers of p, gives:

$$p^0: v_0(x,t)=2k^2\text{sech}^2(kx),$$

$$\begin{aligned}p^1: v_1(x,t)=&\,4480k^9\text{sech}^8(kx)\tanh(kx)t+1120k^5\text{sech}^4(kx)(8k^4\text{sech}^2(kx)\tanh^2(kx)-\\&4k^4\tanh(kx))t-280k^4\text{sech}^4(kx)(32k^5\tanh(kx)-16k^5\text{sech}^2(kx)\tanh^3(kx))t-\\&35840k^{10}\text{sech}^8(kx)\tanh^2(kx)t+4480k^{10}\text{sech}^6(kx)t+168k^3\text{sech}^2(kx)(32k^6\text{sech}^2(kx)\\&\tanh^4(kx)-176k^6\tanh^2(kx)+32k^6(1-\tanh^2(kx)))\tanh(kx)t-28k^2\text{sech}^2(kx)\\&(832k^7\tanh^3(kx)-64k^7\text{sech}^2(kx)\tanh^5(kx)-544k^7\tanh(kx)(1-\tanh^2(kx)))t-\\&1344k^6\text{sech}^4(kx)\tanh^2(kx)t+336k^6\text{sech}^2(kx)t-224k^4(32k^5\tanh(kx)-\\&16k^5\text{sech}^2(kx)\tanh^3(kx))t+64k^7\text{sech}^2(kx)\tanh^5(kx)t-832k^7\tanh^3(kx)t+\\&544\tanh(kx)(1-\tanh^2(kx))t,\end{aligned}$$

$\vdots$

Continuing this process the complete solution $u(x,t)=\lim_{k\to\infty}\varphi_k$ found by means of n-term approximation $\varphi_k=\sum_{n=0}^{k-1}v_n$. The solution $u(x,t)$ in a series form and in a close form by [19] $u(x,t)=2k^2\text{sech}^2(k(x-64k^6t))$. This result can be verified through substitution.

5 Numerical experiments

In this section, we consider the sSK and LsKdV equations for numerical comparisons. Based on the **HPM**, we constructed the solution $u(x,t)$ as $u\approx\varphi_k=\sum_{n=0}^{k-1}v_n$, where $u=\lim_{k\to\infty}\varphi_k$. In this Letter, we demonstrate how the approximate solutions of the sSK and LsKdV equations are close to exact solutions. In order to verify numerically whether the proposed methodology lead to higher accuracy, we can evaluate the numerical solutions using the n-term approximation. Tables 1 and 2 show the difference of the analytical solution and numerical solution of the absolute errors. It is to be note that 3 terms only were used in evaluating the approximate solutions.

We achieved a very good approximation with the actual solution of the equations by using 3 terms only of the decomposition derived above. It is evident that the overall errors can be made smaller by adding new terms of the decomposition series.

Table 1: Numerical results for $|u(x,t)-\varphi_3(x,t)|$ where $u(x,t)=\frac{4k^2}{3}(2-3\tanh^2(k(x-\frac{256k^6}{3}t)))$ when $k=0.1$, for Eq. (4.1)

$t_i \backslash x_i$	0.1	0.2	0.3	0.4	0.5
0.1	9.680871E-5	9.666091E-5	9.634256E-5	9.585464E-5	9.519873E-5
0.2	1.9359368E-4	1.9323749E-4	1.9254064E-4	1.9150556E-4	1.9013561E-4
0.3	2.9035835E-4	2.8973310E-4	2.8859761E-4	2.86955927E-4	2.8481371E-4
0.4	3.8710601E-4	3.8615103E-4	3.8451663E-4	3.8220892E-4	3.7923611E-4
0.5	4.8384002E-4	4.8249459E-4	4.8030098E-4	4.7726770E-4	4.7340589E-4

Table 2: Numerical results for $|u(x,t)-\varphi_3(x,t)|$ where $u(x,t)=2k^2\mathrm{sech}^2(k(x-64k^6t))$ when $k=0.1$ for Eq. (4.3)

$t_i \backslash x_i$	0.1	0.2	0.3	0.4	0.5
0.1	1.5235676E-4	1.5217536E-4	1.5177028E-4	1.5114268E-4	1.5029413E-4
0.2	3.0467663E-4	3.0428498E-4	3.0344638E-4	3.0216288E-4	3.0043790E-4
0.3	4.5696529E-4	4.5633465E-4	4.5503389E-4	4.5306621E-4	4.5043679E-4
0.4	6.0922849E-4	6.0833001E-4	6.0653841E-4	6.0385810E-4	6.0029605E-4
0.5	7.6147198E-4	7.6027667E-4	7.5796562E-4	7.5454412E-4	7.5002106E-4

6 Conclusion

In this work, we successfully apply the homotopy perturbation method to approximate the solution of sSK and LsKdV equations. It gives a simple and a powerful mathematical tool for nonlinear problems. In our work, we use the Maple Package to calculate the series obtained from the iteration method.

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