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ω_0−Nearest Points and ω_0−Farthest Points in Normed Linear Spaces

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Abstract

In this paper we obtain a necessary and a sufficient condition for the set of ω_0-nearest points (ω_0-farthest points) to be non-empty or a singleton set in normed linear spaces. We shall find a necessary and a sufficient condition for an uniquely remotal set to be a singleton set.

Key words: Proximinal sets, Chebyshev sets, Farthest points, Uniquely remotal sets, Remotal sets, ω_0-Nearest point, ω_0-Farthest point.

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1 Introduction

Franchetti and Singer [3] obtained some results on the characterization and existence of farthest points in normed linear spaces in terms of bounded linear functionals.

Let W be a non-empty subset of a normed linear space X. For any $x \in X$, the (possibly empty) set of best approximation x from W is defined by

$$P_W(x) = \{y \in W : \|x - y\| = d(x, W)\},$$

where $d(x, W) = \inf\{\|x - y\| : y \in W\}$.

For $\omega_0 \in W$, we have

$$P_W^{-1}(\omega_0) = \{x \in X : \omega_0 \in P_W(x)\} = \{x \in X : \|x - \omega_0\| = d(x, W)\},$$

which is called ω_0-nearest points set. It is clear that, if $0, \omega_0 \in W$,

$$P_W^{-1}(\omega_0) = \omega_0 + P_W^{-1}(0).$$

Note that if $z \in P_W^{-1}(0)$, then $\alpha z \in P_{\alpha W}^{-1}(0)$ for every scalar α.

The subset W is said to be proximinal if the set $P_W(x)$ is non-empty for every $x \in X$ and the set W is Chebyshev if $P_W(x)$ is a singleton set for every $x \in X$ (see [2-3, 9-10]).

Let W be a non-empty bounded subset of a real normed linear space X and $x \in X$. An element $g_0 \in W$ is called a farthest point to x in W if

$$||g_0 - x|| = \rho(x, W) = \sup_{g \in W} ||g - x||,$$

the (possibly empty) set of farthest points x from W is defined by

$$F_W(x) = \{y \in W : ||y - x|| = \rho(x, W)\}.$$

For $\omega_0 \in W$, we define

$F_W^{-1}(\omega_0) = \{x \in X : \omega_0 \in F_W(x)\} = \{x \in X : \|x - \omega_0\| = \rho(x, W)\}$,

which is called ω_0-farthest points set. It is clear that if $0, \omega_0]inW$,

$$F_W^{-1}(\omega_0) = \omega_0 + F_W^{-1}(0).$$

Note that if $z \in F_W^{-1}(0)$, then $\alpha z \in F_{\alpha W}^{-1}(0)$ for every scalar α.

It is clear that for+ $x \in X$,

$$F_W(x) = (x - F_W^{-1}(0)) \bigcap W.$$

Let W be a bounded set in a normed linear space X. The set W is said to be remotal if the set $F_W(x)$ is non-empty for each $x \in X$, uniquely remotal if the set $F_W(x)$ consist of exactly one element for each $x \in X$,. (see [4, 6-8])
We will use the well-known fact about proximity.

Lemma 1 *[9] Let X be a normed linear space and W be a linear subspace of X. If W is proximinal for $r > 0$, then there exists a $z \in P_W^{-1}(0)$.*

Lemma 2 *[7] Let W be a uniquely remotal subset of a normed space $(X, \| \cdot \|)$. Then a necessary and sufficient condition that W be singleton is that*

$$\| x - q_W(x) \| = \| y - q_W(x) \| \Longrightarrow q_W(x) = q_W(y).$$

2 ω_0-Nearest point sets and ω_0-Farthest point sets

In this section, we will consider, ω_0-nearest point sets and ω_0-farthest point sets and uniquely remotal sets in normed linear spaces.

Theorem 3 *Let X be a normed linear space,* *(i) If W is a non-empty proximinal subset of X and $P_W^{-1}(\omega_0)$ is singleton, then W is Chebyshev.*

(ii)If W is a non-empty remotal bounded subset of X, $\omega_0 \in W$ and $F_W^{-1}(\omega_0)$ is singleton. Then W is uniquely remotal set.

Proof. (i) Suppose $x \in X$ and $g_1, g_2 \in P_W(x)$. Then $x - g_i + \omega_0 \in P_W^{-1}(\omega_0)$, for every $i = 1, 2$. Therefore $g_1 = g_2$.
(ii) The proof is similar to proof of (i)

Theorem 4 *Suppose X is a strictly convex Banach space. If W is a non-empty bounded subset of X and $\omega_0 \in W$. If $F_W^{-1}(\omega_0) \neq \emptyset$ then $P_W^{-1}(\omega_0) \neq \emptyset$.*

Proof. Form [5], if $z \in W$ is a farthest point from an $x \in X$, then z is also a nearest point in W. Now suppose $x \in F_W^{-1}(\omega_0)$, then $\omega_0 \in F_W(x)$, it follows that for some a $z \in X$, we have $\omega_0 \in P_W(z)$, therefore $z \in P_W^{-1}(\omega_0)$.

Example 2.1 *Let X be a normed space and $W = \{x \in X : \|x\| \leq 1\}$. For $x \in X$, it is trivial for every $x \in X$, $d(x, W) = |1 - |x||$, $\delta(x, W) = |1 + |x||$ and $x \in P_W(\frac{x}{\|x\|})$ and $-x \in F_W(\frac{x}{\|x\|})$.*
Also it is clear that $P_W^{-1}(0) = F_W^{-1}(0) = W$

We know that in a normed linear space X, a vector x is said to be Birkhoff orthogonal to a vector y if the inequality $\|x\| \leq \|x + \alpha y\|$ holds for any real number α.

Theorem 5 *Let $(H, , < ., . >)$ be an inner product space, W is a subspace of H. Then $P_W^{-1}(0) = W^{\perp}$ and for $\omega_0 \in W$ we have $P_W^{-1}(\omega_0) = \omega_0 + W^{\perp}$.*

Proof. It is clear that if X is normed linear space, W is a subspace of X and $w \in W$. Then for any $x \in P_W^{-1}(0)$ and $x \perp w$. Therefore $x \in W^{\perp}$. If $x \in W^{\perp}$, then $x \perp w$ for every $w \in W$. Therefore $\|x\| \leq \|x + \alpha w\|$ for $w \in W$. Since W is a subspace, we have $\|x\| = d(x, W)$ and $x \in P^{-1}(0)$. Therefore $P_W^{-1}(0) = W^{\perp}$.

Example 2.2 *(i) Let* $X = \mathbb{R}^2$ *with Euclidean norm, and* $W = \{(x,x) : \ x \in \mathbb{R}\}$ *be a subspace. From Theorem 2.2,* $P_W^{-1}(0) = W^{\perp} =< (-1,1) >$.

(ii) Let $X = \mathbb{R}^2$ *with Euclidean norm, and* $W = \{(x,x) : \ 0 \leq x \leq 1\}$*. Then* $F_W^{-1}(0) = \{(x,y) : \ x \geq 1\}$.

Theorem 6 *Suppose* X *is a normed linear space. (i) If* W *is a non-empty subset of* X *and* $\omega_0 \in W$*. Then* W *is proximinal if and only if* $X = W + P_W^{-1}(\omega_0)$*. (ii) If* W *is a non-empty bounded subset of* X *and* $\omega_0 \in W$*. Then* W *is remotal if and only if* $X = W + F_W^{-1}(\omega_0)$.

Proof. It is clear.

Theorem 7 *Let* X *be a normed linear space.*

(i) W *a subspace of* X *with codimension one, and there exists a* $z \in P_W^{-1}(0)$ *and* $X = W \bigoplus < z >$*, (where* $\bigoplus$ *means that the sum decomposition of each element* $x \in E$ *is unique), then* W *is proximinal.*

(ii) W *a proximinal subspace of* X *and* $P_W^{-1}(0) =< z >$*. Then* W *is Chebyshev.*

(iii) W *a non-empty bounded subset of* X*,* $0 \in W$ *and* W *is remotal, then there exists a* $z \in F_W^{-1}(0)$.

(iv) W *a non-empty bounded subset of* X*,* $0 \in W$*. If there exists a* $z \in F_W^{-1}(0)$ *and* $X = W \bigoplus < z >$*, (where* $\bigoplus$ *means that the sum decomposition of each element* $x \in X$ *is unique.) and* $W = \beta W$ *for every scalar* β*, then* W *is remotal.*

(v) W *a non-empty bounded subset of* X*,* $0 \in W$ *and there exists an unique* $z \in X$ *such that* $F_W^{-1}(0) = \{z\}$*. Then* W *is uniquely remotal*

(vi) X *a reflexive space and has the Kadec-Klee property. For every non-empty bounded subset of* W *of* X *and* $0 \in W$*, the set* $F_W^{-1}(0)$

is compact.

Proof. (i) For arbitrary $x \in X \backslash W$, there exists an unique element $h \in W$ and the scaler α such that $x = h + \alpha z$. In this case $h \in P_W(x)$, and therefore W is proximinal.
(ii) In this case, we show that $X = W \bigoplus P_W^{-1}(0)$. Since if $x \in X$, there exits a $g_0 \in P_W(x)$. Then $x = g_0 + (x - g_0)$ and $X = W + P_W^{-1}(0)$, also $W \bigcap P_W^{-1}(0) = \{0\}$. Now for $x \in X$, suppose $g_1, g_2 \in P_W(x)$. We have $x = g_1 + (x - g_1) = g_2 + (x - g_2)$ and the sum decomposition of each element $x \in X$ is unique. Therefore $g_1 = g_2$.
(iii) Suppose $x \in X \backslash W$, there exists a $g_0 \in F_W(X)$, then $z = x - g_0 \in F^{-1}{}_W(0)$.
(iv) For arbitrary $x \in X \backslash W$, there exists an unique element $h \in W$ and the scaler α such that $x = h + \alpha z$. In this case $h \in F_W(x)$, since $W = \alpha W$, therefore W is remotal.
(v) For $x \in X$, suppose $g_1, g_2 \in F_W(x)$, consider $z_i = x - g_i$. Therefore $z_i \in F_W^{-1}(0)$, for $i = 1, 2$, therefore $z_1 = z_2 = z$, and it follows that $g_1 = g_2$.
(vi) Since X is reflexive, the closed unit ball B_X is weakly compact. Consider the sequence $\{x_n\} \subseteq F_W^{-1}(0)$. We define $y_n = \frac{x_n}{\rho(x_n, W)}$ Therefore $y_n \in B_X$ Therefore there exists a subsequence $\{y_{n_k}\}$ and $y_0 \in B_X$ such that $y_{n_k} \rightharpoonup y_0$. Since X has Kadec-Klee property, $y_{n_k} \longrightarrow y_0$. Also the sequence $\{\rho(x_n, W)\}$ is a bounded sequence and has a convergence subsequence $\{\rho(x_{n_l}, W)\}$ to k. Thus $x_{n_p} \longrightarrow y_0 k$. Then the set $F_W^{-1}(0)$ is compact.

Theorem 8 *Suppose X is a normed linear space.*

(i) If W is a non-empty subspace of X and $\omega_0 \in W$. We have $P_W^{-1}(\omega_0) = X$ if and only if W is singleton and $W = \{\omega_0\}$.

(ii) If W is a non-empty bounded subset of X and $\omega_0 \in W$. We have $F_W^{-1}(\omega_0) = X$ if and only if W is singleton and $W = \{\omega_0\}$.

Proof. (i) If $P_W^{-1}(\omega_0) = X$, then for every $w \in W$, we have $w \in P_W^{-1}(\omega_0)$. Therefore $w = \omega_0$ and $W = \{\omega_0\}$. If $W = \{\omega_0\}$ and $x \in X$, then $d(x, W) = \|x - \omega_0\|$. Therefore $x \in P_W^{-1}(\omega_0)$.

(ii) Suppose $W = \{\omega_0\}$, if $x \in X$, then $q_x = \omega_0$. That is $\|x - \omega_0\| = \rho(x, W)$, therefore $x \in F_W^{-1}(\omega_0)$. It follows that $X = F_W^{-1}(\omega_0)$. If $X = F_W^{-1}(\omega_0)$, it is clear that W is remotal. Also, if for $x \in X$, there exist $w_1, w_2 \in F_W(x)$, then $w_1 = w_2 = \omega_0$. Therefore W is uniquely remotal. From Lemma 1.2, W is singleton. Therefore $W = \{\omega_0\}$.

Theorem 9 *Suppose X is a normed linear space. If W is a non-empty proximinal subspace of X and $\omega_0 \in W$. If $P_W^{-1}(\omega_0)$ is convex. Then W is Chebyshev.*

Proof. Since $P_W^{-1}(\omega_0)$ is convex. The set $P_W^{-1}(0) = P_W^{-1}(\omega_0) - \omega_0$ is convex. From [5], W is Chebyshev.

Theorem 10 *Let $W \subseteq X$ be a proximinal hyperplane and $\omega_0 \in W$. If $P_W(x)$ is compact for each $x \in X$. Then every sequence $\{x_n\}_{n\geq 1} \subseteq S_X$ with $x_n \in P_W^{-1}(\omega_0)$ for each n has a convergent subsequence.*

Proof. Suppose $P_W(x)$ is compact for each $x \in X$. If the sequence $\{x_n\}_{n\geq 1} \subseteq S_X$ with $x_n \in P_W^{-1}(\omega_0)$ for each n. Put $y_n = x_n - \omega_0 \in P_W^{-1}(0)$ and $\|y_n\| = d(x_n, W) = \|x_n\| = 1$. From [Theorem 2.1 ,6, 8], the sequences $\{y_n\}$ and $\{x_n\}$ has a convergent subsequence.

Theorem 11 *Let $W \subseteq X$ be a proximinal hyperplane. If every sequence $\{x_n\}_{n\geq 1} \subseteq S_X$ with $x_n \in P_W^{-1}(0)$ for each n has a convergent subsequence. Then $P_W(x)$ is compact for each $x \in X$.*

Proof. If every sequence $\{x_n\}_{n\geq 1} \subseteq S_X$ with $x_n \in P_W^{-1}(0)$ for each n has a convergent subsequence. From [Theorem 2.1, 6, 8], W is quasi-Chebyshev subspace. It follows that $P_W(x)$ is compact for each $x \in X$.

Let W be a subspace of a normed space X. We define the quotient space X/W to be the set of all sets $x+W$ of W together with the following operations:

$$(x+W)+(y+W)=(x+y)+W,$$

and

$$\lambda(x+W)=\lambda x+W,$$

for all $x, y \in X$ and arbitrary scalar λ. Then, the quotient space X/W is a normed space with the norm $\|x+W\| = inf_{w\in W}\|x-w\|$.

Theorem 12 *Let M be a proximinal subspace of a normed space X, W a proximinal subspace of X containing M. If $P_W^{-1}(0) =< z >$ for $z \in M$, then W/M is Chebyshev .*

Proof. From Lemma [2], W/M is proximinal. Suppose there exist $z_1+M, z_2+M \in P_{W/M}(x+M)$. Since M is proximinal there exist $m_1, m_2 \in M$ such that $d(z_1, M) = \|z_1-m_1\|$ and $d(z_2, M) = \|z_2-m_2\|$. Therefore $z_1-m_1, z_2-m_2 \in P_W^{-1}(0) =< z >$, then $z_1-m_1 = \alpha_1 z$ and $z_2-m_2 = \alpha_2 z$. Therefore $z_1+M = z_2+M$.

For a Banach space X and closed subspace W of X, we denote its unit sphere by S_X. For $x \in X$ with $d(x, W) = 1$, let $Q_W(x) = x - P_W(x)$. It is easy to see that $Q_W(x) = \{z \in S_X : \ f(z) = f(x) \ \ \forall f \in W^{\perp}\}$.
For $f \in X^*$. we define the pre-duality map of X by

$$J_X(f) = \{z \in S_X : \ \ f(z) = \|f\|\}$$

.

Definition 2.1 *[10] Let X be a normed space, W be a subspace of X. Then W is a ω-Chebyshev subspace, if for every $x \in X$, $x+(P_W^{-1}(0)\bigcap S_X)$ is a nonempty and weakly compact set in X.*

Definition 2.2 *[10] A subspace W of a normed space X is called ω-boundedly compact if for every bounded sequence $\{y_n\}$ in W , there exists $x_0 \in W$ and a subsequence $\{y_{n_k}\}$ such that $y_{n_k} \rightharpoonup x_0$.*

In normed space X, suppose the unite sphere with center $\omega_0 \in X$ denoted by

$$S(\omega_0, 1) = \{x \in X : \ \|x - \omega_0\| = 1\}.$$

Theorem 13 *Let X be a normed space, W be a subspace of X and $\omega_0 \in X$. If $x \in X$ and the set $x + (P_W^{-1}(\omega_0) \bigcap S(\omega_0, 1))$ is nonempty and weakly compact, then W is ω-Chebyshev.*

Proof. It is clear, because

$$x + (P_W^{-1}(\omega_0) \bigcap S(\omega_0, 1)) = x + \omega_0 + (P_W^{-1}(0) \bigcap S_X)).$$

Theorem 14 *Let X be a normed space, W be a subspace of X, $\omega_0 \in W$ and codim$W = 1$. Then the following statement are equivalent:*

(i) $x + (P_W^{-1}(\omega_0) \bigcap S(\omega_0, 1))$ is nonempty and weakly compact.

(ii) for every $f \in W^\perp$, $J_X(f)$ is weakly compact.

(iii) for every $x \in X$, $P_W(x)$ is weakly compact.

Proof. $(i) \Rightarrow (ii)$. Since W is ω-Chebyshev, from [Theorem 2.1, 10] for every $f \in W^\perp$, $J_X(f)$ is weakly compact.

$(ii) \Rightarrow (iii)$. Theorem 2.1 of [10].

$(iii) \Rightarrow (i)$. Suppose $x \in X$, and $\{z_n\} \subseteq x + (P_W^{-1}(\omega_0) \bigcap S(\omega_0, 1))$, then $\{z_n - \omega_0 \in x + (P_W^{-1}(0) \bigcap S_X))$. Therefore the sequence $\{z_n - \omega_0\}$ has a weakly convergent subsequence, it follows that $\{z_n\}$ has a weakly convergent subsequence. Also $x + \omega_0 \in X$ and W is ω-Chebyshev, therefore $x + (P_W^{-1}(\omega_0) \bigcap S(\omega_0, 1)) \neq \emptyset$.

Theorem 15 *Let X be a normed space, W be a subspace of X, $\omega_0 \in W$. If $P_W^{-1}(\omega_0)$ is ω-boundedly compact. Then $P_W(x)$ is weakly compact for every $x \in X$.*

Proof. Because $P_W^{-1}(0) = P_W^{-1}(\omega_0) - \omega_0$ is ω-boundedly compact, it follows that from [Theorem 2.3, 10] the set $P_W(x)$ is weakly compact for every $x \in X$.

Corollary 16 *Let X be a normed space, W be a subspace of X, $\omega_0 \in W$. Then $P_W^{-1}(\omega_0)$ is ω-boundedly compact if and only if W is ω-Chebyshev.*

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References

[1] G. C. Ahuja, T. D. Narang and Swaran Trehan, On farthest points, J.Indian Math. Soc. **39**(1975), 293-297.

[2] E. W., Cheney, D. E., Wulbert, The existence and unicity of best approximations. Math. Scand. **24** (1969) 113-140.

[3] J. Fletcher, W. B. Moors, Chebyshev sets. J. Aust. Math. Soc. **98** (2015), 161-231.

[4] C. Franchetti and I. Singer, Deviation and farthest points in normed spaces, Rev. Roum Math. Pures et Appl, **24**(1979), 373-381.

[5] H. Mazaheri and F. M., Maalek Ghaini, Quasi-orthogonality of the best approximant sets. Nonlinear Anal. **65** (2006), 534537.

[6] H. Mohebi, On quasi-Chebyshev subspaces of Banach spaces. J. Approx. Theory 107 (2000), 87-95.

[7] T. D. Narang, On farthest points in metric spaces. J. Korea Soc. Math. Educ. Ser. B Pure Appl. Math. **9** (2002), 1-7.

[8] D. Narayana; T. S. S. R. Rao, Some remarks on quasi-Chebyshev subspaces. J. Math. Anal. Appl. 321 (2006), 193-197.

[9] A. Niknam, On uniquely remotal sets. Indian J. Pure Appl. Math. 15 (1984), 1079-1083.

[10] M. Shams; H. Mazaheri; S. M. Vaezpour, On w-Chebyshev subspaces in Banach spaces. Bull. Korean Math. Soc. 45 (2008), 601-606.

[11] Sangeeta, T. D. Narang, On the farthest points in convex metric spaces and linear metric spaces. Publ. Inst. Math. (Beograd) (N.S.) **95** (109) (2014), 229-238.

[12] I. Singer, Best Approximation in Normed Linear Spaces by Elements of Linear Subspaces, Springer-Verlag, New York, Heidelberg, Berlin, 1970.

[13] I. Singer, The Theory of Best Approximation and Functional Analysis, CBMS Reg. Confer. Series in Applied Math. No.13, SIAM, Philedelphia, 1974.