Mathematics Scientific Journal

Vol. 8, No. 2, (2013), 49-64

The effect of indicial equations in solving inconsistent singular linear system of equations

M. Nikuie[a,*] M. K. Mirnia[b]

[a] *Young Researchers Club, Tabriz Branch, Islamic Azad University, Tabriz, Iran.*

[b] *Department of Computer engineering, Tabriz Branch, Islamic Azad University, Tabriz, Iran.*

Received 23 August 2012; accepted 14 January 2013

Abstract

The index of matrix $A \in C^{n\times n}$ is equivalent to the dimension of largest Jordan block corresponding to the zero eigenvalue of A. In this paper, indicial equations and normal equations for solving inconsistent singular linear system of equations are investigated.

Key words: Indicial equations; Normal equations; Singular linear system; Drazin inverse.

2010 AMS Mathematics Subject Classification : 15A09; 65F20.

* Corresponding Author's E-mail:nikoie_m@yahoo.com(M.Nikuie),Tel:+989149495461

1 Introduction

A matrix has an inverse only if it is square, and even then only if it is nonsingular, or, in other words, if its columns (or rows) are linearly independent. In recent years needs have been felt in numerous areas of applied mathematics for some kind of partial inverse of a matrix that is singular or even rectangular [1]. The inverse of singular and rectangular matrices is called generalized inverses. Linear system of equations play major role in various areas such as fuzzy mathematics [2, 3], differential equations [4], and integral equations [5]. There is many methods for solving fuzzy and crisp Volterra and Fredholm integral equations [6, 7, 8]. In [5] a two-step diagonally-implicit collocation based methods for Volterra integral equations using a systems of equations is given.

The principal application of the generalized inverses is to system of equations that is inconsistent or have a set of solutions [9]. The effect of normal equations in solving such systems is explained in [10, 11]. In this paper, indicial equations for singular linear system of equations is introduced and the effect of indicial equations in solving inconsistent singular linear system of equations is investigated. The purpose of this paper, is give a new approach to the inconsistent singular linear system of equations. In section 2, some preliminaries that we shall use in later are presented. New results on the Drazin inverse and Pseudoinverse are given, in section 3. The title of section 4, is indicial equations and normal equations. Finally, the properties of the introduced concepts are illustrated in the last section.

2 Preliminaries and Basic Definitions

In this section, we present some definitions and simple properties of index of matrix, drazin inverse, pseudoinverse and minimal solutions. For more details, we refer the reader to [12,13,14].

Definition 2.1 *Let $A \in C^{n\times n}$. The index of matrix A is equivalent to the dimension of largest Jordan block corresponding to the zero eigenvalue* *of A and is denoted by $ind(A)$.*

Definition 2.2 *Let $A \in C^{n\times n}$, with $ind(A) = k$. The matrix X of order n is the Drazin inverse of A, denoted by A^D, if X satisfies the following conditions*

$$AX = XA,\ XAX = X,\ A^k XA = A^k$$

When $ind(A) = 1$, A^D is called the group inverse of A, and denoted by A_g.

Theorem 2.1 [15] *Let $A \in C^{n\times n}$, with $ind(A) = k$, $rank(A^k) = r$. We may assume that the Jordan normal form of A has the form as follows*

$$A = P \begin{pmatrix} D & 0 \\ 0 & N \end{pmatrix} P^{-1}$$

where P is a nonsingular matrix, D is a nonsingular matrix of order r, and N is a nilpotent matrix that $N^k = \bar{o}$. Then we can write the Drazin inverse of A in the form

$$A^D = P \begin{pmatrix} D^{-1} & 0 \\ 0 & 0 \end{pmatrix} P^{-1}.$$

When $ind(A) = 1$, obviously, $N = \bar{o}$.

Theorem 2.2 [16] *For any matrix $A \in C^{n\times n}$ the index and Drazin inverse of A exists and is unique.*

Theorem 2.3 [9] *$A^D b$ is a solution of*

$$Ax = b, k = ind(A) \tag{2.1}$$

if and only if $b \in R(A^k)$, and $A^D b$ is an unique solution of (2.1) provided that $x \in R(A^k)$.

Definition 2.3 *An arbitrary complex $m \times n$ matrix A can be factored as*

$$A = PDQ^* \tag{2.2}$$

where P is an $m \times m$ unitary matrix,

$$D = \begin{pmatrix} \sigma_1 & & & & & \\ & \ddots & & & & \\ & & \sigma_r & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix}$$

is an $m \times n$ diagonal matrix, and Q is an $n \times n$ unitary matrix. The factorization in the preceding Definition is called singular value decomposition.

The singular value decomposition is a matrix factorization whose computation is a step in many algorithms. By (2.2) the relation $b = Ax$ can be expressed in terms of $b' = P^*b$ and $x' = Q^*x$

$$b = Ax \Leftrightarrow P^*b = P^*Ax = P^*PDQ^* \Leftrightarrow b' = Dx'$$

Definition 2.4 *The pseudoinverse of a general matrix A is defined by first taking a singular value decomposition (2.2) and then putting*

$$A^+ = QD^+P^*,$$

where

$$D^+ = \begin{pmatrix} \frac{1}{\sigma_1} & & & & & \\ & \ddots & & & & \\ & & \frac{1}{\sigma_r} & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix}.$$

Theorem 2.4 [11] *Corresponding to any matrix A there exists at most one matrix having these four (Penrose) properties*

$$(AX)^* = AX, \quad (XA)^* = XA, \quad XAX = X, \quad AXA = A.$$

Definition 2.5 [11] *Consider a system of equations written in matrix form as $Ax = b$ where A is $m \times n$, x is $n \times 1$, and b is $m \times 1$. The minimal solution of this problem is defined as follows:*

(1). If the system is consistent and has a unique solution, x, then the minimal solution is defined to be x.
(2). If the system is consistent and has a set of solutions, then the minimal solution is the element of this set having the least Euclidean norm.
(3). If the system is inconsistent and has a unique least-squares solution, x, the minimal solution is defined to be x.
(4). If the system is inconsistent and has set of least-squares solutions, then the minimal solution is the element of this set having the least Euclidean norm.

Theorem 2.5 [11] *The minimal solution of the system*

$$Ax = b,$$

is given by the pseudoinverse $x = A^+b$.

Definition 2.6 *A number $\lambda \in C$, is called an eigenvalue of the matrix A if there is a vector $x \neq 0$ such that $Ax = \lambda x$. Any such vector is called an eigenvector of A associated to the eigenvalue λ.*

The set $L(\lambda) = \{x \mid (A - \lambda I)x = 0\}$ forms a linear subspace of C^n, of dimension

$$\rho(\lambda) = n - rank(A - \lambda I).$$

The integer $\rho(\lambda) = dimL(\lambda)$ specifies the maximum number of linearly independent eigenvectors associated with the eigenvalue λ. It is easily seen that $\varphi(\mu) = det(A - \mu I)$ is a nth-degree polynomial of the form

$$\varphi(\mu) = (-1)^n(\mu^n + \alpha_{n-1}\mu^{n-1} + \cdots + \alpha_0).$$

It is called the characteristic polynomial of the matrix A. Its zeros are the eigenvalues of A. If $\lambda_1, \cdots, \lambda_k$ are the distinct zeros of $\varphi(\mu)$, then φ can be represented in the form

$$\varphi(\mu) = (-1)^n(\mu - \lambda_1)^{\sigma_1}(\mu - \lambda_2)^{\sigma_2} \cdots (\mu - \lambda_k)^{\sigma_k}.$$

The integer σ_i, which we also denote by $\sigma(\lambda_i) = \sigma_i$, is called the multiplicity of the eigenvalue λ_i.

3 New results on the Drazin inverse and pseudoinverse

In this section, new results on the Drazin inverse and pseudoinverse are given.

Theorem 3.1 *Let $A \in C^{n\times n}$ be a symmetric matrix with index one. Then $A_g = A^+$.*

Proof. A_g is group inverse of A, then we have

$$AA_g = A_gA, \quad A_gAA_g = A_g, \quad AA_gA = A.$$

$A = A^T$ from [10], $(A_g)^T = A_g$, thus we can write

$$\begin{pmatrix} (AA_g)^T = (A_g)^T A^T = A_gA = AA_g \\ (A_gA)^T = A^T(A_g)^T = AA_g = A_gA. \end{pmatrix}$$

Therefore by theorem 2.4, we have $A_g = A^+$.

Theorem 3.2 *Let $A \in C^{n\times n}$ be a symmetric matrix with index one. Then $(A^+)^T = A^+$.*

Proof. The pseudoinverse of A is A^+, then we have

$$\begin{pmatrix} (A^+A)^T = A^T(A^+)^T = (A^+)^T A^T \\ (AA^+)^T = (A^+)^T A^T = A^T(A^+)^T \\ (A^+)^T A (A^+)^T = (A^+)^T \\ A(A^+)^T A = A. \end{pmatrix}$$

Since pseudoinverse of A is unique, therefore $(A^+)^T = A^+$.

Theorem 3.3 *If $A \in C^{n\times n}$ is a symmetric matrix with index one, and $\lambda \neq 0$ is an eigenvalue of it, then $\frac{1}{\lambda}$ is an eigenvalue of A^+.*

Proof. From $Ax = \lambda x, (x \neq 0)$ we have

$$AA^+x = \lambda A^+x.$$

Then

$$A^+AA^+x = \lambda A^+A^+x.$$

We can get $A^+x = \lambda A^+A^+x$. Now if we set $A^+x = y$ we have $\frac{1}{\lambda}y = A^+y$. Therefore $\frac{1}{\lambda}$ is an eigenvalue of A^+.

Theorem 3.4 *Let $A \in C^{n\times n}$ be a singular matrix. Then A^TA is a singular matrix.*

Proof. We know that $rank(A) = rank(A^T)$ [10], since

$$rank(A^TA) \leq min\{rank(A), \;\; rank(A^T)\},$$

then A^TA is a singular matrix. In addition, let $\lambda_{(1A)} = 0$, then

$$\rho(\lambda_{(1A)}) = n - rank(A),$$

and for matrix A^TA we have

$$\rho(\lambda_{(A^TA)}) = n - rank(A^TA).$$

Therefore by definition 2.1, we have $ind(A) \leq ind(A^T A)$.

Theorem 3.5 *Let A^D be the Drazin inverse of A. Then $(A^D)^n$ for $n \in N$, is the Drazin inverse of A^n .*

Proof. By Definition 2.2, we have

Let $ind(A^D) = l$, then

$$(A^l)^n(A^D)^n(A^n) = (P \begin{pmatrix} D^l & 0 \\ 0 & N^l \end{pmatrix}^n P^{-1}) = (A^l)^n.$$

Therefore, by definition 2.2, $(A^D)^n$, for $n \in N$ is Drazin inverse of A^n.

Theorem 3.6 *Let A^D be the Drazin inverse of A. Then $(A^D)^T$ is the Drazin inverse of A^T.*

Proof. Let $ind(A) = k$ and A^D be Drazin inverse of A . Therefore

$$\begin{pmatrix} (AA^D)^T = (A^D)^T A^T = (A^D A)^T = A^T (A^D)^T, \\ (A^D A A^D)^T = (AA^D)^T (A^D)^T = (A^D)^T (A^T)(A^D)^T = (A^D)^T \end{pmatrix}.$$

By [10], we have $ind(A) = ind(A^T)$. Thus

$$(A^k A^D A)^T = (A^D A)^T (A^k)^T = (A^T)(A^D)^T (A^k)^T = (A^k)^T (A^D)^T (A^T) = (A^k)^T.$$

Therefore, by definition 2.2, $(A^D)^T$ is drazin inverse of A^T.

4 Indicial equations and Normal equations

In this section, the effect of indicial equations and Normal equations in solving inconsistent singular linear system of equations, are investigated.

Consider the inconsistent singular linear system of equations

$$Ax = b, k = ind(A), \tag{4.1}$$

(4.1) has a set of least squares solutions, then the minimal solution is a member of this set that has the least Euclidean norm. In such cases, it is often required to find an x that minimizes the norm of the residual vector $b - Ax$. In the other word, the least square solution of (4.1) is the vector x that makes $\|b - Ax\|$ minimum.

Some properties of indicial equations and Normal equations are listed below:

(1). According to [17] and properties of the Drazin inverse , in order to obtain the Drazin inverse the projection method solves consistent or inconsistent singular linear system (4.1) through solving the consistent singular linear system

$$A^k Ax = A^k b, \quad k = ind(A). \tag{4.2}$$

The system (4.2) is called indicial equations and is a singular consistent system, therefore has a set of solutions [11].

(2). If x is a point such that $A^k(Ax - b) = 0$, then x is a least squares solution of (4.1). Since from $A^k(Ax - b) = 0$, we can not conclude that $b - Ax$ is orthogonal to the column space of A. Therefore x may not be the minimal solution of (4.1).

(3). If x is a point such that $A^T(Ax - b) = 0$, then is a least squares solution of (4.1). The system $A^T Ax = A^T b$ is called normal equations. For inconsistent singular system, normal equations is consistent linear system but have many solutions.

(4). By [11] $x_M = A^+ b = (A^T A)^+(A^T b)$ is the minimal solution of (4.1).

(5). $x_K = (A^k A)^D(A^k b)$ is a least squares solution of system (4.1).

(6). Since $A^T Ax = A^T b$ is a consistent singular linear system of equations and has a set of solutions

$$x_D = (A^T A)^D(A^T b),$$

is a least squares solution of system (4.1).

(7). Let A^2 be a symmetric matrix with index one and $A^2 = MGN^*$ be the singular value decomposition of A^2. Therefore for solving $x_M = x_K$ we have

$$\begin{pmatrix} AAx = Ab \Longleftrightarrow M^*Ab = M^*AAx = M^*MGN^*x \Longleftrightarrow c' = Gx' \\ c' = M^*Ab, \quad x' = N^*x. \end{pmatrix}$$

5 Numerical Examples

In this section, the following examples illustrate our new results.

Example 5.1 *Consider the following symmetric and singular matrix*

$$A = \begin{pmatrix} 2 & 0 & 2 \\ 0 & 5 & 0 \\ 2 & 0 & 2 \end{pmatrix}.$$

The Jordan normal form of A has the form as follows

$$A = P \begin{pmatrix} 5 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{pmatrix} P^{-1}, \ P^{-1} = \begin{pmatrix} 0 & -1 & 0 \\ -\frac{1}{2} & 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{pmatrix}.$$

By Theorem 2.1, we have

$$A^D = P \begin{pmatrix} \frac{1}{5} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 \end{pmatrix} P^{-1} = \begin{pmatrix} \frac{1}{8} & 0 & \frac{1}{8} \\ 0 & \frac{1}{5} & 0 \\ \frac{1}{8} & 0 & \frac{1}{8}. \end{pmatrix}$$

The singular value decomposition of A has the form as follows

$$A = \begin{pmatrix} 0 & -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ 1 & 0 & 0 \\ 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} \begin{pmatrix} 5 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 1 & 0 & 0 \\ 0 & -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{pmatrix}^T .$$

The pseudoinverse of A is

$$A^+ = \begin{pmatrix} 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 1 & 0 & 0 \\ 0 & -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{5} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ 1 & 0 & 0 \\ 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}^T = \begin{pmatrix} \frac{1}{8} & 0 & \frac{1}{8} \\ 0 & \frac{1}{5} & 0 \\ \frac{1}{8} & 0 & \frac{1}{8} \end{pmatrix} .$$

Example 5.2 *Consider the following inconsistent singular linear system*

$$\begin{pmatrix} 2 & 0 & 2 \\ 0 & 5 & 0 \\ 2 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} . \tag{5.1}$$

The minimal solution of (5.1) is

$$x_M = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 2 \\ 0 & 5 & 0 \\ 2 & 0 & 2 \end{pmatrix}^+ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{1}{8} & 0 & \frac{1}{8} \\ 0 & \frac{1}{5} & 0 \\ \frac{1}{8} & 0 & \frac{1}{8} \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{5}{2} \\ \frac{1}{2} \end{pmatrix} .$$

Consider the following consistent singular linear system of equations

$$\begin{pmatrix} 2 & 0 & 2 \\ 0 & 5 & 0 \\ 2 & 0 & 2 \end{pmatrix} \begin{pmatrix} 2 & 0 & 2 \\ 0 & 5 & 0 \\ 2 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 2 \\ 0 & 5 & 0 \\ 2 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} .$$

By singular value decomposition of AA, we have

$$\begin{pmatrix} 0 & -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ 1 & 0 & 0 \\ 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}^T \begin{pmatrix} 8 & 0 & 8 \\ 0 & 25 & 0 \\ 8 & 0 & 8 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 & -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ 1 & 0 & 0 \\ 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}^T \begin{pmatrix} 2 & 0 & 2 \\ 0 & 5 & 0 \\ 2 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix},$$

and

$$\begin{pmatrix} 25 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ 1 & 0 & 0 \\ 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}^T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 & -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ 1 & 0 & 0 \\ 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}^T \begin{pmatrix} 2 & 0 & 2 \\ 0 & 5 & 0 \\ 2 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix},$$

because

$$\begin{pmatrix} 0 & -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ 1 & 0 & 0 \\ 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}^T \begin{pmatrix} 2 & 0 & 2 \\ 0 & 5 & 0 \\ 2 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 10 \\ -8\sqrt{2} \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 25 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 0 \end{pmatrix} x' = \begin{pmatrix} 10 \\ -8\sqrt{2} \\ 0 \end{pmatrix}.$$

We can get

$$x' = \begin{pmatrix} \frac{2}{5} \\ -\frac{\sqrt{2}}{2} \\ 0 \end{pmatrix}.$$

Therefore

$$x_D = \begin{pmatrix} \frac{1}{2} \\ \frac{5}{2} \\ \frac{1}{2} \end{pmatrix}.$$

Example 5.3 *Consider the following inconsistent singular linear system*

$$\begin{pmatrix} 1 & 3 & 1 \\ 2 & 0 & -2 \\ 1 & 3 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}. \tag{5.2}$$

The minimal solution of (5.2) is

$$x_M = \begin{pmatrix} 1 & 3 & 1 \\ 2 & 0 & -2 \\ 1 & 3 & 1 \end{pmatrix}^{+} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{1}{22} & \frac{1}{4} & \frac{1}{22} \\ \frac{3}{22} & 0 & \frac{3}{22} \\ \frac{1}{22} & -\frac{1}{4} & -\frac{1}{22} \end{pmatrix} = \begin{pmatrix} \frac{15}{22} \\ \frac{6}{11} \\ -\frac{7}{22} \end{pmatrix},$$

because

$$\begin{pmatrix} 1 & 3 & 1 \\ 2 & 0 & -2 \\ 1 & 3 & 1 \end{pmatrix}^{+} = \begin{pmatrix} -\frac{\sqrt{11}}{\sqrt{22}} & 0 & -\frac{\sqrt{11}}{\sqrt{22}} \\ 0 & -1 & 0 \\ -\frac{\sqrt{11}}{\sqrt{22}} & 0 & \frac{\sqrt{11}}{\sqrt{22}} \end{pmatrix} \begin{pmatrix} \sqrt{22} & 0 & 0 \\ 0 & \sqrt{8} & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -\frac{\sqrt{11}}{11} & -\frac{\sqrt{2}}{2} & -3\frac{\sqrt{22}}{22} \\ -3\frac{\sqrt{11}}{11} & 0 & -\frac{\sqrt{22}}{11} \\ -\frac{\sqrt{11}}{11} & \frac{\sqrt{2}}{2} & -3\frac{\sqrt{22}}{22} \end{pmatrix}^{T}.$$

The normal equation of (5.2) is

$$\begin{pmatrix} 1 & 3 & 1 \\ 2 & 0 & -2 \\ 1 & 3 & 1 \end{pmatrix}^{T} \begin{pmatrix} 1 & 3 & 1 \\ 2 & 0 & -2 \\ 1 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 6 & 6 & -2 \\ 6 & 18 & 6 \\ -2 & 6 & 6 \end{pmatrix} = P \begin{pmatrix} 22 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 0 \end{pmatrix} P^{-1},$$

where

$$P^{-1} = \begin{pmatrix} \frac{1}{11} & \frac{3}{11} & \frac{1}{11} \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{9}{22} & -\frac{3}{11} & \frac{9}{22} \end{pmatrix}.$$

Thus, we have

$$x_D = \begin{pmatrix} 6 & 6 & -2 \\ 6 & 18 & 6 \\ -2 & 6 & 6 \end{pmatrix}^D \begin{pmatrix} 8 \\ 12 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{129}{1936} & \frac{3}{242} & -\frac{113}{1936} \\ \frac{3}{242} & \frac{9}{242} & \frac{3}{242} \\ -\frac{113}{1936} & \frac{3}{242} & \frac{129}{1936} \end{pmatrix} \begin{pmatrix} 8 \\ 12 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{15}{22} \\ \frac{6}{11} \\ -\frac{7}{22} \end{pmatrix}.$$

Indicial equations of (5.2) is

$$\begin{pmatrix} 1 & 3 & 1 \\ 2 & 0 & -2 \\ 1 & 3 & 1 \end{pmatrix}^3 = \begin{pmatrix} 16 & 12 & -8 \\ 0 & 0 & 0 \\ 16 & 12 & -8 \end{pmatrix} = P \begin{pmatrix} 8 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} P^{-1},$$

where

$$P^{-1} = \begin{pmatrix} 2 & \frac{3}{2} & -1 \\ -1 & \frac{3}{2} & 1 \\ -1 & 0 & 1 \end{pmatrix}.$$

Therefore

$$x_K = \begin{pmatrix} 16 & 12 & -8 \\ 0 & 0 & 0 \\ 16 & 12 & -8 \end{pmatrix}^D \begin{pmatrix} 8 \\ 0 \\ 8 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} & \frac{3}{16} & -\frac{1}{8} \\ 0 & 0 & 0 \\ \frac{1}{4} & \frac{3}{16} & -\frac{1}{8} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

6 Conclusions

In this paper, the effect of indicial equations in finding minimal solution for inconsistent singular linear system of equations is investigated.

Acknowledgment

We would like to express our gratitude to the Prof S. Abbasbandy Editor in Chief of Mathematics Scientific Journal for his editorial help and the reviewers for their insightful valuable comments and suggestions.

References

[1] A. Ben, T. Greville, Generalized Inverses: Theory and applications, USA, 2001.

[2] S. Abbasbandy, M. Alavi, A new method for solving symmetric fuzzy linear systems, Mathematics Scientific Journal (Islamic Azad University-Arak Branch), 1 (2005) 55-62.

[3] A. Jafarian, S. Measoomy Nia, Utilizing a new feed-back fuzzy neural network for solving a system of fuzzy equations, Communications in Numerical Analysis, ID cna-00096 (2012).

[4] M. Saravi, E. Babolian, M. Rastegari, Systems of linear differential equations and collocation method , Mathematics Scientific Journal(Islamic Azad University-Arak Branch), 5 (2010) 79-87.

[5] D. Conte, R. D'Ambrosio, B. Paternoster, Two-step diagonally-implicit collocation based methods for Volterra Integral Equations, Applied Numerical Mathematics, 62 (2012) 1312-1324.

[6] A. Shahsavaran, Numerical solution of linear Volterra and Fredholm integro differential equations using Haar wavelets, J. Mathematics Scientific Journal, 6 (2010) 85-96.

[7] M. Tavassoli Kajani, S. Mahdavi, Numerical solution of nonlinear integral equations by Galerkin methods with hybrid Legendre and Block-Pulse functions, Mathematics Scientific Journal (Islamic Azad University-Arak Branch), 7 (2011) 97-105.

[8] M. Mosleh, M. Otadi, Fuzzy Fredholm integro-differential equations with artificial neural networks, Communications in Numerical Analysis, ID cna-00128, (2012).

[9] S. Campbell, C. Meyer, Generalized Inverses of Linear Transformations, Pitman London,1979.

[10] B. Detta, Numerical Linear Algebra and Application, Thomson, 1994.

[11] D. Kincaid, W. Cheney, Numerical analysis mathematics of scientific Computing California,1990.

[12] L. Lin, Y. Wei, N. Zhang, Convergence and quotient convergence of iterative methods for solving singular linear equations with index one, Linear Algebra and its Applications, 203 (2008) 35-45.

[13] M. Nikuie, M. K. Mirnia, A method for solving singular linear system of equations, Journal of Operational Research and its Applications (Journal of Applied Mathematics), Appl Math Islamic Azad University-Lahijan Branch, 7 (2010) 79-90.

[14] L. Trefethen, D. Bau, Numerical linear algebra, SIAM, 1997.

[15] L. Zheng, A characterization of the Drazin inverse, Linear Algebra and its Application, 335 (2001) 183-188.

[16] M. Nikuie, M. K. Mirnia, Y. Mahmoudi, Some results about the index of matrix and Drazin inverse, Mathematical Sciences Quarterly Journal (Islamic Azad University-Karaj Branch), 4 (2010) 283-294.

[17] Y. Wei, J. Zhou, A two-step algorithm for solving singular linear systems with index one, Applied Mathematics and Computation, 174 (2006) 252-270.