Legendre wavelet method for solving Hammerstein integral equations of the second kind

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Abstract

An efficient method, based on the Legendre wavelets, is proposed to solve the second kind Fredholm and Volterra integral equations of Hammerstein type. The properties of Legendre wavelet family are utilized to reduce a nonlinear integral equation to a system of nonlinear algebraic equations, which is easily handled with the well-known Newton's method. Examples assuring efficiency of the method and its superiority are presented.

Key words: Legendre wavelets; Fredholm-Hammerstein integral equations; Volterra-Hammerstein integral equations; Newton's method; Operational matrix.
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1 Introduction

The Hammerstein equation is a general model to study semi-linear boundary value problems (BVPs). Actually two-point boundary value problems are reduced to a Hammerstein integral equation through defining a suitable Green's function. The general form of this equation is stated as follows:

$$y(s) = x(s) + \int_\Omega k(s,t)g(t,y(t))dt, \tag{1.1}$$

where the kernel $k(s,t)$ typically arises as the Green's function of a differential operator. As well as reformulation of BVPs, it appears in nonlinear physical phenomena such as electro-magnetic fluid dynamics [1], modeling inhibitory networks in biology [2] and many other scientific fields [3]. There is a wide literature dealing with the Hammerstein equations, including solution existence and solution strategies both numerical and analytic. The equation (1) was first considered by Hemmerstein in early 1930's [4]. Existence of the solution is studied in many papers from different points of view [2,5–9].

A good survey of classic numerical methods has been collected by Atkinson [10]. The interested reader can refer to [3] for a brief introduction and useful references on the subject. Different approaches have also been applied to handle these family of equation for more information see [11,12]. However, here, we focus on methods which are related, to some extend, to our work.

Kumar and Sloans [13] used an equivalent representation of equation (1) and applied collocation techniques to approximate the solution. Using this new representation some (pseudo-)spectral techniques based on orthogonal polynomials have been used to solve Hammerstein equations [14,15]. Wavelet approaches which are efficient techniques to handle integral equations, have also been applied to solve the Hammerstein equation (1). A Petrov-Galerkin approach is discussed in [16]. Different families of wavelets including Daubechies [17], Legendre [18], Chebyshev [19] and rationalized Haar [20] have been successfully applied to Hammerstein equations.

Here we are concerned with the second kind Hammerstein integral equations, both Fredholm and Volterra, which are respectively represented as follows:

Fredholm-Hammerstein:

$$y(s) = x(s) + \int_0^1 k(s,t)g(t,y(t))\,dt, \qquad s \in [0,1), \qquad (1.2)$$

Volterra-Hammerstein:

$$y(s) = x(s) + \int_0^s k(s,t)g(t,y(t))\,dt, \qquad s \in [0,1), \qquad (1.3)$$

where x, k and g are knows functions, with $g(t,y(t))$ nonlinear in y.

The paper is outlined as follows:
To make the paper self-contained, second section reviews some features of Legendre family of wavelets including definitions, function approximation with this family and also operational matrix of integration is addressed to ease referencing in subsequent sections. In sections 3 and 4, the method is respectively applied to Fredholm-Hammerstein and Volterra-Hammerstein integral equations of the second kind, i.e. equations (2) and (3), here the basic ideas are presented. Finally, section 5 is devoted to examples to assure applicability of the method, some comparisons are addressed here to show superiority of the method.

2   The family of Legendre wavelets

This section is devoted to review basic concepts of the Legendre family of wavelets.

2.1   Definitions

In recent years, wavelets have found their way into many different fields of science and engineering. Wavelets are a class of functions constructed

from dilation and translation of a single function called the mother wavelet $\psi(t)$, we have the following family of continuous wavelets as [21]:

$$\psi_{a,b}(t) = \frac{1}{\sqrt{|a|}}\psi(\frac{t-b}{a}), a, b \in \mathbb{R}, a \neq 0 \tag{2.1}$$

Legendre wavelets $\psi_{n,m}(t) = \psi(k, \widehat{n}, m, t)$, have four arguments, $\widehat{n} = 2n - 1$, $n = 1, 2, \ldots, 2^{k-1}$, k can assume any positive integer, m is degree of the Legendre polynomials and t denotes the variable, which is defined on $[0, 1)$ as follows:

$$\psi_{n,m}(t) = \begin{cases} \sqrt{m + \frac{1}{2}}2^{\frac{k}{2}}L_m(2^k t - \widehat{n}), & \frac{\widehat{n}-1}{2^k} \leq t < \frac{\widehat{n}+1}{2^k}, \\ 0, & \text{otherwise,} \end{cases} \tag{2.2}$$

where $n = 1, 2, \ldots, 2^{k-1}$, $m = 0, 1, \ldots, M - 1$.

Here $L_m(t)$, are Legendre polynomials of degree m which are orthogonal with respect to the weight function $\omega(t) = 1$ on $[-1, 1]$.

2.2 Function approximation

Any function $x(s) \in L^2[0, 1)$ can be expanded into Legendre wavelet series as [22]:

$$x(s) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{n,m}\psi_{n,m}(s), \tag{2.3}$$

where the coefficients $c_{n,m}$ are given by

$$c_{n,m} = (x(s), \psi_{n,m}(s)). \tag{2.4}$$

In (2.4), $(\cdot, \cdot)$ denotes the inner product of the function space $L^2[0, 1)$. If the infinite series in (2.3) is truncated, then (2.3) can be written as

$$x(s) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m}\psi_{n,m}(s) = C^T\Psi(t), \tag{2.5}$$

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where C and $\Psi(t)$ are square matrices of size $2^{k-1}M$, given by

$$C = [c_{10}, c_{11}, \ldots, c_{1,M-1}, c_{20}, \ldots, c_{2,M-1}, \ldots, c_{2^{k-1}0}, \ldots, c_{2^{k-1},M-1}]^T. \tag{2.6}$$

$$\Psi = [\psi_{10}, \psi_{11}, \ldots, \psi_{1,M-1}, \psi_{20}, \ldots, \psi_{2,M-1}, \ldots, \psi_{2^{k-1}0}, \ldots, \psi_{2^{k-1},M-1}]^T. \tag{2.7}$$

Similarly, a function of two variables $k(s,t) \in L^2([0,1) \times [0,1))$ can be expanded into Legendre wavelet series as

$$k(s,t) \simeq k^*(s,t) = \Psi^T(s)K\Psi(t) \tag{2.8}$$

where K is an $2^{k-1}M \times 2^{k-1}M$ matrix with entries
$K_{ij} = \left(\Psi_i(t), (k(t,s), \Psi_j(s)) \right)$ for $i,j = 1, 2, \cdots, 2^{k-1}M$ [23].

2.3 The operational matrices

Let $\Psi(t)$ be the Legendre wavelets vector defined in (2.7). The integration of vector $\Psi(t)$ is given by

$$\int_0^t \Psi(\tau)\, d\tau \simeq P\Psi(t), \qquad t \in [0,1), \tag{2.9}$$

where P is a $2^{k-1}M \times 2^{k-1}M$ operational matrix for integration and is obtained as

$$P = \frac{1}{2^k} \begin{bmatrix} L & F & F & \ldots & F \\ 0 & L & F & \ldots & F \\ 0 & 0 & L & \ldots & F \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & L \end{bmatrix}, \tag{2.10}$$

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where F and L are $M \times M$ matrices given by [22]

$$F = \begin{bmatrix} 2 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}, \quad (2.11)$$

and

$$L = \begin{bmatrix} 1 & \frac{\sqrt{3}}{3} & 0 & 0 & \dots & 0 & 0 \\ -\frac{\sqrt{3}}{3} & 0 & \frac{\sqrt{3}}{3\sqrt{5}} & 0 & \dots & 0 & 0 \\ 0 & -\frac{\sqrt{5}}{5\sqrt{3}} & 0 & \frac{\sqrt{5}}{5\sqrt{7}} & \dots & 0 & 0 \\ 0 & 0 & -\frac{\sqrt{7}}{7\sqrt{5}} & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & \frac{\sqrt{2M-3}}{(2M-3)\sqrt{2M-1}} \\ 0 & 0 & 0 & 0 & \dots & -\frac{\sqrt{2M-1}}{(2M-1)\sqrt{2M-3}} & 0 \end{bmatrix}. \quad (2.12)$$

The family $\psi_{n,m}(t)$ forms an orthonormal basis for $L^2[0,1]$, that is,

$$\int_0^1 \psi_{n,m}(t)\psi_{n',m'}(t)\, dt = \delta_{n,n'}\delta_{m,m'}, \quad (2.13)$$

where δ is the Kronecker δ-function, then the integration of the product of two Legendre wavelets vector functions is obtained as:

$$\int_0^1 \Psi(s)\Psi^T(s)\, ds = I, \quad (2.14)$$

where I is $2^{k-1}M \times 2^{k-1}M$-dimensional identity matrix.

The $2^{k-1}M$-square Legendre wavelets matrix is defined as:

$$H = \left[\Psi(\tfrac{1}{2^k M}) \ \Psi(\tfrac{3}{2^k M}) \ \dots \ \Psi(\tfrac{2^k M-1}{2^k M}) \right]. \quad (2.15)$$

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The following property of the product of two Legendre wavelet vector functions will also be used:

$$\Psi(t)\Psi^T(t)C = \tilde{C}\Psi(t), \tag{2.16}$$

where C is defined in (9) and $\tilde{C}$ is called the coefficient matrix and is given in [24] as

$$\tilde{C} = \begin{bmatrix} \widetilde{C_1} & 0 & \dots & 0 \\ 0 & \widetilde{C_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \widetilde{C_n} \end{bmatrix}, \tag{2.17}$$

where $\widetilde{C_i}$, $(i = 1, 2, \dots n)$ are $m \times m$ matrices.

3 Solution of nonlinear Fredholm equations of the second kind

Consider the following nonlinear Fredholm integral equation of the second kind:

$$y(s) = x(s) + \int_0^1 k(s,t)g(t,y(t))\, dt, \qquad s \in [0,1), \tag{3.1}$$

where $x(s) \in L^2[0,1)$, $k(s,t) \in L^2[0,1) \times [0,1)$ and $g(t,y(t))$ is a nonlinear function of the unknown function $y(s)$.
We have defined the function $z(t)$ as

$$\begin{aligned} z(t) &= g(t,y(t)) \\ &= g(t,x(t) + \int_0^1 k(t,u)g(u,y(u))\, du), \\ &= g(t,x(t) + \int_0^1 k(t,u)z(u)\, du), \end{aligned} \tag{3.2}$$

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According to Eqs.(2.8) and (2.5), expand functions z, x and k in terms of Legendre wavelets as

$$z(t) \simeq Z^T \Psi(t), \tag{3.3}$$

and

$$k(s,t) \simeq \Psi^T(s)K\Psi(t), \qquad x(s) \simeq X^T \Psi(s). \tag{3.4}$$

Substituting Eqs.(3.4) and (3.3) into Eq.(3.2) and applying Eq.(2.14) one has

$$
\begin{aligned}
Z^T \Psi(t) &= g(t, X^T \Psi(t) + \int_0^1 \Psi^T(t)K\Psi(u)\Psi^T(u)Z\,du) \\
&\simeq g(t, X^T \Psi(t) + \Psi^T(t)KZ).
\end{aligned}
\tag{3.5}
$$

In order to construct the approximations for $z(t)$ we collocate Eq.(3.5) with Newton-Cotes points $t_i = \frac{2i-1}{2^k M}$, $i = 1, 2, \ldots 2^{k-1}M$, and by using Eq. (2.15) we obtain

$$Z^T He_i \simeq g(t_i, X^T He_i + e_i^T H^T KZ), \qquad i = 1, 2, \ldots 2^{k-1}M. \tag{3.6}$$

After replacing $\simeq$ with $=$, we have a nonlinear system that can be solved with Newton's method for the unknown vector Z.

Considering (3.6), the required approximation to the solution y of Eq.(3.1) in Legendre wavelets is given by

$$
\begin{aligned}
y(s) &= x(s) + \int_0^1 k(s,t)z(t)\,ds \\
&\simeq X^T \Psi(s) + \int_0^1 \Psi^T(s)K\Psi(t)\Psi^T(t)Z\,dt \\
&\simeq (X^T + Z^T K^T)\Psi(s) \simeq Y^T \Psi(s)
\end{aligned}
\tag{3.7}
$$

where $Y = X^T + Z^T K^T$.

4 Solution of nonlinear Volterra equations of the second kind

Consider the following integral equation:

$$y(s) = x(s) + \int_0^s k(s,t)g(t,y(t))\, dt, \qquad s \in [0,1), \qquad (4.1)$$

where $x(s) \in L^2[0,1), k(s,t) \in L^2[0,1) \times [0,1)$ and $g(t,y(t))$ is a nonlinear function of unknown function $y(s)$.
Here, also, we define $z(t)$ as

$$
\begin{aligned}
z(t) &= g(t,y(t)) \\
&= g(t, x(t) + \int_0^t k(t,u)g(u,y(u))\, du) \\
&= g(t, x(t) + \int_0^t k(t,u)z(u)\, du). \qquad (4.2)
\end{aligned}
$$

According to Eqs.(2.8) and (2.5) we expand the functions z, x and k in terms of Legendre wavelets as

$$z(t) \simeq Z^T \Psi(t), \qquad (4.3)$$

and

$$k(s,t) \simeq \Psi^T(s)K\Psi(t), \qquad x(s) \simeq X^T \Psi(s). \qquad (4.4)$$

Substituting Eqs.(4.4) and (4.3) into Eq.(4.2) and applying Eq.(2.16) one has

$$
\begin{aligned}
Z^T \Psi(t) &= g(t, X^T \Psi(t) + \int_0^t \Psi^T(t)K\Psi(u)\Psi^T(u)Z\, du) \\
&\simeq g(t, X^T \Psi(t) + \Psi^T(t)K \int_0^t \Psi(u)\Psi^T(u)Z\, du) \\
&\simeq g(t, X^T \Psi(t) + \Psi^T(t)K \int_0^t \widetilde{Z}\Psi(u)\, du). \qquad (4.5)
\end{aligned}
$$

The integrals of (4.5) can be obtained by multiplying the operational matrix of integration of (2.9) as follows:

$$Z^T \Psi(t) \simeq g(t, X^T \Psi(t) + \Psi^T(t)K\widetilde{Z}P\Psi(t)). \qquad (4.6)$$

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By approximating $\Psi^T(t)K\widetilde{Z}P\Psi(t)$ in Eq.(4.6) in terms of Legendre wavelets we achieve

$$\Psi^T(t)K\widetilde{Z}P\Psi(t) = \widehat{Z}\Psi(t). \qquad (4.7)$$

We can achieve $\widehat{Z}$ by a way like $\widetilde{C}$ and we see that each element $\widehat{Z}$ is obtained by the sum of column elements of $K\widetilde{Z}P$ with respect to coefficient $\widetilde{C}$ in Eq.(2.16). So

$$Z^T\Psi(t) \simeq g(t, X^T\Psi(t) + \widehat{Z}\Psi(t)). \qquad (4.8)$$

Just like the Fredholm case, we construct the approximations for $z(t)$ collocating Eq.(4.8) with the $2^{k-1}M$ collocation points, $t_i = \frac{2i-1}{2^kM}$, $i = 1, 2, \ldots 2^{k-1}M$,

$$Z^T He_i \simeq g(t_i, X^T He_i + \widehat{Z}He_i), \qquad i = 1, 2, \ldots 2^{k-1}M.$$

After replacing $\simeq$ with $=$, we have a nonlinear system that can be solved with Newton's method for the unknown vector Z.

Considering (3.6), the required approximation to the solution y of Eq.(4.1) in Legendre wavelets is given by

$$\begin{aligned} y(s) &= x(s) + \int_0^s k(s,t)z(t)\, ds \\ &\simeq X^T\Psi(s) + \int_0^s \Psi^T(s)K\Psi(t)\Psi^T(t)Z\, dt \\ &\simeq (X^T + \widehat{Z})\Psi(s) = Y^T\Psi(s), \end{aligned}$$

where $Y = X^T + \widehat{Z}$.

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5 Numerical examples

Example 1: Consider the nonlinear Fredholm integral equation of the second kind as follows:

$$y(s) = x(s) + \int_0^1 (s-t)y^2(t)\,dt, \qquad 0 \le s \le 1, \qquad (5.1)$$

where

$$x(s) = ln(s+1) + 2ln(2)(1 - sln(2) + 2s) - 2s - \frac{5}{4}, \qquad (5.2)$$

and the exact solution is $y(s) = ln(s+1)$ [25].

In Table 1, we have reported the absolute errors of our method, with $k = 2, M = 15$, and the homotopy analysis method (HAM) [25]. It is easily concluded that it is efficient and superior to HAM.

Table 1. Errors in Example 1

s	Our method	HAM [25]
0.0	0.8733291867e-12	2.56186467595e-08
0.1	0.6161599009e-12	2.53309949127e-08
0.2	0.2554345624e-12	2.19432191028e-08
0.3	0.4911071549e-12	1.45529119865e-08
0.4	0.9847678228e-13	1.06476171213e-08
0.5	0.8460454559e-12	1.84023705163e-08
0.6	0.1638689184e-11	6.84976264597e-09
0.7	0.1591504706e-11	6.08723849100e-10
0.8	0.2009281630e-11	3.55107188090e-09
0.9	0.2072231275e-11	2.81965084610e-09
1.0	0.2941535904e-11	4.16376155700e-10

Example 2: Consider the nonlinear Fredholm integral equation of the

second kind as follows:

$$y(s) = 1 + se - e^s + \int_0^1 k(s,t)e^{y(t)} \, dt, \qquad 0 \le s \le 1, \qquad (5.3)$$

where

$$k(s,t) = \begin{cases} -t(1-s), \ s \le t, \\ -s(1-t), \ t \le s, \end{cases} \qquad t,s \in [0,1]. \qquad (5.4)$$

Its exact solution is $y(s) = s$. We applied the Legendre method according to section 3, with $k = 2$ and $M = 11$. Table 2 illustrates the errors and it is seen that the approximate solution coincides with the exact solution for its (almost) 14 significant digits.

Example 3: Consider the following Fredholm integral equation of the second kind

$$y(s) = e^s - \frac{(1+2e^3)s}{9} + \int_0^1 sty(t)^3 \, dt, \qquad 0 \le s \le 1, \qquad (5.5)$$

where the exact solution is $y(s) = e^s$. It has been handled by a Wavelet-Galerkin approach in [26]. Table 3. reports the errors of the Wavelet-Galerkin method and the method proposed in section 3 and it confirms that our method competes well with other approaches.

Example 4: Consider the following Fredholm integral equation of the second kind

$$y(s) = e^{s+1} - \int_0^1 e^{s-2t} y(t)^3 \, dt, \qquad 0 \le s \le 1, \qquad (5.6)$$

with the exact solution $y(s) = e^s$. We have solved this equation with the Legendre method described in section 3 by choosing $k = 1$ and $M = 16$. Comparing the errors of our method and the Haar wavelet approach [27], which is illustrated in Table 4. shows that our method is superior to Haar family.

Example 5: Consider the following equation

$$y(s) = \sin(\pi s) + \frac{1}{5} \int_0^1 \cos(\pi s) \sin(\pi t) y(t)^3 \, dt, \qquad 0 \le s \le 1, \qquad (5.7)$$

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Table 2. Errors in Example 2

s	Error of approximation
0	0.3762958374e-15
0.0625	0.06249999999999995
0.1250	0.1249999999999997
0.1875	0.1874999999999999
0.2500	0.2499999999999996
0.3125	0.3124999999999997
0.3750	0.3749999999999995
0.4375	0.4374999999999998
0.5000	0.4999999999999959
0.5625	0.5624999999999985
0.6250	0.6250000000000000
0.6875	0.6874999999999988
0.7500	0.7500000000000004
0.8125	0.8124999999999990
0.8750	0.8750000000000004
0.9375	0.9374999999999989
1.0000	0.9999999999999965

where

$$y(s) = \sin(\pi s) + \frac{1}{3}(20 - \sqrt{391})\cos(\pi s), \qquad (5.8)$$

is the exact solution. The Legendre based method of section 3 is applied to this equation with $k = 2$ and $M = 15$ and Table 5. shows the absolute error of SE and DE-Sinc methods [28] and the results of Legendre wavelet method. It is seen that the used method is more convenient than the Sinc methods.

Example 6: Consider the nonlinear Volterra integral equation of the

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Table 3. Error comparisons for Example 3

s	Error of approximation		
0.0	1.000000	0.9999538	0.999956
0.2	1.221403	1.221467	1.221391
0.4	1.491825	1.491994	1.491845
0.6	1.822119	1.822376	1.822157
0.8	2.225541	2.225821	2.225517
1.0	2.718282	2.718574	2.718217

Table 4. Error comparisons for Example 4

s	Errors of our method	Errors of method [27]
0.1	6e-06	8e-03
0.2	7e-06	9e-03
0.3	8e-06	1e-02
0.4	9e-06	1e-02
0.5	1e-05	2e-02
0.6	1e-05	1e-02
0.7	1e-05	1e-02
0.8	1e-05	1e-02
0.9	1e-05	1e-02

second kind as follows:

$$y(s) = e^{-s} + \int_0^s e^{-s+t}(y(t) + e^{-y(t)})\, dt, \qquad 0 \le s \le 1, \qquad (5.9)$$

with the exact solution $y(s) = ln(s + e)$.

Example 7: Consider the following Volterra type integral equation of the second kind

$$y(s) = e^s - \frac{1}{2}(e^{2s} - 1) + \int_0^t y(t)^2\, dt, \qquad 0 \le s \le 1, \qquad (5.10)$$

where $y(s) = e^s$ is the exact solution [27]. Here the Legendre wavelets

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Table 5. Error comparisons for Example 5

s	Legendre wavelet method	SE-Sinc method	DE-Sinc method
0.0	3.39e-10	4.15e-07	1.4e-10
0.1	2.98e-10	1.33e-05	1.29e-08
0.2	3.80e-10	6.85e-06	2.05e-08
0.3	3.84e-10	6.22e-06	1.25e-08
0.4	4.80e-10	2.19e-06	2.41e-09
0.5	5.38e-10	2.15e-07	0.00e-00
0.6	5.45e-10	5.85e-06	9.54e-09
0.7	4.68e-10	1.02e-05	2.31e-08
0.8	7.64e-10	8.25e-06	2.33e-08
0.9	2.99e-11	1.66e-05	1.61e-08
1.0	2.43e-10	4.14e-07	1.4e-010

method is applied according to section 4 with $k = 2$ and $M = 8$, and again it is seen that the Legendre method is superior to Haar wavelet approach [27], see Table 7.

6   Conclusion

The Legendre wavelet family have proved its efficiency in handling different families of functional equations, here this family is used for solving Hammerstein equations. The Legendre matrix of integration is applied to the equation after rewriting the equation according to Kumar and Sloan [13]. Then the nonlinear equation is reduced to a system of nonlinear equations which is easily handled by Newton's method. The results show that the method is superior to some other methods like Haar wavelet and rationalized Haar and Sinc methods. Also, when equal number of terms is used to expand the solution the method competes well enough with methods like homotopy analysis method.

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Table 6. Error comparisons for Example 6

s	Legendre wavelet method	SE-Sinc method	DE-Sinc method	
0	1.012410	1.002718	1.000630	1
0.0625	1.027253	1.022463	1.022406	1.022732
0.1250	1.044094	1.043699	1.044988	1.044959
0.1875	1.062735	1.066242	1.068789	1.066703
0.2500	1.082990	1.088767	1.086951	1.087983
0.3125	1.104695	1.117714	1.109553	1.108821
0.3750	1.127692	1.140411	1.126028	1.129233
0.4375	1.151843	1.157493	1.147332	1.149236
0.5000	1.173849	1.169304	1.162342	1.168848
0.5625	1.216191	1.197647	1.182438	1.188082
0.6250	1.250489	1.218966	1.196168	1.206953
0.6875	1.277540	1.233995	1.215157	1.225475
0.7500	1.298073	1.243355	1.227795	1.243659
0.8125	1.312754	1.270816	1.245856	1.261519
0.8750	1.322193	1.290882	1.257384	1.279066
0.9375	1.326946	1.304338	1.274662	1.296310
1.0000	1.327525	1.311900	1.285329	1.313262

References

[1] K. E. Atkinson, *The Numerical Solution of Integral Equations of The Second Kind*, Cambridge University Press, Cambridge, 1997.

Table 7. Error comparisons for Example 7

s	Legendre method	Haar method
0.1	3e-03	1e-03
0.2	6e-03	1e-03
0.3	1e-02	2e-03
0.4	2e-03	3e-03
0.5	5e-03	3e-03
0.6	6e-02	8e-03
0.7	8e-02	2e-02
0.8	2e-02	7e-02
0.9	1e-01	1e-01

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