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BMO Space and its relation with wavelet theory

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Abstract

The aim of this paper is $a)$ if $\sum_{k=1}^{\infty} a_k^2 < \infty$ then $\sum_{k=1}^{\infty} a_k r_k(x) \in BMO$ that $\{r_k(x)\}$ is Rademacher system. $b)$ $\sum_{k=1}^{\infty} a_k \omega_{n_k}(x) \in BMO, 2^k \le n_k < 2^{k+1}$ that $\{\omega_n(x)\}$ is Walsh system. $c)$ If $|a_k| < \frac{1}{k}$ then $\sum_{k=1}^{\infty} a_k \omega_k(x) \in BMO$.

Keywords: BMO space, wavelets, Orthonormal system, Rademacher system, Walsh system, Haar system.

1 Introduction

1.1 The space of bounded mean oscillation functions

***Definition* 1.1.** ([3],[5],[6],[7],[8])A locally integrable function f will be said to belong to BMO if the inequality

$$\frac{1}{|B|} \int_B |f(x) - f_B| dx \le A \tag{1.1}$$

holds for all balls B; here $|B|$ is volume of B and $f_B = |B|^{-1} \int_B f dx$ denotes the mean value of f over the ball B. The inequality (1) asserts that over any ball B, the average oscillation of f is bounded.

The smallest bound A in (1) is called the norm of f in this space, and is denoted by $||f||_{BMO}$.

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Theorem 1.1. *([5],[6])Suppose that f is in BMO. Then*
(a) For any $p < \infty$, f is locally in L^p, and

$$\frac{1}{|B|}\int_B |f - f_B|^p dx \leq c_p \|f\|^p_{BMO}, \tag{1.2}$$

for all balls B.
(b) There exist positive constants c_1 and c_2 so that, for every $\alpha > 0$ and every ball B,

$$|\{x \in B : |f(x) - f_B| > \alpha\}| \leq c_1 e^{-c_2\alpha/\|f\|_{BMO}}.|B|. \tag{1.3}$$

***Definition* 1.2.** ([5])For n=1,2,3,..., the nth Rademacher function is defined by

$$r_n(x) = \begin{cases} 1, & \text{if i odd and } x \in ((i-1)/2^n, i/2^n) = \Delta_n^i; \\ -1, & \text{if i even and } x \in ((i-1)/2^n, i/2^n) = \Delta_n^i. \end{cases} \tag{1.4}$$

In addition, it will be convenient to suppose that $r_0(x) = 1$ for $x \in (0,1)$ and that $r_n(i/2^n) = 0$ for $i = 0, 1, ..., 2^n$; n=0,1,....Then we can give a more intensive definition of the Rademacher functions by the formula

$$r_n(x) = \text{sgnsin} 2^n \pi x, x \in [0,1], n = 0, 1, ... \tag{1.5}$$

If n is a positive integer,

$$n = \sum_{k=0}^{\infty} \theta_k 2^k = \sum_{k=0}^{k(n)} \theta_k 2^k, \quad k(n) = [\log_2 n], \quad \theta_{k(n)}(n) = 1.$$

***Definition* 1.3.** ([5])The Walsh system is the system $W = \{\omega_n(x)\}_{n=0}^{\infty}, x \in [0,1]$, where $\omega_0(x) = 1$ and, for $n \geq 1$,

$$\omega_n(x) = \prod_{k=0}^{\infty} [r_{k+1}(x)]^{\theta_k} = r_{k(n)+1}(x) \prod_{k=0}^{k(n)-1} [r_{k+1}(x)]^{\theta_k},$$

where $r_k(x)$, k=1,2,..., are the Rademacher functions.

1.2 Quasi-orthogonal expansions

***Definition* 1.4.** ([6])A binary interval or dyadic interval is an interval of the form $((i-1)/2^k, i/2^k)$, where $i = 1, \ldots, 2^k$, $k = 0, 1, ...$

Our orthogonal decompositions (more precisely, "quasi-orthogonal" decompositions) will be given in terms of a family of "bump" functions; each such function will be associated to a dyadic cube. We fix our notation as follows: the letter Q will be reserved for a dyadic cube, and $B = B_Q$ will be the ball with the same center and twice the diameter (thus $B_Q \supset Q$); similarly the ball B_j will be associated to Q_j, etc.

For each dyadic cube Q, we will be given a function ϕ_j, supported in B_Q, that satisfies certain natural size, regularity, and moment conditions. We shall assume that

$$|D^\alpha \phi_Q| \leq \frac{l(Q)^{-|\alpha|}}{|Q|^{1/2}}, \int x^\alpha \phi_Q(x)dx = 0, \quad 0 \leq |\alpha| \leq n \tag{1.6}$$

with $l(Q)$ denoting the length of a side of the cube $Q \subset R^n$.
We shall be dealing with functions f that can be represented in the form

$$f = \sum_Q a_Q \phi_Q, \tag{1.7}$$

where a_Q is a suitable collection of constants, and the summation in (7) is carried over all dyadic cubes.
Various extensions of the same ideas are possible, giving also characterizations of many other function spaces besides BMO, leading in addition to what are now known as "wavelet" decompositions.

Theorem 1.2. *([6])(a) Suppose the coefficients a_Q satisfy the inequalities*

$$\sum_{Q \subset Q_0} |a_Q|^2 \leq A|Q_0| \tag{1.8}$$

for all dyadic cubes Q_0, where the summation in (8) is taken over all dyadic subcubes of Q_0. Then the series (7) gives an $f \in BMO$ in the sense that

$$\lim_{\rho_1 \to 0, \rho_2 \to \infty} \sum_{\rho_1 \leq l(Q) \leq \rho_2} a_Q \phi_Q = f$$

exists in the weak topology of BMO.
(b) Conversely, suppose $f \in BMO$. Then there is a collection of functions ϕ_Q and a collection of coefficients a_Q that satisfy (6) and (8) respectively, so that f is representable as the sum (7), in the sense asserted in part (a).

The smallest A for which (8) holds is comparable with $||f||^2_{BMO}$.
Remark. A simplified version of the system a_Q occurs in the dyadic context, and is given by the Haar basis. We describe the situation in one dimension. Suppose h is the function supported in the unit interval $[0, 1]$ that equals 1 in the left half and -1 in the right half. For any dyadic interval Q, set

$$h_Q = 2^{j/2} h(2^j x - k), Q = [k2^{-j}, (k+1)2^J].$$

While the h_Q satisfy only the size condition $|h_Q| \leq |Q|^{-1/2}$ and the moment condition $\int h_Q dx = 0$ (and not the full conditions (6)), they have the compensating merit of forming a complete orthonormal basis for $L^2(R^1)$. For $f = \sum a_Q h_Q$, the property

$$\sum_{Q \subseteq Q_0} |a_Q|^2 \leq c|Q_0|$$

is then equivalent with f being in BMO in the dyadic sense.
Corollary 1.1. Let f is a function on $[0,1]$, then $f \in BMO$ if and only if for every dyadic interval $J \subseteq [0,1]$ the inequality

$$\sum_{I \subseteq J} |f_I|^2 \leq A|J|$$

be satisfied, that I is dyadic.
Proof. If $\chi_I(x)$ be the Haar function associated with the dyadic interval I and the Haar coefficient over I of f is

$$f_I = (f, \chi_I) := \int_I f(x)\chi_I(x)dx,$$

then from Theorem 1.2. the corollary is immediate.

2 Main results

Theorem 2.1. *If* $\sum_{k=1}^{\infty} a_k^2 < \infty$ *then*

$$\sum_{k=1}^{\infty} a_k r_k(x) \in BMO$$

that $\{r_k(x)\}$ *is Rademacher system.*

Proof. Let $f(x) = \sum_{k=1}^{\infty} a_k r_k(x)$ then for every dyadic I with $|I| = \frac{1}{2^n}$:

$$f_I = \int_0^1 f.\chi_I dx = \int_0^1 (\sum_{k=1}^{\infty} a_k r_k(x)).\chi_I(x)dx = \sum_{k=1}^{\infty} a_k \int_0^1 r_k(x).\chi_I(x)dx = a_n 2^{\frac{n}{2}}.|I| = a_n \frac{1}{2^{\frac{n}{2}}}$$

therefore if $|J| = \frac{1}{2^m}$ then

$$\sum_{I \subseteq J} |f_I|^2 = |f_J|^2 + |f_{J_1^{(1)}}|^2 + |f_{J_1^{(2)}}|^2 + \ldots + \sum_{i=1}^{2^k} |f_{J_k^{(i)}}|^2 + \ldots$$

$$\Rightarrow \sum_{I \subseteq J} |f_I|^2 = |a_m|^2.\frac{1}{2^m} + 2|a_{m+1}|^2.\frac{1}{2^{m+1}} + \ldots + 2^k|a_{m+k}|^2.\frac{1}{2^{m+k}} + \ldots \leq \|\{a_n\}\|_2^2|J| = A|J|$$

Now corollary 1.1 implies that $f \in BMO$.

Theorem 2.2. *If* $\sum_{k=1}^{\infty} a_k^2 < \infty$ *then*

$$\sum_{k=1}^{\infty} a_k \omega_{n_k}(x) \in BMO$$

that $\{\omega_n(x)\}$ *is Walsh system and* $2^k \leq n_k < 2^{k+1}$.

Proof. Let $f(x)=\sum_{k=1}^{\infty} a_k\omega_{n_k}(x)$ then for every dyadic I with $|I|=\frac{1}{2^n}$:

$$f_I=\int_0^1 f.\chi_I dx=\int_0^1 (\sum_{k=1}^{\infty} a_k\omega_{n_k}(x)).\chi_I(x)dx=\sum_{k=1}^{\infty} a_k\int_0^1 r_1^{\alpha_1}(x).r_2^{\alpha_2}(x).\cdots .r_k^{\alpha_k}(x).r_{k+1}(x).\chi_I dx$$

where $\alpha_1,\alpha_2,\ldots,\alpha_k\in\{0,1\}$. Now $r_1^{\alpha_1}(x).r_2^{\alpha_2}(x).\ldots.r_k^{\alpha_k}(x).r_{k+1}(x).\chi_I$ is equal $\frac{1}{\sqrt{|I|}}$ if $x\in I$ and elsewhere is equal zero, then $|f_I|=|a_n|.2^{\frac{n}{2}}.|I|=\frac{|a_n|}{2^{\frac{n}{2}}}$.
Therefore if $|J|=\frac{1}{2^m}$ the following equality is satisfied:

$$\sum_{I\subseteq J}|f_I|^2=|f_J|^2+|f_{J_1^{(1)}}|^2+\sum_{i=1}^{2^k}|f_{J_k^{(i)}}|^2+\ldots=|a_m|^2.\frac{1}{2^m}+2|a_{m+1}|^2.\frac{1}{2^{m+1}}+\ldots+2^k|a_{m+k}|^2.\frac{1}{2^{m+k}}+\ldots$$

and finally

$$\sum_{I\subseteq J}|f_I|^2=\frac{1}{2^m}(|a_m|^2+|a_{m+1}|^2+\ldots)\leq\|\{a_n\}\|_2^2|J|=A|J|.$$

Now corollary 1.1 implies that $f\in BMO$.

Theorem 2.3. *If $|a_k|<\frac{1}{k}$ then*

$$\sum_{k=1}^{\infty} a_k\omega_k(x)\in BMO$$

that $\{\omega_n(x)\}$ is Walsh system.

Proof. Let $f(x)=\sum_{k=1}^{\infty} a_k\omega_k(x)$ then for every dyadic I with $|I|=\frac{1}{2^{k_0}}$:

$$f_I=\int_0^1 f.\chi_I dx=\int_0^1 (\sum_{n=0}^{\infty}\sum_{k=2^n}^{2^{n+1}-1} a_k\omega_k(x)).\chi_I(x)dx=\int_0^1 (\sum_{k=2^{k_0}}^{2^{k_0+1}-1} a_k\omega_k(x)).\chi_I(x)dx$$

then

$$|f_I|\leq\frac{1}{2^{k_0}}\sum_{k=2^{k_0}}^{2^{k_0+1}-1}|\int_0^1\omega_k(x).\chi_I(x)dx|\leq\frac{1}{2^{k_0}}=|I|$$

$$\Rightarrow\sum_{I\subseteq J}|f_I|^2\leq\sum_{I\subseteq J}|I|^2=|J|^2+2.(\frac{|J|}{2})^2+\cdots+2^n.(\frac{|J|}{2^n})^2+\cdots=|J|^2(1+\frac{1}{2}+\frac{1}{4}+\cdots)=A|J|.$$

Now corollary 1.1 implies that $f\in BMO$.

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