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Approximate fixed point theorems for Geraghty-contractions

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Abstract

The purpose of this paper is to obtain necessary and sufficient conditions for existence approximate fixed point on Geraghty-contraction. In this paper, definitions of approximate -pair fixed point for two maps T_α, S_α, and their diameters are given in a metric space.

Key words: Approximate fixed point; Approximate-pair fixed point; Geraghty-contraction.

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1 Introduction

In 1973, Geraghty [2] introduced the Geraghty-contraction and proved the fixed point property for it. In 2006, MăDălina Berinde [1] proved the approximate fixed point property for various types of well known generalized contractions on metric spaces.

In this paper, starting from the article of Zhang, Su, Cheng [3], we study Geraghty-contraction on partially ordered metric spaces, and we give some qualitative and quantitative results regarding approximate fixed points of such contraction mapping.

Throughout this article, we denote by Γ the functions $\beta : [0, \infty) \to [0, 1)$ satisfying the following condition:

$$\beta(t_n) \to 1 \Rightarrow t_n \to 0.$$

Definition 1.1 *[2] Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be a Geraghty-contraction if there exists $\beta \in \Gamma$ such that for any $x, y \in X$,*

$$d(Tx, Ty) \leq \beta(d(x, y))d(x, y).$$

Theorem 1.1 *[2] Let (X, d) be a complete metric space, and let $T : X \to X$ be an operator. Suppose that there exists $\beta \in \Gamma$ such that for any $x, y \in X$,*

$$d(Tx, Ty) \leq \beta(d(x, y))d(x, y).$$

Then T has a unique fixed point.

In 2012, Caballero et al. considered another contraction condition also give a generalization of Theorem 1.2 by considering a non-self mapping, and they get the following theorems.

Definition 1.2 *[4] Let A, B be two nonempty subsets of a metric space (X, d). A mapping $T : A \to B$ is said to be a Geraghty-contraction if there exists $\beta \in \Gamma$ such that for any $x, y \in A$,*

$$d(Tx, Ty) \leq \beta(d(x, y))d(x, y).$$

In [1], the author defined the approximate fixed point property for self mapping on metric spaces.

Definition 1.3 *[1] Let (X, d) be a metric space, $\epsilon > 0$ and $T : X \to X$ be a map. Then $x_0 \in X$ is $\epsilon-$fixed point for T if $d(Tx_0, x_0) < \epsilon$.*

Definition 1.4 *[1] In this paper we will denote the set of all $\epsilon-$ fixed points of T, for a given ϵ, by :*

$$F_\epsilon(T) = \{x \in X \mid x \ is \ an \ \epsilon - fixed \ point \ of \ T\}.$$

Definition 1.5 *[1] Let (X, d) be a metric space and $T : X \to X$ be a map. Then T has the approximate fixed point property if*

$$\forall \epsilon > 0, \ F_\epsilon(T) \neq \varnothing.$$

Definition 1.6 *[5] Let $(X, \|.\|)$ be a completely norm space and $T : X \to X$, and $T_\alpha : X \to X$ be a map as follow:*

$$T_\alpha = \alpha I + (1 - \alpha)T, \ 0 < \alpha < 1.$$

Then $x_0 \in X$ is $\epsilon-$fixed point for T_α if $\|T_\alpha x_0 - x_0\| < \epsilon$.

Remark 1.1 *[5] In this paper we will denote the set of all $\epsilon-$ fixed points of T_α, for a given ϵ, by :*

$$F_\epsilon(T_\alpha) = \{x \in X \mid x \ is \ an \ \epsilon - fixed \ point \ of \ T_\alpha\}.$$

2 $\epsilon-$ fixed point in Geraghty-contraction for T and T_α maps

In this section, we give some results on $\epsilon-$ fixed point in Geraghty-contraction and its diameter.

Theorem 2.1 *Let (X, d) be a metric space and $T : X \to X$ be a map, $x_0 \in X$ and $\epsilon > 0$. If $d(T^n(x_0), T^{n+k}(x_0)) \to 0$ as $n \to \infty$ for some $k > 0$, then T^k has an $\epsilon-$ fixed point.*

Proof: . *Since* $d(T^n(x_0), T^{n+k}(x_0)) \to 0$ *as* $n \to \infty$, $\epsilon > 0$

$$\exists n_0 > 0 \;\; s.t. \;\; \forall n \geq n_0 \quad d(T^n(x_0), T^{n+k}(x_0)) < \epsilon.$$

Then

$$d(T^{n_0}(x_0), T^k(T^{n_0}(x_0)) < \epsilon,$$

therefore $T^{n_0}(x_0)$ *is an* $\epsilon-$ *fixed point of* T^k. ■

Theorem 2.2 *Let* (X, d) *be a metric space and* $T : X \to X$ *a Geraghty-contraction map. Then:*

$$\forall \epsilon > 0, \;\; F_\epsilon(T) \neq \varnothing.$$

Proof: *Let* $\epsilon > 0$, $x \in X$.

$$\begin{aligned} d(T^n(x), T^{n+1}(x)) &= d(T(T^{n-1}(x)), T(T^n(x))) \\ &\leq \beta(d(T^{n-1}(x), T^n(x)))d(T^{n-1}(x), T^n(x)) \\ &\leq ..... \\ &\leq (\beta(d(T^{n-1}(x), T^n(x))))^{n-1} d(T(x), T^2(x)) \\ &\leq (\beta(d(T^{n-1}(x), T^n(x))))^{n} d(x, Tx). \end{aligned}$$

But $\beta \in \Gamma$ *Therefore*

$$Lim_{n\to\infty} d(T^n(x), T^{n+1}(x)) = 0, \;\; \forall x \in X.$$

Now by Theorem 2.3 it follows that $F_\epsilon(T) \neq \varnothing, \forall \epsilon > 0$. ■

Theorem 2.3 *Let* (X, d) *be a metric space and* $T : X \to X$ *a Geraghty-contraction map. If* $F_\epsilon(T)$, *the set of Approximate fixed point of* T, *is nonempty then the mapping*

$$T_\alpha = \alpha I + (1 - \alpha)T, \;\; 0 < \alpha < 1$$

satisfy in Geraghty-contraction and $F_\epsilon(T) = F_\epsilon(T_\alpha)$. *Moreover* $d(T^n_\alpha(x), T^{n+k}_\alpha(x)) \to 0$ *as* $n \to \infty$, *for some* $k > 0$, $\epsilon > 0$.

Proof: *By the definition of* $F_\epsilon(T)$, $F_\epsilon(T) = F_\epsilon(T_\alpha)$. *Also, since* T *satisfy in Geraghty-contraction and* I *is identify function, it follows that* T_α *satisfy in Geraghty-contraction. Now, we prove* $d(T_\alpha^n(x_0), T_\alpha^{n+k}(x_0)) \to 0$ *as* $n \to \infty$. *Suppose* $x \in X$ *now, observe first that* $d(T_\alpha x, T_\alpha^2 x) \leq \beta(d(x, T_\alpha x))d(x, T_\alpha x)$ *and, by induction, that* $d(T_\alpha^n x, T_\alpha^{n+1} x) \leq (\beta(d(x, T_\alpha x)))^n d(x, T_\alpha x)$. *Thus, for any* n *and any* $k > 0$, *we have*

$$\begin{aligned} d(T_\alpha^n(x), T_\alpha^{n+k}(x)) &\leq \sum_{i=n}^{n+k-1} d(T_\alpha^i(x), T_\alpha^{i+1}(x)) \\ &\leq ((\beta(d(x, T_\alpha x)))^n + \cdots + (\beta(d(x, T_\alpha x)))^{n+k-1})d(x, T_\alpha x) \\ &\leq \frac{(\beta(d(x, T_\alpha x)))^n}{1 - (\beta(d(x, T_\alpha x)))} d(x, T_\alpha x). \end{aligned}$$

But $\beta \in \Gamma$ *Therefore* $d(T_\alpha^n(x_0), T_\alpha^{n+k}(x_0)) \to 0$ *as* $n \to \infty$. ■

Corollary 2.1 *Let* (X, d) *be a metric space and* $T : X \to X$ *a Geraghty-contraction map. If* $F_\epsilon(T)$, *the set of Approximate fixed point of* T, *is nonempty then the mapping*

$$T_\alpha = \alpha I + (1 - \alpha)T, \ 0 < \alpha < 1$$

satisfy in Geraghty-contraction and $F_\epsilon(T) = F_\epsilon(T_\alpha)$. *Then:*

$$\forall \epsilon > 0, \ F_\epsilon(T_\alpha) \neq \varnothing.$$

Proof: *By Theorem 2.4 it follows that* $F_\epsilon(T) \neq \varnothing, \forall \epsilon > 0$, *Therefore*

$$F_\epsilon(T_\alpha) \neq \varnothing, \forall \epsilon > 0.$$

■

Definition 2.1 *Let* $T : X \to X$, *be a map and* $\epsilon > 0$. *We define diameter* $F_\epsilon(T)$ *by*

$$diam(F_\epsilon(T)) = \sup\{d(x, y) : \quad x, y \in F_\epsilon(T)\}.$$

Theorem 2.4 *Let $T : X \to X$, and $\epsilon > 0$. If $T : X \to X$ a Geraghty-contraction map. Then*

$$diam(F_\epsilon(T) \leq \frac{2\epsilon}{1-\beta(d(x,Tx))}, \quad \beta \in \Gamma.$$

Proof. *If $x, y \in F_\epsilon(T)$, then*

$$\begin{aligned} d(x,y) &\leq d(x,Tx) + d(Tx,Ty) + d(Ty,y) \\ &\leq \epsilon_1 + \beta(d(x,Tx))d(x,y) + \epsilon_2. \end{aligned}$$

put $\epsilon = Max\{\epsilon_1, \epsilon_2\}$, therefore $d(x,y) \leq \frac{2\epsilon}{1-\beta(d(x,Tx))}$. Hence $diam(F_\epsilon(T)) \leq \frac{2\epsilon}{1-\beta(d(x,Tx))}$. ■

Definition 2.2 *Let $T : X \to X$ a map,*

$$T_\alpha = \alpha I + (1-\alpha)T, \ 0 < \alpha < 1$$

a map and $\epsilon > 0$. We define diameter $F_\epsilon(T_\alpha)$ by

$$diam(F_\epsilon(T_\alpha)) = \sup\{d(x,y) : \quad x, y \in F_\epsilon(T_\alpha)\}.$$

Theorem 2.5 *Let $T : X \to X$, and $\epsilon > 0$. If $T : X \to X$ a Geraghty-contraction map and $T_\alpha : X \to X$ be a map as follow:*

$$T_\alpha = \alpha I + (1-\alpha)T, \ 0 < \alpha < 1$$

Then

$$diam(F_\epsilon(T_\alpha) \leq \frac{2\epsilon}{1-\beta(d(x,T_\alpha x))}.$$

Proof. *If $x, y \in F_\epsilon(T_\alpha)$, then*

$$\begin{aligned} d(x,y) &\leq d(x,T_\alpha x) + d(T_\alpha x,T_\alpha y) + d(T_\alpha y,y) \\ &\leq \epsilon_1 + \beta(d(x,T_\alpha x))d(x,y) + \epsilon_2. \end{aligned}$$

put $\epsilon = Max\{\epsilon_1, \epsilon_2\}$, therefore $d(x,y) \leq \frac{2\epsilon}{1-\beta(d(x,T_\alpha x))}$. Hence $diam(F_\epsilon(T_\alpha)) \leq \frac{2\epsilon}{1-\beta(d(x,T_\alpha x))}$. ■

3 Approximate-pair fixed point and (T_α, S_α)

In this section we will consider the existence of approximate fixed points for two maps $T_\alpha : A \cup B \to A \cup B$, $S_\alpha : A \cup B \to A \cup B$, where

$$T_\alpha = \alpha I + (1-\alpha)T,\ S_\alpha = \alpha I + (1-\alpha)S,\ 0 < \alpha < 1,$$

and $T : A \cup B \to A \cup B$, $S : A \cup B \to A \cup B$.

In 2011, Mohsenalhosseini et al. considered the existence of approximate best proximity points for two maps $T : A \cup B \to A \cup B$, $S : A \cup B \to A \cup B$ and they get the following theorems.

Definition 3.1 *[6] Let A and B be nonempty subsets of a metric space (X, d) and $T : A \cup B \to A \cup B$, $S : A \cup B \to A \cup B$ be two maps such that $T(A) \subseteq B$, $S(B) \subseteq A$. A point (x,y) in $A \times B$ is said to be an approximate-pair fixed point for (T, S) in X, if there exists $\epsilon > 0$*

$$d(Tx, Sy) \leq d(A, B) + \epsilon.$$

We say that the pair (T,S) has the approximate-pair fixed property in X if

$$P^a_{(T,S)}(A, B) \neq \emptyset,$$

where

$$P^a_{(T,S)}(A, B) = \{(x, y) \in A \times B :\ d(Tx, Sy) \leq d(A, B) + \epsilon\ \ for\ some\ \epsilon > 0\}.$$

Theorem 3.1 *[6] Let A and B be nonempty subsets of a metric space (X, d) and $T : A \cup B \to A \cup B$, $S : A \cup B \to A \cup B$ be two maps such that $T(A) \subseteq B$, $S(B) \subseteq A$. If, for every $(x, y) \in A \times B$,*

$$d(T^n(x), S^n(y)) \to d(A, B)$$

then (T, S) has the approximate-pair fixed property.

Definition 3.2 *Let A and B be nonempty subsets of a metric space (X,d) and $T_\alpha : A \cup B \to A \cup B$, $S_\alpha : A \cup B \to A \cup B$ be two Geraghty-contraction maps such that $T_\alpha(A) \subseteq B$, $S_\alpha(B) \subseteq A$. A point (x,y) in*

$A \times B$ is said to be an approximate-pair fixed point for (T_α, S_α) in X, if there exists $\epsilon > 0$

$$d(T_\alpha x, S_\alpha y) \leq d(A, B) + \epsilon.$$

We say that the pair (T_α, S_α) has the approximate-pair fixed property in X if

$$P^{\epsilon}_{(T_\alpha, S_\alpha)}(A, B) \neq \emptyset,$$

where

$$P^{\epsilon}_{(T_\alpha,S_\alpha)}(A, B) = \{(x, y) \in A \times B : \ d(T_\alpha x, S_\alpha y) \leq d(A, B) + \epsilon \ \textit{ for some } \epsilon > 0\}.$$

Theorem 3.2 *Let A and B be nonempty subsets of a metric space (X, d) and $T_\alpha : A \cup B \to A \cup B$, $S_\alpha : A \cup B \to A \cup B$ be two maps such that $T_\alpha(A) \subseteq B$, $S_\alpha(B) \subseteq A$. If, for every $(x, y) \in A \times B$,*

$$d(T^n_\alpha(x), S^n_\alpha(y)) \to d(A, B)$$

then (T_α, S_α) has the approximate-pair fixed property .

Proof. For $\epsilon > 0$, Suppose $(x, y) \in A \times B$. Since

$$d(T^n_\alpha(x), S^n_\alpha(y)) \to d(A, B)$$

$$\exists n_0 > 0 \ s.t. \ \forall n \geq n_0 : \ d(T^n_\alpha(x), S^n_\alpha(y)) < d(A, B) + \epsilon$$

Then $d(T_\alpha(T^{n-1}_\alpha(x), S_\alpha(S^{n-1}_\alpha(y)) < d(A, B) + \epsilon$ for every $n \geq n_0$. Put $x_0 = T^{n_0-1}_\alpha(x)$ and $y_0 = S^{n_0-1}_\alpha(y))$. Hence $d(T_\alpha(x_0), S_\alpha(y_0)) \leq d(A, B)+\epsilon$ and $P^a_{(T_\alpha,S_\alpha)}(A, B) \neq \emptyset$. ■

Definition 3.3 *Let $T_\alpha : A \cup B \to A \cup B$, $S_\alpha : A \cup B \to A \cup B$ be continues maps such that $T_\alpha(A) \subseteq B$, $S_\alpha(B) \subseteq A$. We define diameter $P^{\epsilon}_{(T_\alpha,S_\alpha)}(A, B)$ by,*

$$diam(P^a_{(T_\alpha,S_\alpha)}(A, B)) = \sup\{d(x, y) : \ d(T_\alpha x, S_\alpha y) \leq \epsilon + d(A, B) \ \textit{ for some } \ \epsilon > 0\}.$$

Example 3.6. *Suppose $A = \{(x, 0) : \ 0 \leq x \leq 1\}$, $B = \{(x, 1) : \ 0 \leq x \leq 1\}$, $T(x, 0) = T(x, 1) = (\frac{1}{2}, 1)$ and $S(x, 1) = S(x, 0) = (\frac{1}{2}, 0)$. Then $d(T(x, 0), S(y, 1)) = 1$ and $diam(P^{\epsilon}_{(T_\alpha,S_\alpha)}(A, B)) = diam(A \times B) = \sqrt{2}$.*

Theorem 3.3 *Let* $T_\alpha : A\cup B \to A\cup B$, $S_\alpha : A\cup B \to A\cup B$ *be continues maps such that* $T_\alpha(A) \subseteq B$, $S_\alpha(B) \subseteq A$. *If, there exists a* $k \in [0,1]$,

$$d(x, T_\alpha x) + d(S_\alpha y, y) \le kd(x, y).$$

Then

$$diam(P^{\epsilon}_{(T_\alpha,S_\alpha)}(A,B)) \le \frac{\epsilon}{1-k} + \frac{d(A,B)}{1-k} \ for\ some\ \epsilon > 0.$$

Proof. If $(x,y) \in P^{\epsilon}_{(T_\alpha,S_\alpha)}(A,B)$, then

$$\begin{aligned} d(x,y) &\le d(x,T_\alpha x) + d(T_\alpha x, S_\alpha y) + d(S_\alpha y, y) \\ &\le \epsilon + kd(x,y) + d(A,B). \end{aligned}$$

Therefore $d(x,y) \le \frac{\epsilon}{1-k} + \frac{d(A,B)}{1-k}$. Then $diam(P^{a}_{(T_\alpha,S_\alpha)}(A,B)) \le \frac{\epsilon}{1-k} + \frac{d(A,B)}{1-k}$. ■

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