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Hilbert modules over pro-C*-algebras

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Abstract

In this paper, we generalize some results from Hilbert C*-modules to pro-C*-algebra case. We also give a new proof of the known result that $l^2(A)$ is a Hilbert module over a pro-C*-algebra A.

Key words: Pro-C*-algebra, σ-C*-algebra, Hilbert modules, Bounded module maps, Inverse system.

1 Introduction

Hilbert modules over pro-C*-algebras are the generalization of Hilbert C*-modules by allowing the inner product to take values in a pro-C*-algebra. A.Mallios in [10] and N.C.Phillips in [11] studied such spaces independently. The Hilbert modules over pro-C*-algebras are also studied

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in [4] , [5]. Pro-C*-algebras are applied to relativistic quantum mechanics (see [1], [2]). Therefore, it is useful to develop the theory of Hilbert modules over pro-C*-algebras as well.

In the present paper, the notion of a Hilbert module over a pro-C*-algebra is discussed and some new results are obtained for these spaces. We also present a new proof of the known result that $l^2(A)$ is a Hilbert A-module.

We refer the reader to papers [3], [11] for more details on pro-C*-algebras and [6], [7], [11], [12] for Hilbert modules over pro-C*-algebras.

The paper is organized as follows. In section 2, we recall the basic definitions and some results about the inverse limit of an inverse system of the topological vector spaces. In section 3, we bring some definitions and basic properties of the pro-C*-algebras and give several examples of such spaces. In section 4, we deal with the Hilbert modules over pro-C*-algebras. Also, we generalize the polar decomposition property from Hilbert C*-modules to pro-C*-algebra case. In section 5, we present some results about bounded operators on Hilbert pro-C*-modules.

2 Preliminaries

In this section, we recall some facts about the inverse limit of an inverse directed system of topological vector spaces.

Let $\{\phi_\alpha : X \to X_\alpha\}$ be a family of linear maps from a vector space X to topological vector spaces X_α. The projective topology induced on X by this family is the weakest topology on X such that each of the maps ϕ_α is continuous. It is easy to show that:

Proposition 1 *The projective topology induced on X by a family of linear maps, as above, is the unique t.v.s. topology τ on X such that a linear map ψ from a t.v.s. Y to (X, τ) is continuous iff $\phi_\alpha \, o \, \psi : Y \to X_\alpha$ is continuous for every α.*

If $\{X_\alpha\}$ is a family of topological vector spaces, then by the above result, the cartesian product $X = \prod X_\alpha$ is a t.v.s. and the linear maps over X induced his topology are the projections $\prod X_\alpha \to X_\alpha$.

Definition 2 *A family $\{X_\alpha, \phi_{\alpha\beta}\}$ where α and β belong to a directed set $\mathcal{A}$, X_α is a t.v.s. for each $\alpha \in \mathcal{A}$, $\{\phi_{\alpha\beta} : X_\beta \to X_\alpha\}$ is a set of continuous linear maps for each pair $\alpha, \beta \in \mathcal{A}$ with $\alpha < \beta$ and $\phi_{\alpha\beta}\, o\, \phi_{\beta\gamma} = \phi_{\alpha\gamma}$ whenever $\alpha < \beta < \gamma$, is called an inverse directed system of t.v.s.'s. The projective limit (or inverse limit), $\varprojlim_\alpha X_\alpha$, of such system is the subspace of the cartesian product $\prod X_\alpha$ consisting of elements $\{x_\alpha\}$ which satisfy ;*

$$\phi_{\alpha\beta}(x_\beta) = x_\alpha \quad \textit{for} \quad \alpha < \beta.$$

Note that the inverse limit, $\varprojlim_\alpha X_\alpha$, is a closed subspace of $\prod X_\alpha$ and has the projective topology induced by the family of maps $\{\phi_\alpha : \varprojlim_\alpha X_\alpha \to X_\alpha\}$ where ϕ_α is the inclusion $\varprojlim_\alpha X_\alpha \to \prod X_\alpha$ followed by the projection on X_α.
If Y is a t.v.s., we say that a system of continuous linear maps $\{\psi_\alpha : Y \to X_\alpha\}$ is compatible with the inverse directed system $\{X_\alpha, \phi_{\alpha\beta}\}$ if $\psi_\alpha = \phi_{\alpha\beta}\, o\, \psi_\beta$ for all $\alpha < \beta$.
Note also that the system of maps $\{\phi_\alpha : \varprojlim_\alpha X_\alpha \to X_\alpha\}$ is compatible with $\{X_\alpha, \phi_{\alpha\beta}\}$. We have the following result.

Proposition 3 *If Y is a t.v.s. and $\{\psi_\alpha\}$ is a system of continuous linear maps compatible with an inverse directed system $\{X_\alpha, \phi_{\alpha\beta}\}$, then there is a unique continuous linear map $\psi : Y \to \varprojlim_\alpha X_\alpha$ such that $\psi_\alpha = \phi_\alpha\, o\, \psi$ for each α.*

Proof. The system $\{\psi_\alpha\}$ determines a continuous linear map of Y into $\prod X_\alpha$ by Prop 2.1 . The compatibility condition ensures that the image of this map lies in $\varprojlim_\alpha X_\alpha$. □

Proposition 4 *The inverse limit of a system of complete t.v.s.'s is com-*

plete.

Proof. This follows immediately if we can first show that the cartesian product of a family of complete t.v.s.'s is complete since the projective limit is a closed subspace of the cartesian product. However, a filter base in a cartesian product is clearly cauchy iff it is cauchy in each coordinate and is convergent iff it is convergent in each coordinate. Now, the proposition follows. □

The inverse limit of topologigal algebras is defined as similar, only the continuous linear maps will be replaced by appropriate continuous homomorphisms. Thus, we note that the results just stated for *t.v.s.*'s are also valid in these categories.

3 Pro-C*-algebras

Recall that a pro-C*-algebra is a complete Hausdorff topological $*$- complex algebra A whose topology is determined by its continuous C*-seminorms in the sense that a net $\{a_\lambda\}$ converges to 0 iff $p(a_\lambda) \to 0$ for any continuous C*-seminorm p on A.

A σ-C*-algebra is a pro-C*-algebra if its topology is determined by only countably many C*-seminorms.

Let A be a unital pro-C*-algebra and let $a \in A$. Then the spectrum $\mathrm{sp}(a)$ of $a \in A$ is the set $\{\lambda \in \mathbb{C} \ : \ \lambda 1_A - a \text{ is not invertible}\}$. If A is not unital, then the spectrum is taken with respect to its unitization $\tilde{A}$.

If A^+ denotes the set of all positive elements of A, then A^+ is a closed convex cone such that $A^+ \cap (-A^+) = 0$. We denote by $S(A)$, the set of all continuous C*-seminorms on A. For $p \in S(A)$, we put $\ker(p) = \{a \in A : p(a) = 0\}$; which is a closed ideal in A. For each $p \in S(A)$, $A_p = A/\ker(p)$ is a C*-algebra in the norm induced by p which defined as ;

$$\|a + \ker(p)\|_{A_p} = p(a) \quad , \quad p \in S(A) \ ,$$

and we have $A = \varprojlim_{p} A_p$. (see [11])

The canonical map from A onto A_p for $p \in S(A)$ will be denoted by π_p, and the image of $a \in A$ under π_p will be denoted by a_p. Hence $l^2(A_p)$ is a Hilbert A_p-module (see [4]) with the norm defined as ;

$$\|(\pi_p(a_i))_{i\in\mathbb{N}}\|_p = [\ p(\textstyle\sum_{i\in\mathbb{N}} a_i {a_i}^*)\ ]^{1/2} \quad , \quad p \in S(A) \ , \ (\pi_p(a_i))_{i\in\mathbb{N}} \in l^2(A_p) .$$

The connecting maps of the inverse system $\{A_p\}_{p\in S(A)}$ will be denoted by π_{pq}, whenever $p, q \in S(A),\ p \leq q$ and we have :

$$\pi_{pq} : A_q \to A_p \quad , \quad \pi_{pq}(a_q) = a_p \ .$$

Example 3.1 Every C*-algebra is a pro-C*-algebra.

Example 3.2 A closed $*$-subalgebra of a pro-C*-algebra is a pro-C*-algebra.

Example 3.3 ([11]) Let X be a locally compact Hausdorff space and let $A = C(X)$ denotes all continuous complex-valued functions on X with the topology of uniform convergence on compact subsets of X, then A is a pro-C*-algebra.

Example 3.4 ([11]) A product of C*-algebras with the product topology is a pro-C*-algebra.

Proposition 5 *If* $\sum_{i=1}^{\infty} a_i$ *is a convergent series in a pro-C*-algebra* A *and* $a_i \geq 0$ *for* $i \in \mathbb{N}$, *then it converges unconditionally.*

Proof. For $n \in \mathbb{N}$, let $S_n = \sum_{i=1}^{n} a_i$. Then for any $\varepsilon \geq 0$ and $p \in S(A)$, there is a positive integer N_p such that for $m, n \geq N_p$;

$$p(\textstyle\sum_{i=m}^{n} a_i) \leq \varepsilon \ .$$

For a permutation σ of $\mathbb{N}$, we define $S'_n = \sum_{i=1}^{n} a_{\sigma(i)}$. Let $k \in \mathbb{N}$ such that

$$\{1, 2, ..., N_p\} \subseteq \{\sigma(1), \sigma(2), ..., \sigma(k)\} \ .$$

Then $S'_n - S_n$ for $n \geq k$, do not have any a_i for $1 \leq i \leq N_p$. Hence for $n \geq k$,

$$p(S'_n - S_n) \leq \varepsilon \ .$$

Thus for $S = \sum_{i=1}^{\infty} a_i$ and $n \geq k$, we have ,

$$p(S'_n - S) \leq p(S'_n - S_n) + p(S_n - S) \leq 2\varepsilon \ .$$

This means that $\lim_{n\to\infty} S'_n = S$. □

Recall that an approximate identity of a pro-C*-algebra A is an increasing net $\{e_\lambda\}_{\lambda\in\Lambda}$ of positive elements such that

(i) $p(e_\lambda) \leq 1$ for all $p \in S(A)\,,\ \lambda \in \Lambda$

(ii) $\lim_\lambda (a - ae_\lambda) = \lim_\lambda (a - e_\lambda a) = 0$ for any $a \in A$.

It is shown in [3] that every pro-C*-algebra has an approximate identity.

4 Hilbert pro-C*-modules

We begin with some facts about Hilbert modules over pro-C*-algebra from [11].

Definition 6 *A pre-Hilbert module over pro-C*-algebra A is a complex vector space E which is also a left A-module compatible with the complex algebra structure, equipped with an A-valued inner product $\langle .,. \rangle : E \times E \to A$ which is $\mathbb{C}$-and A-linear in its first variable and satisfies the following conditions:*
(i) $\langle x, y\rangle^* = \langle y, x\rangle$
(ii) $\langle x, x\rangle \geq 0$
(iii) $\langle x, x\rangle = 0$ *iff* $x = 0$
for every $x, y \in E$. We say that E is a Hilbert A-module (or Hilbert pro-C-module over A) if E is complete with respect to the topology determined by the family of seminorms*

$$\bar{p}_E(x) = \sqrt{p(\langle x, x\rangle)} \qquad x \in E\ ,\ p \in S(A)\ .$$

If E is a Hilbert A-module and $p \in S(A)$, then $\ker(\bar{p}_E) = \{x \in E : p(\langle x, x\rangle) = 0\}$ is a closed submodule of E and $E_p = E/\ker(\bar{p}_E)$ is a Hilbert A_p-module with the scalar product

$$a_p.(x + \ker(\bar{p}_E)) = ax + \ker(\bar{p}_E) \quad , \quad a \in A \ , \ x \in E$$

and the following inner product:

$$\langle\ x + \ker(\bar{p}_E)\ ,\ y + \ker(\bar{p}_E)\ \rangle = \langle x, y\rangle_p \qquad x, y \in E\ .$$

The caconical map from E onto E_p is denoted by σ_p and the image of x in E under σ_p is denoted by x_p for $p \in S(A)$.

Example 4.1 If A is a pro-C*-algebra, then it is a Hilbert A-module with respect to the inner product defined by

$$\langle a, b\rangle = ab^* \quad , \quad a, b \in A\ .$$

For each $p, q \in S(A)$ with $p \leq q$, there is a canonical surjective linear map $\sigma_{p,q} : E_q \to E_p$ such that $\sigma_{p,q}(x_q) = x_p$ for $x \in E$. Then

$$\{E_p; A_p; \sigma_{p,q}\ , \quad p, q \in S(A),\ p \leq q\}\ ,$$

is an inverse system of Hilbert C*-modules in the following sense:

(i) $\sigma_{p,q}(a_q x_q) = \pi_{p,q}(a_q)\sigma_{p,q}(x_q)$
(ii) $\langle\ \sigma_{p,q}(x_q), \sigma_{p,q}(y_q)\ \rangle = \pi_{p,q}(\langle x_q, y_q\rangle)$
(iii) $\sigma_{p,q} o\ \sigma_{q,r} = \sigma_{p,q}$
(iv) $\sigma_{p,p} = id_{E_p}$
for every $x, y \in E, a \in A$ and $p, q, r \in S(A)$ with $p \leq q \leq r$.([11])

By Proposition 4.4 of [11], we have $E \cong \varprojlim_p E_p$ and $\varprojlim_p E_p$ is a Hilbert $(\varprojlim_p A_p)$-module with the following product:

$$(a_p)_{p\in S(A)}.(x_p)_{p\in S(A)} = ((ax)_p)_{p\in S(A)}$$

and the inner product:

$$\langle\ (x_p)_p\ ,\ (y_p)_p\ \rangle = (\langle x, y\rangle_p)_p$$

for all $a \in A$ and $x, y \in E$. Moreover, $\varprojlim_p E_p$ has a topology determined by the family of seminorms

$$\tilde{p}((x_q)_{q\in S(A)}) = \|x_p\|_{E_p} = \bar{p}_E(x)\ .$$

We recall that an element a in A (x in E) is bounded if

$$\|a\|_\infty = \sup\{p(a)\ ;\ p \in S(A)\} < \infty$$

$$(\|x\|_\infty = \sup\{\bar{p}_E(x)\ ;\ p \in S(A)\} < \infty\ )$$

The set of all bounded elements in A (in E) will be denoted by $b(A)$ ($b(E)$). We know that $b(A)$ is a C*-algebra in the C*-norm $\|.\|_\infty$, and $b(E)$ is a Hilbert $b(A)$-module.(see Proposition 1.11 of [11] and Theorem 2.1 of [12])

Let $M \subset E$ be a closed submodule of a Hilbert A-module E, and let

$$M^\perp = \{y \in E\ :\ \langle x, y\rangle = 0\quad for\ all\ x \in M\}\ .$$

Note that the inner product in a Hilbert modules is separately continuous, hence $M^\perp$ is a closed submodule of the Hilbert A-module E. Also a closed submodule M in a Hilbert A-module E is called orthogonally complementable if $E = M \oplus M^\perp$. A closed submodule M in a Hilbert A-module E is called topologically complementable if there exists a closed submodule N in E such that $M \oplus N = E$, $N \cap M = \{0\}$.

Let $l^2(A)$ be the set of all sequences $(a_n)_{n\in\mathbb{N}}$ of elements of a pro-C*-algebra A such that the series $\sum_{i=1}^\infty a_i a_i{}^*$ is convergent in A.

Proposition 7 *Let A be a pro-C*-algebra. Then $l^2(A)$ is a pre-Hilbert module over A with respect to the pointwise operations and the following inner product*

$$\langle\, (a_i)_{i\in\mathbb{N}}\ ,\ (b_i)_{i\in\mathbb{N}}\,\rangle = \sum_{i=1}^{\infty} a_i {b_i}^*\ .$$

Proof. It is not difficult to check that $l^2(A)$ is a left A-module. We show that the inner product on $l^2(A)$ is well defined. Since $\sum_{i=1}^{\infty} a_i {a_i}^*$ and $\sum_{i=1}^{\infty} b_i {b_i}^*$ are convergent in A, so for $\varepsilon > 0$ and $p \in S(A)$, there is a positive integer N such that for $m, n \geq N$,

$$p(\textstyle\sum_{i=m}^{n} a_i {a_i}^*) < \varepsilon \quad , \quad p(\sum_{i=m}^{n} b_i {b_i}^*) < \varepsilon\ .$$

By Cauchy-Bunyakovskii inequality in Hilbert module over pro-C*-algebra A (Lemma 2.1 of [12]), we can write

$$p(\langle\, (a_i)_{i=m}^n\, , \,(b_i)_{i=m}^n\, \rangle) \leq$$
$$p(\langle\, (a_i)_{i=m}^n\, , \,(a_i)_{i=m}^n\, \rangle)^{1/2}\ .\ p(\langle\, (b_i)_{i=m}^n\, , \,(b_i)_{i=m}^n\, \rangle)^{1/2}.$$

Therefore, if $m, n \geq N$, we have

$$p(\textstyle\sum_{i=m}^{n} a_i {b_i}^*) \leq \sqrt{\varepsilon}\, .\, \sqrt{\varepsilon} = \varepsilon\ .$$

Hence, $\sum_{i=1}^{\infty} a_i {b_i}^*$ converges in A and clearly $\langle .,. \rangle$ is an inner product on $l^2(A)$. □

Now, we show that $l^2(A)$ is complete with respect to the topology determined by the family of seminorms

$$\bar{p}(\{a_i\}_i) = p(\textstyle\sum_{i=1}^{\infty} a_i {a_i}^*)^{1/2} \qquad \{a_i\}_i \in l^2(A)\ ,\ p \in S(A)\ .$$

Lemma 8 *$\{a_i\}_{i\in\mathbb{N}} \in l^2(A)$ if and only if $\{\pi_p(a_i)\}_{i\in\mathbb{N}} \in l^2(A_p)$ for each $p \in S(A)$.*

Proof. Let $m \leq n$. Then for each $p \in S(A)$ we have

$$\|\textstyle\sum_{i=m}^{n} \langle \pi_p(a_i), \pi_p(a_i)\rangle\|_{A_p} = \|\pi_p(\sum_{i=m}^{n} \langle a_i, a_i\rangle)\|_{A_p} = p(\sum_{i=m}^{n} \langle a_i, a_i\rangle)\ .$$

Thus, the sequence of partial sums of series $\sum_{i\in\mathbb{N}} \langle \pi_p(a_i), \pi_p(a_i)\rangle$ is cauchy in $l^2(A_p)$, for each $p \in S(A)$ iff the sequence of partial sums of series $\sum_{i\in\mathbb{N}} \langle a_i, a_i\rangle$ is cauchy in $l^2(A)$ and so the proof is complete . □

Lemma 9 *Suppose that for any $p \in S(A)$, $\phi_p : l^2(A) \to l^2(A_p)$ be such* *that* $\phi_p(\{a_i\}_{i\in\mathbb{N}}) = \{\pi_p(a_i)\}_{i\in\mathbb{N}}$. *Then, the sequence* $\{f_k\}_{k\in\mathbb{N}}$ *is cauchy (convergent) in* $l^2(A)$ *if and only if* $\{\phi_p(f_k)\}_{k\in\mathbb{N}}$ *is cauchy (convergent) in* $l^2(A_p)$ *for each* $p \in S(A)$.

Proof. Let $f_k = (a_{ki})_{i\in\mathbb{N}}$, $k \in \mathbb{N}$, $p \in S(A)$, $m \le n$. Then

$$p\langle f_n - f_m , f_n - f_m\rangle = p\langle (a_{ni} - a_{mi})_{i\in\mathbb{N}} , (a_{ni} - a_{mi})_{i\in\mathbb{N}}\rangle$$

$$= p(\textstyle\sum_{i\in\mathbb{N}}\langle a_{ni} - a_{mi} , a_{ni} - a_{mi}\rangle) .$$

On the other hand, if $\|.\|_p$ is the induced norm by the inner product on $l^2(A)$, then for each $p \in S(A)$ and $m \le n$, we have :

$$\|\phi_p(f_n) - \phi_p(f_m)\|_p^2 = \|\phi_p(f_n - f_m)\|_p^2$$

$$= \|\phi_p(a_{ni} - a_{mi})_{i\in\mathbb{N}}\|_p^2$$

$$= \|\pi_p(a_{ni} - a_{mi})_{i\in\mathbb{N}}\|_p^2$$

$$= p(\textstyle\sum_{i\in\mathbb{N}}\langle a_{ni} - a_{mi} , a_{ni} - a_{mi}\rangle) .$$

Hence, if $\{\bar{p} : p \in S(A)\}$ denotes the set of all continuous seminorms on $l^2(A)$, then :

$$\bar{p}(f_n - f_m) = \|\phi_p(f_n) - \phi_p(f_m)\|_p .$$

Therefore, the sequence $\{f_k\}_{k\in\mathbb{N}}$ in $l^2(A)$ is cauchy iff $\{\phi_p(f_k)\}_{k\in\mathbb{N}}$ is cauchy in $l^2(A_p)$ for each $p \in S(A)$.
On the other hand, we can write :

$$\|\phi_p(f_k)\|_p^2 = \|(\pi_p(a_{ki}))_{i\in\mathbb{N}}\|_p^2 = p(\textstyle\sum_{i\in\mathbb{N}}\langle a_{ki} , a_{ki}\rangle) = p(\langle f_k, f_k\rangle) = [\bar{p}(f_k)]^2 .$$

This means that the sequence $\{f_k\}_{k\in\mathbb{N}}$ is convergent in $l^2(A)$ if and only if the sequence $\{\phi_p(f_k)\}_{k\in\mathbb{N}}$ is convergent in $l^2(A_p)$ for each $p \in S(A)$.□

Corollary 4.1 *$l^2(A)$ is complete and so is a Hilbert A-module .*

Proof. It is enough to prove the completeness of $l^2(A)$. Since, A_p is a C*-algebra for each $p \in S(A)$, we conclude that $l^2(A_p)$ is complete (see [9]), for each $p \in S(A)$. So the proof follows from Lemmas 4.1 and 4.2 . □

Let E_i for $i \in \mathbb{N}$ be a Hilbert A-module with the topology induced by the family of continuous seminorms $\{\bar{p}_i\}_{p\in S(A)}$ defined as :

$$\bar{p}_i(x) = \sqrt{p(\langle x, x\rangle)} \qquad (x \in E_i) \ .$$

Direct sum of $\{E_i\}_{i\in\mathbb{N}}$ is defined as follows :

$$\bigoplus_{i\in\mathbb{N}} E_i = \{(x_i)_{i\in\mathbb{N}} \ \ : \ \ x_i \in E_i \ \ , \ \ \textstyle\sum_{i=1}^{\infty}\langle x_i, x_i\rangle \ \textit{is convergent in } A\}$$

It has been shown (see [8 , Example 3.2.3]) that the direct sum $\bigoplus_{i\in\mathbb{N}} E_i$ is a Hilbert A-module with A-valued inner product $\langle x, y\rangle = \sum_{i=1}^{\infty}\langle x_i, y_i\rangle$, where $x = (x_i)_{i\in\mathbb{N}}$ and $y = (y_i)_{i\in\mathbb{N}}$ are in $\bigoplus_{i\in\mathbb{N}} E_i$ with pointwise operations and the topology determined by the familly of seminormes

$$\bar{p}(x) = \sqrt{p(\langle x, x\rangle)} \qquad x \in \bigoplus_{i\in\mathbb{N}} E_i \ \ , \ \ p \in S(A) \ .$$

The direct sum of a countable copies of a Hilbert module E is denoted by H_E . If E is a Hilbert A-module, then we denote by $A.E$ the closure in E of the linear span of all the elements of the form $a.x$, for $x \in E$ and $a \in A$.

Proposition 10 *We have $A.E = E$.*

Proof. Let $\{e_\lambda\}_\lambda$ be an approximate identity of A. Then for any $x \in E$ and $p \in S(A)$

$$\begin{aligned}
\bar{p}_E(x - e_\lambda x)^2 &= p(\langle x - e_\lambda x, x - e_\lambda x\rangle) \\
&= p(\langle x, x\rangle - e_\lambda\langle x, x\rangle - \langle x, x\rangle e_\lambda + e_\lambda\langle x, x\rangle e_\lambda) \\
&= p([\langle x, x\rangle - e_\lambda\langle x, x\rangle] - [\langle x, x\rangle - e_\lambda\langle x, x\rangle]e_\lambda)
\end{aligned}$$

$$\leq p[\langle x,x\rangle - e_\lambda \langle x,x\rangle] + p[\langle x,x\rangle - e_\lambda\langle x,x\rangle]p(e_\lambda)$$

$$= (1+p(e_\lambda))p(\langle x,x\rangle - e_\lambda\langle x,x\rangle) \to 0 \ .$$

Hence the elements of the form $e_\lambda.x$ are dense in E . □

Proposition 11 *Let A be a unital pro-C*-algebra and E is a Hilbert A-module. Then for any $x \in E$,*

$$x = \lim_{\varepsilon\to 0} \langle x,x\rangle(\langle x,x\rangle + \varepsilon)^{-1}.x$$

Proof. Let $a = \langle x,x\rangle$. Then, by the spectral theorem (see [11, Proposition 1.9]), for any $p \in S(A)$

$$\bar{p}_E(\langle x,x\rangle(\langle x,x\rangle+\varepsilon)^{-1}.x - x)^2 = \bar{p}_E([\langle x,x\rangle(\langle x,x\rangle+\varepsilon)^{-1} - 1].x)^2$$

$$= p(\langle[\langle x,x\rangle(\langle x,x\rangle+\varepsilon)^{-1} - 1].x \ , \ [\langle x,x\rangle(\langle x,x\rangle+\varepsilon)^{-1} - 1].x\rangle)$$

$$= p(\langle[a(a+\varepsilon)^{-1} - 1].x \ , \ [a(a+\varepsilon)^{-1} - 1].x\rangle)$$

$$= p([a(a+\varepsilon)^{-1} - 1]a[a(a+\varepsilon)^{-1} - 1])$$

$$= p(a.[a(a+\varepsilon)^{-1} - 1]^2)$$

$$= p[a^3(a+\varepsilon)^{-2} - 2a^2(a+\varepsilon)^{-1} + a]$$

$$= \sup\{|t^3(t+\varepsilon)^{-2} - 2t^2(t+\varepsilon)^{-1} + t| \ \ : \ \ t \in \mathrm{sp}(a_p)\}$$

$$\leq \tfrac{7}{2}\varepsilon \ ,$$

since the following inequalities hold under the condition $t \geq 0$:

$$|t^3(t+\varepsilon)^{-2} - 2t^2(t+\varepsilon)^{-1} + t| \leq |t^3(t+\varepsilon)^{-2} - t| + 2|t^2(t+\varepsilon)^{-1} - t| \ ,$$

$$|t^3(t+\varepsilon)^{-2} - t| = |t(\frac{-\varepsilon^2 - 2\varepsilon t}{(t+\varepsilon)^2})| = \varepsilon|\frac{\varepsilon t + 2t^2}{(t+\varepsilon)^2}| \leq \varepsilon(\frac{1}{2}+2) = \frac{5}{2}\varepsilon \ ,$$

$$|t^2(t+\varepsilon)^{-1} - t| = |\frac{t\varepsilon}{t+\varepsilon}| < \varepsilon \ . \qquad \square$$

The following statment is a polar decomposition for Hilbert pro-C*-modules.

Proposition 12 *Let E be a Hilbert A-module , $x \in E$, and $0 < \alpha < 1/2$. Then there exists an element $z \in E$ such that ,*

$$x = \langle x, x\rangle^{\alpha}.z$$

Proof. For $n \in \mathbb{N}$, put

$$g_n(\lambda) = \begin{cases} n^{\alpha} \text{ if } \lambda \leq 1/n \\ \lambda^{-\alpha} \text{ if } \lambda > 1/n \ . \end{cases}$$

Then, by the spectral theorem, for each $p \in S(A)$,

$$\bar{p}_E([g_n(\langle x, x\rangle) - g_m(\langle x, x\rangle)].x) = p(\langle x, x\rangle[g_n(\langle x, x\rangle) - g_m(\langle x, x\rangle)]^2)^{1/2}$$

$$= (\sup\{|\lambda(g_n(\lambda) - g_m(\lambda))^2| \ : \ \lambda \in \mathrm{sp}(\langle x, x\rangle_p)\})^{1/2}.$$

Therefore, the sequence $\{g_n(\langle x, x\rangle).x\}$ is a cauchy sequence, so converges to some $z \in E$. Then ,

$$\bar{p}_E(\langle x, x\rangle^{\alpha}.z - x) = \lim_{n\to\infty} \bar{p}_E(\langle x, x\rangle^{\alpha} g_n(\langle x, x\rangle).x - x)$$

$$= \lim_{n\to\infty} \bar{p}_E([\langle x, x\rangle^{\alpha} g_n(\langle x, x\rangle) - 1].x)$$

$$= \lim_{n\to\infty} \sup\{|(\lambda^{\alpha} g_n(\lambda) - 1)^2 \lambda| \ : \ \lambda \in \mathrm{sp}(\langle x, x\rangle_p)\}^{1/2}$$

$$= 0 \ .$$

This completes the proof . $\square$

5 Operators on Hilbert modules

Let A be a pro-C*-algebra and let E and F be two Hilbert A-modules. An A-module map $T : E \to F$ is said to bounded if for each $p \in S(A)$, there is $C_p > 0$ such that :

$$\bar{p}_F(Tx) \leq C_p.\bar{p}_E(x) \qquad (x \in E) \ ,$$

where $\bar{p}_E$, respectively $\bar{p}_F$, are continuous seminormes on E, respectively F. A bounded A-module map from E to F is called an operator from E to F. We denote the set of all operators from E to F by $Hom_A(E, F)$ and we set $End_A(E) = Hom_A(E, E)$.

Let $T \in Hom_A(E, F)$. We say T is adjointable if there exists an operator $T^* \in Hom_A(F, E)$ such that :

$$\langle Tx, y\rangle = \langle x, T^*y\rangle$$

for all $x \in E$, $y \in F$. We denote by $Hom^*_A(E, F)$, the set of all adjointable operators from E to F and $End^*_A(E) = Hom^*_A(E, E)$.

By a little modification in the proof of Lemma 3.2 of [12], we have the following result :

Proposition 13 *Let $T : E \to F$ and $T^* : F \to E$ be two maps such that the equality*

$$\langle x, T^*y\rangle = \langle Tx, y\rangle$$

holds for all $x \in E$, $y \in F$. Then $T \in Hom^*_A(E, F)$.

It is easy to see that for any $p \in S(A)$, the map defined by

$$\hat{p}(T) = \sup\{\ \bar{p}_F(Tx) \ \ : \ \ x \in E \ , \ \bar{p}_E(x) \leq 1\} \ \ , \ \ T \in Hom_A(E, F) \ ,$$

is a seminorm on $Hom_A(E, F)$. Moreover $Hom_A(E, F)$ with the topology determined by the family of seminorms $\{\hat{p}\}_{p \in S(A)}$ is a complete locally convex space ([7 , Proposition 3.1]).

By Proposition 4.7 of [11], we have the canonical isomorphism ,

$$Hom_A(E,F) \cong \varprojlim_p Hom_{A_p}(E_p,F_p) \ .$$

Consequently, $End_A^*(E)$ is a pro-C*-algebra for any Hilbert A-module E and its topology is obtained by $\{\hat{p}\}_{p\in S(A)}$.

Let $T \in End_A^*(E)$ and $p \in S(A)$. Define :

$$T_p : E_p \to E_p \qquad , \qquad {T_p}^* : E_p \to E_p$$

$$T_p(x+\ker(\bar{p}_E)) = Tx + \ker(\bar{p}_E) \qquad {T_p}^*(x+\ker(\bar{p}_E)) = T^*x + \ker(\bar{p}_E)$$

for all $x \in E$. Then we have :

$$\Big\langle x+\ker(\bar{p}_E) \ , \ T_p(y+\ker(\bar{p}_E))\Big\rangle = \Big\langle {T_p}^*(x+\ker(\bar{p}_E)) \ , \ y+\ker(\bar{p}_E)\Big\rangle$$

for all $x, y \in E$. By Proposition 5.1, we have $T_p \in End_{A_p}^*(E_p)$ and the map $T \to T_p$ for each $p \in S(A)$ is a $*$-homomorphism from the pro-C*-algebra $End_A^*(E)$ to the C*-algebra $End_{A_p}^*(E_p)$. Moreover,

$$\|T_p\| = \hat{p}(T) \ .$$

Note that, $End_A^*(E) \cong \varprojlim_p End_{A_p}^*(E_p)$, (see [11 , Proposition 4.7]). Hence T is a positive element of $End_A^*(E)$ if and only if T_p is a positive element of $End_{A_p}^*(E_p)$ for any $p \in S(A)$. Note also that T is a positive element of $End_A^*(E)$ if and only if $\langle Tx, x\rangle \geq 0$ for any element $x \in E$. ([12 , Proposition 3.2])

Lemma 14 *Let X be a Hilbert module over C*-algebra B, $S \in End_B^*(X)$ and S be a positive element of $End_B^*(X)$. Then for each $x \in X$,*

$$\langle Sx, x\rangle \leq \|S\| \langle x, x\rangle \ .$$

Proof. Since S is a positive element in $End_B^*(X)$, we have, $S \leq \|S\|I$, where I is the identity element in $End_B^*(X)$. Hence $S - \|S\|I \geq 0$, and then

$$\Big\langle (\|S\|I - S)x \ , \ x \Big\rangle \geq 0 \quad , \quad \forall x \in X \ .$$

Therefore, we have :

$$\langle Sx, x\rangle \leq \|S\| \langle x, x\rangle,$$

for all $x \in X$. □

Remark 5.1. Note that if $T \in End_B^*(X)$, then T^*T is a positive element in $End_B^*(X)$. Thus, we can write :

$$\langle Tx, Tx\rangle = \langle T^*Tx, x\rangle \leq \|T^*T\| \langle x, x\rangle = \|T\|^2 \langle x, x\rangle,$$

for all $x \in X$.

Definition 15 *Let E and F be two Hilbert modules over pro-C^*-algebra A. Then the operator $T : E \to F$ is called uniformly bounded, if there exists $C > 0$ such that for each $p \in S(A)$,*

$$\bar{p}_F(Tx) \leq C \bar{p}_E(x) \quad , \quad \forall x \in E. \tag{5.1}$$

The number C in (5.1) is called an upper bound for T and we set:

$$\|T\|_\infty = \inf\{C \ : \ C \textit{ is an upper bound for } T\}.$$

Clearly, in this case we have:

$$\hat{p}(T) \leq \|T\|_\infty \quad , \quad \forall p \in S(A).$$

Proposition 16 *Let E be a Hilbert module over pro-C^*-algebra A and T be an invertible element in $End_A^*(E)$ such that both are uniformly bounded. Then for each $x \in E$,*

$$\|T^{-1}\|_\infty^{-2} \langle x, x\rangle \leq \langle Tx, Tx\rangle \leq \|T\|_\infty^2 \langle x, x\rangle.$$

Proof. Recall that for each $p \in S(A)$, the space $End^*_{A_p}(E_p)$ is a C*-algebra and T_p belong to this space with the norm defined by:

$$\|T_p\|_p = \hat{p}_E(T).$$

Therefore by Remark 5.1, for each $p \in S(A)$ and $x \in E$,

$$\begin{aligned}\langle Tx, Tx\rangle_p &= \langle (Tx)_p, (Tx)_p\rangle \\ &= \langle T_p(x_p), T_p(x_p)\rangle \\ &\leq \|T_p\|_p^2 \langle x_p, x_p\rangle \\ &= \hat{p}_E(T)^2 \langle x, x\rangle_p \\ &\leq \|T\|_\infty^2 \langle x, x\rangle_p.\end{aligned}$$

By Remark 2.2 of [3], we have:

$$\langle Tx, Tx\rangle \leq \|T\|_\infty^2 \langle x, x\rangle \quad , \quad \forall x \in E. \tag{5.2}$$

On the other hand, by replacing T^{-1} and y instead of T and x in (5.2), we obtain:

$$\langle T^{-1}y, T^{-1}y\rangle \leq \|T^{-1}\|_\infty^2 \langle y, y\rangle.$$

Let $x \in E$ such that $Tx = y$. Then, we can conclude:

$$\langle x, x\rangle \leq \|T^{-1}\|_\infty^2 \langle Tx, Tx\rangle.$$

because T is an invertible operator, it can be concluded that: $\|T^{-1}\|_\infty > 0$ and hence:

$$\|T^{-1}\|_\infty^{-2} \langle x, x\rangle \leq \langle Tx, Tx\rangle \quad , \quad \forall x \in E. \square$$

Let $N \subseteq M$ be a closed submodule of a Hilbert module M. Then, in general, the equality $M = N + N^\perp$ dose not hold, as the following example shows.

Example 5.1. ([8]) Let $A = C[a, b]$ be the pro-c*-algebra of all continuous fanctions on the segment $[a, b]$. consider in the Hilbert A-module $M = A$, the submodule $N = C_0(a, b)$ of functions that vanish at the end points of the sigment. Then obviously, $N^\perp = \{0\}$.

Let N and M be two closed submodules in a Hilbert module E such that $E = M \oplus N$. We denote by P_M, the projection onto M along N .

Proposition 17 *Let M be an orthogonally complemented submodule of a Hilbert A-module E. Then $P_M \in End_A^*(E)$.*

Proof. Let $x, y \in E$. Then, there exist unique elements $a, b \in M$ and $a', b' \in M^\perp$ such that, $x = a + a'$, $y = b + b'$. Therefore

$$\langle P_M(x) \, , \, y \rangle = \langle a \, , \, b + b' \rangle = \langle a \, , \, b \rangle \ .$$

On the other hand,

$$\langle x \, , \, P_M(y) \rangle = \langle a + a' \, , \, b \rangle = \langle a \, , \, b \rangle \ .$$

By Lemma 3.2 of [12], we have $P_M = {P_M}^*$. Using Proposition 5.1, we conclude $P_M \in End_A^*(E)$. □

Proposition 18 *Let M be an orthogonally complemented submodule of a Hilbert A-module E and let $T \in End_A^*(E)$ be an ivertible operator such that $T^*TM \subseteq M$. Then we have:*

$$T(M^\perp) = (TM)^\perp \quad , \quad P_{TM} = TP_M T^{-1} \ .$$

Proof. Let $u \in M$ and $v \in M^\perp$. Since $T^*Tu \in M$, then we have $\langle Tu, Tv \rangle = \langle T^*Tu, v \rangle = 0$. Thus $T(M^\perp) \subseteq (TM)^\perp$. On the other hand if $y \in (TM)^\perp$, then there exists $x \in E$ such that $y = Tx$. Let $x = m + n$ for some $m \in M$ and $n \in M^\perp$, then we have

$$\langle y, Tm \rangle = \langle Tx, Tm \rangle = \langle Tm + Tn, Tm \rangle = \langle Tm, Tm \rangle + \langle Tn, Tm \rangle = 0.$$

Since $\langle Tn, Tm \rangle = 0$, we have $\langle Tm, Tm \rangle = 0$ and then $Tm = 0$. Thus $y = Tn$, and we have $(TM)^\perp \subseteq T(M^\perp)$. Let $x \in E$. Since $E = M + M^\perp$, so we have $x = u + v$, $u \in M$, $v \in M^\perp$. Hence, $Tx = Tu + Tv$. On the other hand, we have, $TM^\perp = (TM)^\perp$. Thus $Tu \in TM$ and $Tv \in (TM)^\perp$. Therefore :

$$P_{TM}(Tx) = Tu \quad , \quad TP_M(x) = Tu \ .$$

This completes the proof . $\square$

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