Theory of Approximation and Applications

Vol. 10, No. 1, (2014), 1-12

On the modification of the preconditioned AOR iterative method for linear system

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Received 10 March 2012; accepted 1 October 2013

Abstract

In this paper, we will present a modification of the preconditioned AOR-type method for solving the linear system. A theorem is given to show the convergence rate of modification of the preconditioned AOR methods that can be enlarged than the convergence AOR method.

Key words: AOR iterative method; Preconditioner; Z-matrix; Convergence.
2010 AMS Mathematics Subject Classification : 65H20 ; 35R11.

1 Introduction

Consider the linear system as the following

$$Ax = b, \tag{1.1}$$

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where $A = (a_{ij})$ is an $n \times n$ square and nonsingular matrix and x and b are n-dimensional vectors. The linear system appears in many scientific problems [1-13]. So the problem of solving Eq.(1.1) is important in numerical linear algebra. When the condition number of A is very large, the system of Eq.(1.1) is ill-posed and small changes in elements of A can make large changes to the obtained response. To eliminate the recent issue, a preconditioned technique would be useful. Kohno *et al.* in [4] have been considered a preconditioner $P_\alpha = I + S_\alpha$, where S_α is given by

$$S_\alpha = \begin{pmatrix} 0 & -\alpha_1 a_{12} & 0 & ... & 0 \\ 0 & 0 & -\alpha_2 a_{23} & ... & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \ddots & -\alpha_{n-1} a_{n-1,n} \\ 0 & 0 & 0 & ... & 0 \end{pmatrix}, \tag{1.2}$$

and α_i, $i = 1, 2, ..., n-1$, are nonnegative real numbers. Kotakemorie *et al.* in [2] proposed $P_\beta = I + \beta U$ as the preconditioned matrix, where β is a positive real number. Wu *et al.* presented preconditioned AOR iterative methods with two different preconditioners in [12], Also these preconditioned methods presented by Kohno *et al.* in [4] and Kotakemori in [5]. Gauss type preconditioning methods for nonnegative matrices and M-matrix linear systems are applied by Zhang in [14] . A new preconditioned AOR method for Z-matrices presented in [11] by Wang *et al.* as the following

$$P_\beta = I + K_\beta = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ -\beta_1 a_{12} & 1 & ... & 0 & 0 \\ 0 & -\beta_2 a_{23} & \vdots & \vdots & \vdots \\ \vdots & \vdots & \ddots & 1 & 0 \\ 0 & 0 & \dots & -\beta_{n-1} a_{n-1,n} & 1 \end{pmatrix}, \tag{1.3}$$

where β_i, $i = 1, 2, ..., n-1$ are nonnegative real numbers. In this paper, we will present the preconditioned AOR iterative method with

$$P_{\alpha\beta} = I + S_{\alpha\beta} = \begin{pmatrix} 1 & -\alpha_1 a_{12} & 0 & \dots & 0 & 0 \\ -\beta_1 a_{12} & 1 & -\alpha_2 a_{23} & \dots & 0 & 0 \\ 0 & -\beta_2 a_{23} & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & 1 & -\alpha_{n-1} a_{n-1,n} \\ 0 & 0 & 0 & \dots & -\beta_{n-1} a_{n-1,n} & 1 \end{pmatrix}, \tag{1.4}$$

where α_i, β_i, $i = 1, 2, ..., n-1$, are nonnegative real numbers. We will show that the rate of convergence of this preconditioned can be faster than the rate of convergence of the AOR method.

2 Preliminaries

For solving the linear system Eq.(1.1), if we split A into $A = M - N$ with the non-singular matrix M, the basic iterative method can be expressed with

$$x^{(i+1)} = M^{-1} N x^{(i)} + M^{-1} b, \quad i = 0, 1, 2, ..., \tag{2.1}$$

at which iterative method is convergent to the unique solution $x = A^{-1} b$ for each initial value $x^{(0)}$ if and only if $\rho(M^{-1}N) < 1$.
For simplicity, we let $A = I - L - U$ where I is the identity matrix, $-L$, $-U$ are strictly lower and strictly upper triangular part of A, respectively.
Definition 2.1. The accelerated over-relaxation AOR method is

$$x^{(i+1)} = L_{\sigma,\omega} x^{(i)} + (I - \sigma L)^{-1} \omega b, \quad i = 0, 1, 2, ..., \tag{2.2}$$

where

$$L_{\sigma,\omega} = (I - \sigma L)^{-1} [(1-\omega) I + (\omega - \sigma) L + \omega U], \tag{2.3}$$

is iteration matrix and σ, ω are real parameters with $\omega \neq 0$, [3].
The original system Eq.(1.1) may be transform into the preconditioned

form as follows

$$PAx = Pb. \tag{2.4}$$

Then the corresponding basic iterative method can be defined by

$$x^{(i+1)} = {M_p}^{-1} N_p x^{(i)} + {M_p}^{-1} Pb, \quad i = 0, 1, 2, ..., \tag{2.5}$$

where $PA = M_p - N_p$ is a splitting of PA.

Definition 2.2.

(i) A matrix A is a Z-matrix if $a_{ij} \leq 0$, $i, j = 1, 2, ..., n$, $i \neq j$, [13].
(ii) A nonsingular Z-matrix is called an M-matrix if $A^{-1} \geq 0$, [7,13].

Definition 2.3. If A be a real matrix, $A = M - N$ is called a splitting of A if M be a nonsingular matrix. The splitting is called M-splitting if and only if M is an M-matrix and $N \geq 0$, [12].

Lemma 2.1. Let $A \geq 0$

(i) If $\alpha x \leq Ax$ for some positive vector x, $x \neq 0$, then $\alpha \leq \rho(A)$.
(ii) If $Ax \leq \beta x$ for some positive vector x, then $\rho(A) \leq \beta$. Moreover, if A is irreducible and if $0 \neq \alpha x \leq Ax \leq \beta x$ for some nonnegative vector, then $\alpha \leq \rho(A) \leq \beta$, and x is a positive vector [10].

Lemma 2.2. let $A = M - N$ be an $M-$splitting of A then $\rho(M^{-1}N) < 1$ if and only if A is a nonsingular $M-$matrix [5].
Lemma 2.3. Let A be a $Z-$matrix, then A is a nonsingular $M-$matrix if and only if there is a positive vector x such that $Ax >> 0$, [1].

3 AOR method with the modification of the preconditioner $I + S_{\alpha\beta}$

In this section, we consider a preconditioned form

$$P_{\alpha\beta} Ax = P_{\alpha\beta} b, \tag{3.1}$$

with the preconditioner $P_{\alpha\beta} = I + S_{\alpha\beta}$, i.e.,

$$A_{\alpha\beta}x = b_{\alpha\beta}, \tag{3.2}$$

where $A_{\alpha\beta} = P_{\alpha\beta}A$ and $b_{\alpha\beta} = P_{\alpha\beta}b$.
We use the AOR method for solving Eq.(3.2) and have the corresponding preconditioned AOR iterative method with the following iterative matrix

$$\bar{L}_{\sigma,\omega} = (D_{\alpha\beta} - \sigma L_{\alpha\beta})^{-1}[(1-\omega)D_{\alpha\beta} + (\omega - \sigma)L_{\alpha\beta} + \omega U_{\alpha\beta}, \tag{3.3}$$

where $D_{\alpha\beta}$ is diagonal matrix and $-L_{\alpha\beta}$, $-U_{\alpha\beta}$ are strictly lower and strictly upper triangular matrices which are obtained by splitting $A_{\alpha\beta}$, respectively. The main result is given as follows:
Theorem 3.1. Let $A = [a_{ij}]$ is an $n \times n$ nonsingular Z-matrix, assume that $0 \le \sigma \le \omega \le 1,\ \omega \neq 0$ and $\alpha_i, \beta_i \in [0, 1]$, $i = 1, 2, ..., n-1$,

(i) If $\rho(L_{\sigma,\omega}) < 1$, then

$$\rho(\bar{L}_{\sigma,\omega}) \le \rho(L_{\sigma,\omega}) < 1.$$

(ii) Let A be irreducible, let

$$a_{i,i+1}a_{i+1,i} < 1, i = 1, 2, ..., n-1,$$

then

$$\rho(\bar{L}_{\sigma,\omega}) = \rho(L_{\sigma,\omega}) < 1,$$

or

$$\rho(\bar{L}_{\sigma,\omega}) \ge \rho(L_{\sigma,\omega}) > 1.$$

Proof: Let

$$
\begin{aligned}
M &= \frac{1}{\omega}(I - \sigma L),\\
N &= \frac{1}{\omega}[(1-\omega)I + (\omega - \sigma)L + \omega U],\\
E_{\alpha\beta} &= \frac{1}{\omega}(D_{\alpha\beta} - \sigma L_{\alpha\beta}),\\
F_{\alpha\beta} &= \frac{1}{\omega}[(1-\omega)D_{\alpha\beta} + (\omega - \sigma)L_{\alpha\beta} + \omega U_{\alpha\beta}],\\
M_{\alpha\beta} &= \frac{1}{\omega}(I + S_{\alpha\beta})(I - \sigma L),\\
N_{\alpha\beta} &= \frac{1}{\omega}(I + S_{\alpha\beta})[(1-\omega)I + (\omega - \sigma)L + \omega U],
\end{aligned}
$$

where σ, ω are defined in definition Eq.(2.1), $-L$, $-U$ are strictly lower and strictly upper triangular part of A, respectively. $D_{\alpha\beta}$, $-L_{\alpha\beta}$, $-U_{\alpha\beta}$ are the diagonal, strictly lower and strictly upper triangular matrices obtained from $A_{\alpha\beta}$, respectively.
Then, we have

$$A = M - N, \quad A_{\alpha\beta} = E_{\alpha\beta} - F_{\alpha\beta} = M_{\alpha\beta} - N_{\alpha\beta}.$$

(i) Obviously, since A is a nonsingular $Z-$matrix and $\omega \neq 0$, $0 \leq \sigma \leq \omega \leq 1$, then $M = \frac{1}{\omega}(I - \sigma L)$ is a nonsingular $M-$matrix and $N \geq 0$, then A can be splitted as an $M-$splitting as the following

$$A = M - N = \frac{1}{\omega}(I - \sigma L) - \frac{1}{\omega}[(1-\omega)I + (\omega - \sigma)L + \omega U].$$

If $\rho(L_{\sigma,\omega}) < 1$, it implies of Lemma 2.2, that A is a nonsingular $M-$matrix, then by using Lemma 2.3 there is a positive vector x such that $Ax \geq 0$, hence we have $A_{\alpha\beta}x = (I + S_{\alpha\beta})Ax \geq 0$. Similarly, $A_{\alpha\beta}$ is also a nonsingular $M-$ matrix. The entries of $A_{\alpha\beta}$ are

$$
\begin{aligned}
\bar{a}_{ij} &= a_{ij} - \alpha_i a_{i,i+1} a_{i+1,j} - \beta_{i-1} a_{i-1,i} a_{i-1,j} \quad for \quad 1 < i < n,\\
\bar{a}_{ij} &= a_{ij} - \alpha_i a_{i,i+1} a_{i+1,j} \quad for \quad i = 1,\\
\bar{a}_{ij} &= a_{ij} - \beta_{i-1} a_{i-1,i} a_{i-1,j} \quad for \quad i = n.
\end{aligned}
\tag{3.4}
$$

The entries of matrix $D_{\alpha\beta} = diag(\bar{d}_{11}, \bar{d}_{22}, ..., \bar{d}_{nn})$ are

$$
\begin{aligned}
\bar{d}_{ii} &= 1 - \alpha_i a_{i,i+1} a_{i+1,j} - \beta_{i-1} a_{i-1,i} a_{i-1,j} \quad when \quad 1 < i < n, \\
\bar{d}_{ii} &= 1 - \alpha_i a_{i,i+1} a_{ij} \quad for \quad i = 1, \\
\bar{d}_{ii} &= 1 - \beta_{i-1} a_{i-1,i} a_{i-1,j} \quad for \quad i = n.
\end{aligned} \tag{3.5}
$$

let $A_{\alpha\beta}$ be a nonsingular $M-$matrix, so $\bar{d}_{ii} > 0$. So $D_{\alpha\beta}$ is an invertible positive diagonal matrix. We know that $L_{\alpha\beta} \geq 0$, this implies that $E_{\alpha\beta}$ can be a $Z-$matrix. Suppose $\sigma D_{\alpha\beta}^{-1} L_{\alpha\beta} \geq 0$ is a strictly lower triangular matrix it yields $\rho(\sigma D_{\alpha\beta}^{-1} L_{\alpha}) = 0 < 1$, we have $(I - \sigma D_{\alpha\beta}^{-1} L_{\alpha\beta})^{-1} \geq 0$, then

$$
E_{\alpha\beta} = (I - \sigma D_{\alpha\beta}^{-1} L_{\alpha\beta})^{-1} D_{\alpha\beta}^{-1} \geq 0. \tag{3.6}
$$

Therefore $E_{\alpha\beta}$ is a nonsingular $M-$matrix.
Obviously, we know that $U_{\alpha\beta}$ and $F_{\alpha\beta} \geq 0$. Hence, we prove that $A_{\alpha\beta} = E_{\alpha\beta} - F_{\alpha\beta}$ is an $M-$splitting. Using Lemma 2.2, it yields $\rho(\bar{L}_{\sigma,\omega}) = \rho(E_{\alpha\beta}^{-1} F_{\alpha\beta}) < 1$, since $A_{\alpha\beta} = E_{\alpha\beta} - F_{\alpha\beta}$ and $A = M - N$ are both $M-$splitting and $M_{\alpha\beta}^{-1} N_{\alpha\beta} = M^{-1} N$, therefore, both splitting $A_{\alpha\beta} = E_{\alpha\beta} - F_{\alpha\beta}$ and $A_{\alpha\beta} = M_{\alpha\beta} - N_{\alpha\beta}$ are nonnegative.
On the other hand, let $D_{\alpha\beta} - L_{\alpha\beta} = I - L - S_{\alpha\beta} L$, $L_{\alpha\beta} = D_{\alpha\beta} - I + L + S_{\alpha\beta} L$, we have

$$
\begin{aligned}
M_{\alpha\beta} - E_{\alpha\beta} &= \frac{1}{\omega}(I + S_{\alpha\beta})(I - \sigma L) - \frac{1}{\omega}(D_{\alpha\beta} - \sigma L_{\alpha\beta}) \\
&= \frac{1}{\omega}(I + S_{\alpha\beta} - \sigma L - \sigma S_{\alpha\beta} L - D_{\alpha\beta} + \sigma L_{\alpha\beta}) \\
&= \frac{1}{\omega}[I + S_{\alpha\beta} - \sigma L - \sigma S_{\alpha\beta} L - D_{\alpha\beta} + \sigma(D_{\alpha\beta} - I + L + S_{\alpha\beta})] \\
&= \frac{1}{\omega}[(1 - \sigma)(I - D_{\alpha\beta}) + S_{\alpha\beta}] \geq 0.
\end{aligned} \tag{3.7}
$$

So

$$
A_{\alpha\beta}^{-1} M_{\alpha\beta} - A_{\alpha\beta}^{-1} E_{\alpha\beta} = A_{\alpha\beta}^{-1}(M_{\alpha\beta} - E_{\alpha\beta}) \geq 0,
$$

then we get

$$
A_{\alpha\beta}^{-1} M_{\alpha\beta} \geq A_{\alpha\beta}^{-1} E_{\alpha\beta} \geq 0,
$$

we have $\rho(E_{\alpha\beta}^{-1} F_{\alpha\beta}) \leq \rho(M_{\alpha\beta}^{-1} N_{\alpha\beta})$, [8]. That is

$$
\rho(\bar{L}_{\sigma,\omega}) \leq \rho(L_{\sigma,\omega}) < 1.
$$

(ii) Let $A = I - L - U$ be irreducible. Suppose

$$\begin{aligned} L_{\sigma,\omega} &= (I-\sigma L)^{-1}[(1-\omega)I + (\omega-\sigma)L + \omega U] \\ &= (1-\omega)I + \omega(1-\sigma)L + \omega U + H, \end{aligned} \tag{3.8}$$

with

$$H = (I-\sigma L)^{-1}\sigma L[\omega(1-\sigma)L + \omega U] \geq 0.$$

$L_{\sigma,\omega}$ is a nonnegative and irreducible matrix. There exists a positive vector x, such that [10]

$$L_{\sigma,\omega} = \vartheta x,$$

where $\rho(L_{\sigma,\omega})$ is denoted by ϑ. Using Eq.(3.8), we obtain the identity as the following

$$[(1-\omega)I + (\omega-\sigma)L + \omega U]x = \vartheta(I-\sigma L)x. \tag{3.9}$$

By manipulating Eq.(3.9), we get

$$[(1-\omega-\vartheta)I + (\omega-\sigma+\vartheta\sigma)L + \omega U]x = 0, \tag{3.10}$$

and

$$(\vartheta-1)(I-\sigma L)x = \omega(L+U-I)x. \tag{3.11}$$

Let $S_{\alpha\beta}\ L = D_1 + L_1$, $S_{\alpha\beta}\ U = D_2 + U_1$, where D_1, L_1, are the diagonal and lower triangular parts of $S_{\alpha\beta}\ L$ and D_2, U_1 are the diagonal and upper triangular parts of $S_{\alpha\beta}\ U$, respectively.
Hence,

$$\begin{aligned} A_{\alpha\beta} &= D_{\alpha\beta} - L_{\alpha\beta} - U_{\alpha\beta} \\ &= (I - L - S_{\alpha\beta}\ L) - (U - S_{\alpha\beta} + S_{\alpha\beta}\ U) \qquad (3.12) \\ &= (I - D_1 - D_2) - (L + L_1) - (U - S_{\alpha\beta} + U_1), \qquad (3.13) \end{aligned}$$

where

$$D_{\alpha\beta} = I - D_1 - D_2, \quad L_{\alpha\beta} = L + L_1, \quad U_{\alpha\beta} = U - S_{\alpha\beta} + U_1.$$

By Eqs.(3.10) and (3.11), we have

$$\begin{aligned}
\bar{L}_{\sigma,\omega}x - \vartheta x &= (D_{\alpha\beta} - \sigma L_{\alpha\beta})^{-1}[(1-\omega)D_{\alpha\beta} + (\omega-\sigma)L_{\alpha\beta} + \omega U_{\alpha\beta} - \vartheta(D_{\alpha\beta} - \sigma L_{\alpha\beta})]x \\
&= (D_{\alpha\beta} - \sigma L_{\alpha\beta})^{-1}[(1-\omega-\vartheta)(I - D_1 - D_2) + (\omega - \sigma + \sigma\vartheta)(L + L_1) + \\
&\quad \omega(U - S_{\alpha\beta} + U_1)]x \\
&= (D_{\alpha\beta} - \sigma L_{\alpha\beta})^{-1}\{[(1-\omega-\vartheta) + (\omega - \sigma - \sigma\vartheta)L + \omega U] \\
&+ [-(1-\omega-\sigma)(D_1 + D_2) + (\omega - \sigma + \sigma\vartheta)L_1 + \omega(U_1 - S_{\alpha\beta})]\}x \\
&= (D_{\alpha\beta} - \sigma L_{\alpha\beta})^{-1}[(\vartheta - 1)(D_1 + D_2) + \omega(D_1 + D_2) + \sigma(\vartheta - 1)L_1 + \omega L_1 \\
&+ \omega(U_1 - S_{\alpha\beta})]x \\
&= (D_{\alpha\beta} - \sigma L_{\alpha\beta})^{-1}[(\vartheta - 1)(D_1 + D_2) + \sigma(\vartheta - 1)L_1 + \omega(S_{\alpha\beta}(L + U) - S_{\alpha\beta})]x \\
&= (D_{\alpha\beta} - \sigma L_{\alpha\beta})^{-1}[(\vartheta - 1)(D_1 + D_2) + \sigma(\vartheta - 1)L_1 + (\vartheta - 1)S_{\alpha\beta}(I - \sigma L)]x \\
&= (D_{\alpha\beta} - \sigma L_{\alpha\beta})^{-1}[(\vartheta - 1)(1 - \sigma)D_1 + (\vartheta - 1)D_2 + (\vartheta - 1)S_{\alpha\beta}]x,
\end{aligned}$$

here $0 \leq \sigma < 1$, $S_{\alpha\beta} \geq 0$, $D_1, D_2 \geq 0$. Using Eq.(3.6), we have $D_{\alpha\beta} - \sigma L_{\alpha\beta}$ is an $M-$matrix.

If $\vartheta < 1$, then $\bar{L}_{\sigma,\omega}x - \vartheta x \leq 0$, so $\bar{L}_{\sigma,\omega}x \leq \vartheta x$. By using Lemma 2.1, we get

$$\rho(\bar{L}_{\sigma,\omega}) \leq \rho(L_{\sigma,\omega}) < 1.$$

If $\vartheta > 1$, then $\bar{L}_{\sigma,\omega}x - \vartheta x \geq 0$, so $\bar{L}_{\sigma,\omega}x \geq \vartheta x$. By using Lemma 2.1, we get

$$\rho(\bar{L}_{\sigma,\omega}) \geq \rho(L_{\sigma,\omega}) > 1.$$

Corollary 3.1. Let $A = [a_{ij}] \in R^{n\times n}$ be a nonsingular $M-$matrix. Suppose that

$$0 \leq \alpha_i \leq 1, \quad i = 1, 2, ..., n-1,$$

then for $\omega \neq 0$, $0 \leq \sigma \leq \omega \leq 1$, it yields

$$\rho(\bar{L}_{\sigma,\omega}) \leq \rho(L_{\sigma,\omega}) < 1.$$

Remark 3.1. We have given some inequalities of spectral radius of iteration matrices. The spectral radius of the AOR method also depends upon the choice of the parameters α_i, β_i, $i = 1, 2, ..., n-1$.

Example 3.1. Let the coefficient matrix A of Eq.(1.1) is given by

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

If we choose $\omega = 1$ and $\sigma = 0.5$, We get $\rho(L_{0.5}, 1) = 0.70$. By choosing $\alpha_1 = \frac{1}{2}$, $\beta_1 = \frac{1}{3}$, we get $\rho(\bar{L}_{0.5,1}) = 0.57$. It shows that $\rho(\bar{L}_{0.5,1}) \leq \rho(L_{0.5,1})$.

Example 3.2. Let the coefficient matrix A of Eq.(1.1) is given by

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -\frac{1}{2} & 0 & 1 \end{bmatrix},$$

If we choose $\omega = 1$ and $\sigma = 0.6$, we get $\rho(L_{0.6,1}) = 0.75$. By choosing $\alpha_1 = \beta_1 = 0$, $\alpha_2 = 1$, $\beta_2 = \frac{1}{2}$, we get $\rho(\bar{L}_{0.6,1}) = 0.62$. It shows that $\rho(\bar{L}_{0.6,1}) \leq \rho(L_{0.6,1})$.

Example 3.3. Let the coefficient matrix A of Eq.(1.1) is given by

$$A = \begin{bmatrix} 1 & -0.2 & -0.1 & -0.4 & -0.2 \\ -0.2 & 1 & -0.3 & -0.1 & -0.6 \\ -0.3 & -0.2 & 1 & -0.1 & -0.6 \\ -0.1 & -0.1 & -0.1 & 1 & -0.01 \\ -0.2 & -0.3 & -0.4 & -0.3 & 1 \end{bmatrix},$$

If we choose $\omega = 1$ and $\sigma = 0.5$, we get $\rho(L_{0.5,1}) = 0.97$. By choosing $\alpha_1 = \beta_1 = 0$, $\alpha_2 = \alpha_3 = \frac{1}{3}$, $\beta_2 = \beta_3 = \frac{1}{7}$, $\alpha_4 = \beta_4 = 0$, we get $\rho(\bar{L}_{0.5,1}) = 0.90$. It shows that $\rho(\bar{L}_{0.5,1}) \leq \rho(L_{0.5,1})$.

Let the coefficient matrix A of Eq.(1.1) is given by

$$A = \begin{bmatrix} 1 & -0.4 & -0.1 & 0 & -0.2 & -0.1 \\ -0.05 & 1 & -0.1 & 0 & 0 & 0 \\ 0 & -0.05 & 1 & -0.45 & -0.1 & -0.2 \\ -0.1 & -0.1 & -0.1 & 1 & -0.2 & -0.25 \\ 0 & -0.1 & 0 & -0.05 & 1 & -0.1 \\ -0.25 & -0.15 & -0.1 & 0 & -0.1 & 1 \end{bmatrix},$$

If we choose $\omega = 1$ and $\sigma = 0.5$, we get $\rho(L_{0.5,1}) = 0.45$.
If we choose $\omega = 1$ and $\sigma = 0.6$, we get $\rho(L_{0.6,1}) = 0.43$.
By choosing $\alpha_i = \beta_i = \frac{1}{2}$, $i = 1, ..., 5$, we get $\rho(\bar{L}_{0.5,1}) = 0.35$ and

$\rho(\bar{L}_{0.6,1}) = 0.33$.
By choosing $\alpha_i = \frac{1}{2}$, $\beta_i = \frac{1}{3}$, $i = 1, ..., 5$, we get $\rho(\bar{L}_{0.5,1}) = 0.35$ and $\rho(\bar{L}_{0.6,1}) = 0.34$. It implies that $\rho(\bar{L}_{0.5,1}) \leq \rho(L_{0.5,1})$ and $\rho(\bar{L}_{0.6,1}) \leq \rho(L_{0.6,1})$.

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