Theory of Approximation and Applications, 2020, Volume 14, No. 2: 105-113, DOR: 20.1001.1.25382217.2020.14.2.8.2

A Strong Convergence Process for Multi-valued Quasi Nonexpansive Mappings in CAT(0) spaces

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Article Info

Keywords
Quasi nonexpansive mapping
CAT(0) spaces
fixed point
Hemicompact map.

Article History
Received: 2020 February 9
Accepted: 2020 July 21

Abstract

A new iterative is proposed for finding a common fixed points of multi-valued quasi nonexpansive mappings and the strong convergence the scheme is proved in CAT(0) spaces. The strong convergence theorem for hemicompact map shown is also.

1 Introduction

The study of CAT(0) spaces was initiated by W.A.Kirk [7]. He show that every nonexpansive single-valued mapping defined on a bounded closed convex subset of a complete CAT(0) space always has a fixed point. The fixed point theorems in CAT(0) spaces has applications in graph theory, biology, and computer science(see [1, 4, 5, 9]). Dhompongsa et al in [3]obtained some convergence theorems for different iterations for nonexpansive single-valued mappings in CAT(0) spaces. Many authors introduced and studied kinds of iterative for single and multi-valued mappings in Hilbert spaces (see [6, 8, 10, 11, 12]).
The purpose of this article is study the iterative scheme define as follow:

Let D be a closed convex subset of a complete CAT(0) space. Let the multi-valued $T_1, T_2 : D \to CB(D)$ be quasi nonexpansive map with $F(T_1) \bigcap F(T_2) \neq \emptyset$. Suppose $\{x_n\}$ is generated iteratively by $x_1 \in D$,

$$\begin{aligned} y_n &= \alpha_n z_n \oplus (1-\alpha_n)x_n, \\ x_{n+1} &= \beta_n \acute{z}_n \oplus (1-\beta_n)x_n. \end{aligned} \tag{1.1}$$

for all $n \geq 1$, where $\{\alpha_n\}, \{\beta_n\} \in [a, b]$ are real sequences in $a, b \in (0, 1)$ and $z_n \in T_1 x_n, \acute{z}_n \in T_2 y_n$. We show that the sequence $\{x_n\}$ is strongly convergence to common fixed point T_1 and T_2.

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2 CAT(0) Spaces

Let (X,d) be a metric space. A geodesic path joining $x \in X$ to $y \in X$ (or more briefly, a geodesic from x to y) is a map γ from a closed interval $[0,l] \subset R$ to X such that $\gamma(0)=x, \gamma(l)=y$, and $d(\gamma(t), \gamma(\acute{t}) = |t-\acute{t}|$, for all $t, \acute{t} \in [0,l]$. In particular, γ is an isometry and $d(x,y)=l$. The image γ is called a geodesic (or metric) segment joining x and y. When it is unique, this geodesic is denoted by $[x,y]$. The space (X,d) is said to be a geodesic space if every two points of X are joined by a geodesic, and X is said to be uniquely geodesic if there is exactly one geodesic joining x to y, for each $x, y \in X$. A subset $Y \subset X$ is said to be convex if Y includes every geodesic segment joining any two of its points. A geodesic triangle $\Delta(x_1,x_2,x_3)$ in a geodesic metric space (X,d) consists of three points in X (the vertices of Δ) and a geodesic segment between each pair of vertices (the edges of Δ). A comparison triangle for geodesic triangle $\Delta(x_1,x_2,x_3)$ in (X,d) is a triangle $\overline{\Delta}(x_1,x_2,x_3) := \Delta(\overline{x_1},\overline{x_2},\overline{x_3})$ in the Euclidean plane R^2 such that $d_{R^2}(\overline{x_i},\overline{x_j}) = d(x_i,x_j)$, for $i,j \in \{1,2,3\}$. A geodesic metric space is said to be a CAT(0) space [2] if all geodesic triangles of appropriate size satisfy the following comparison axiom. Let Δ be a geodesic triangle in X and let $\overline{\Delta}$ be a comparison triangle for Δ. Then, Δ is said to satisfy the CAT(0) inequality if for all $x, y \in \Delta$ and all comparison points $\overline{x}, \overline{y} \in \overline{\Delta}$, $d(x,y) \leq d_{R^2}(\overline{x},\overline{y})$. It is known that in a CAT(0) space, the distance function is convex [2]. Complete CAT(0) spaces are often called Hadamard spaces. Finally, we observe that if x, y_1, y_2 are points of a CAT(0) space and if y_0 is the midpoint of the segment $[y_1,y_2]$, which we will denote by $\frac{y_1 \bigoplus y_2}{2}$, then the CAT(0) inequality implies

$$d(x, \frac{y_1 \bigoplus y_2}{2})^2 \leq \frac{1}{2}d(x,y_1)^2 + \frac{1}{2}d(x,y_2)^2 - \frac{1}{4}d(y_1,y_2)^2. \tag{2.1}$$

A geodesic metric space is a CAT(0) space if and only if it satisfies inequality (2.1)(which is known as the CN inequality).
Let X be a complete CAT(0) space and $\{x_n\}$ be a bounded sequence in X. For $x \in X$ set

$$r(x,\{x_n\}) = \limsup_{n\to\infty} d(x,x_n)$$

the asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = inf\ \{r(x,\{x_n\}) :\ x \in X\}$$

and the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the

$$A(\{x_n\}) = \{x \in X :\ r(x,\{x_n\}) = r(\{x_n\})\}.$$

Let(X,d) be a geodesic space and D be a nonempty convex subset of complete CAT(0) space X. The set D is called Proximinal if for each $x \in X$, there exists an element $y \in D$ such that $d(x,y) = d(x,D)$, where $d(x,D) = inf\ \{d(x,z) :\ z \in D\}$.
The families of nonempty closed bounded subsets, and nonempty proximinal bounded subsets of D, is denoted by $CB(D)$ and $P(D)$, respectively.
The Hausdroff metric on $CB(D)$ is defined by

$$H(A,B) = Max\ \{sup\ _{x\in A}d(x,B),\ sup_{y\in B}d(y,A)\}$$

for $A, B \in CB(D)$.
An element $p \in D$ is called a fixed point of multi-valued $T : D \to CB(D)$ if $p \in Tp$. The set of fixed points of T is denoted by $F(T)$.
Also, The multi-valued mapping $T : D \to CB(D)$ is called

(1): Quasi nonexpansive, if $F(T) \neq$ and $H(Tx, Tp) \leq d(x, p)$ for all $x \in D$ and $p \in F(T)$.

(2): L-Lipschitzian, if there exists a constant $L > 0$ such that

$$H(Tx, Ty) \leq L\, d(x, y)$$

for all $x, y \in D$.

(3): Hemicompact, if for any sequence $\{x_n\}$ in D such that $d(x_n, Tx_n) \to 0$ as $n \to \infty$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to p \in D$.

(4): Two multi-valued maps $T_1, T_2 : D \to CB(D)$ are satisfied condition II if there is a non decreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$, $f(r) > 0$ for $r \in (0, \infty)$ such that

$$\sum_{i=1}^{2} d(x, T_i x) \geq f(d(x, \bigcap_{i=1}^{2} F(T_i)))$$

The following Lemma will be useful for proving the main results in this paper:

Lemma 2.1. *([3]) Let (X,d) be a CAT(0) space. For $x, y \in X$ and $t \in [0, 1]$, there exists a unique point $z \in [x, y]$ such that*

$$d(x, z) = td(x, y) \quad and \quad d(y, z) = (1 - t)d(x, y),$$

we use the notation $(1 - t)x \oplus ty$ for the unique z.

Lemma 2.2. *([3]) Let (X,d) be a CAT(0) space. Then*

$$d((1 - t)x \oplus ty, z)^2 \leq (1 - t)d(x, z)^2 + td(y, z)^2 - t(1 - t)d(x, y)^2,$$

for all $t \in [0, 1]$ and $x, y, z \in X$.

3 Strong convergence theorems

Here our main result is presented.

Theorem 3.1. *Let X be a complete CAT(0) space and D a nonempty closed convex subset X and $T_1, T_2 : D \to CB(D)$ be a quasi nonexpansive multi-valued maps with $F(T_1) \bigcap F(T_2) \neq \emptyset$ such that $T_1 p = \{p\}$ and $T_2 p = \{p\}$ for each $p \in F(T_1) \bigcap F(T_2)$. Suppose $x_1 \in D$ and $\{x_n\}$ is defined by (1.1). Then $\lim_{n\to\infty} d(x_n, p)$ exists.*

Proof. Let $p \in F(T_1) \bigcap F(T_2)$, we have

$$\begin{aligned} d(y_n,p) &= d(\alpha_n z_n \oplus (1-\alpha_n)x_n, p) \\ &\leq \alpha_n d(z_n,p) + (1-\alpha_n)d(x_n,p) \\ &= \alpha_n d(z_n, T_1p) + (1-\alpha_n)d(x_n,p) \\ &\leq \alpha_n H(T_1x_n, T_1p) + (1-\alpha_n)d(x_n,p). \end{aligned}$$

Since T_1 is quasi nonexpansive

$$d(y_n,p) \leq \alpha_n d(x_n,p) + (1-\alpha_n)d(x_n,p) = d(x_n,p). \tag{3.1}$$

So

$$\begin{aligned} d(x_{n+1},p) &= d(\beta_n z\acute{}_n) \oplus (1-\beta_n)x_n, p) \\ &\leq \beta_n d(z\acute{}_n,p) + (1-\beta_n)d(x_n,p) \\ &= \beta_n d(z\acute{}_n, T_2p) + (1-\beta_n)d(x_n,p) \\ &\leq \beta_n H(T_2y_n, T_2p) + (1-\beta_n)d(x_n,p). \end{aligned}$$

T_2 is quasi nonexpansive

$$d(x_{n+1},p) \leq \beta_n d(y_n,p) + (1-\beta_n)d(x_n,p).$$

Now, by (3.1) we have

$$d(x_{n+1},p) \leq \beta_n d(x_n,p) + (1-\beta_n)d(x_n,p) = d(x_n,p)$$

this implies that the sequence $\{d(x_n,p)\}$ is decreasing and bounded. Then $\lim_{n\to\infty} d(x_n,p)$. □

Theorem 3.2. *Let X be a complete CAT(0) space and D a nonempty closed convex subset X and $T_1 : D \to CB(D)$ be a quasi nonexpansive multi-valued map and $T_2 : D \to CB(D)$ be a quasi nonexpansive and $L-$Lipchitzian multi-valued map. Moreover, $F(T_1) \bigcap F(T_2) \neq \emptyset$ and $T_1p = \{p\}, T_2p = \{p\}$ for each $p \in F(T_1) \bigcap F(T_2)$. If $\{T_1, T_2\}$ satisfies condition II, then the sequence $\{x_n\}$ generated by (1.1) convergence strongly to common fixed point T_1 and T_2.*

Proof. Let $p \in F(T_1) \bigcap F(T_2)$, we have

$$\begin{aligned} d(y_n,p)^2 &= d(\alpha_n z_n \oplus (1-\alpha_n)x_n, p)^2 \\ &\leq \alpha_n d(z_n,p)^2 + (1-\alpha_n)d(x_n,p)^2 - \alpha_n(1-\alpha_n)d(z_n,x_n)^2 \\ &= \alpha_n d(z_n,T_1p)^2 + (1-\alpha_n)d(x_n,p)^2 - \alpha_n(1-\alpha_n)d(z_n,x_n)^2 \\ &\leq \alpha_n H(T_1x_n,T_1p)^2 + (1-\alpha_n)d(x_n,p)^2 - \alpha_n(1-\alpha_n)d(z_n,x_n)^2 \end{aligned}$$

since T_1 is quasi nonexpansive

$$d(y_n,p)^2 \leq d(x_n,p)^2 - \alpha_n(1-\alpha_n)d(z_n,x_n)^2. \tag{3.2}$$

It follows that

$$\begin{aligned} d(x_{n+1},p)^2 &= d(\beta_n \acute{z_n} \oplus (1-\beta_n)x_n, p)^2 \\ &\leq \beta_n d(\acute{z_n},p)^2 + (1-\beta_n)d(x_n,p)^2 - \beta_n(1-\beta_n)d(\acute{z_n},x_n)^2 \\ &= \beta_n d(\acute{z_n},T_2p)^2 + (1-\beta_n)d(x_n,p)^2 - \beta_n(1-\beta_n)d(\acute{z_n},x_n)^2 \\ &\leq \beta_n H(T_2y_n,T_2p)^2 + (1-\beta_n)d(x_n,p)^2 - \beta_n(1-\beta_n)d(\acute{z_n},x_n)^2 \end{aligned}$$

since T_2 is quasi nonexpansive

$$d(x_{n+1},p)^2 \leq \beta_n d(y_n,p)^2 + (1-\beta_n)d(x_n,p)^2 - \beta_n(1-\beta_n)d(\acute{z_n},x_n)^2.$$

The inequality (3.2) implies that

$$d(x_{n+1},p)^2 \leq d(x_n,p)^2 - \beta_n\alpha_n(1-\alpha_n)d(z_n,x_n)^2 - \beta_n(1-\beta_n)d(\acute{z_n},x_n)^2. \tag{3.3}$$

Therefore

$$\begin{aligned} a^2(1-b)d(z_n,x_n)^2 + a(1-b)d(\acute{z_n},x_n)^2 &\leq \beta_n\alpha_n(1-\alpha_n)d(z_n,x_n)^2 + \beta_n(1-\beta_n)d(\acute{z_n},x_n)^2 \\ &\leq d(x_n,p)^2 - d(x_{n+1},p)^2. \end{aligned}$$

But, we have

$$\sum_{n=1}^{\infty} a^2(1-b)d(z_n,x_n) < \infty \quad and \quad \sum_{n=1}^{\infty} a(1-b)d(\acute{z_n},x_n) < \infty.$$

This implies that

$$\lim_{n\to\infty} d(z_n,x_n) = 0, \lim_{n\to\infty} d(\acute{z_n},x_n) = 0. \tag{3.4}$$

Additionally, since

$$d(x_n,T_1x_n) \leq d(x_n,z_n) \to 0 \quad as \quad n\to\infty$$

hence

$$\lim_{n\to\infty} d(x_n,T_1x_n) = 0. \tag{3.5}$$

Moreover,

$$\begin{aligned} d(y_n,z_n) &= d(\alpha_n z_n \oplus (1-\alpha_n)x_n, z_n) \\ &\leq \alpha_n d(z_n,z_n) + (1-\alpha_n)d(x_n,z_n) = (1-\alpha_n)d(x_n,z_n) \end{aligned}$$

then

$$\lim_{n\to\infty} d(y_n, z_n) = 0. \tag{3.6}$$

It follows from (3.4) and (3.6)

$$d(x_n, y_n) \leq d(y_n, z_n) + d(z_n, x_n) \to 0, as \ \ n \to \infty.$$

We have

$$\begin{aligned} d(x_n, T_2x_n) &\leq d(x_n, T_2y_n) + d(T_2y_n, T_2x_n) \\ &\leq d(x_n, T_2y_n) + H(T_2y_n, T_2x_n) \end{aligned}$$

since T_2 is $L-$Lipschitzian

$$d(x_n, T_2x_n) \leq d(x_n, z\acute{}_n) + L\, d(y_n, x_n) \to 0 as \ \ n \to \infty.$$

This implies that

$$\lim_{n\to\infty} d(x_n, T_2x_n) = 0. \tag{3.7}$$

By condition (II) we conclude $\lim\limits_{n\to\infty} d(x_n, F(T_1) \bigcap F(T_2)) = 0$, i.e., there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and sequence p_k in $F(T_1) \bigcap F(T_2)$ such that

$$d(x_{n_k}, p_k) \leq \frac{1}{2^k} \quad \ for \ \ all \ \ k.$$

From

$$d(x_{n_{k+1}}, p_k) \leq d(x_{n_k}, p_k) < \frac{1}{2^k}$$

it follows that

$$d(p_{k+1}, p_k) \leq d(x_{n_{k+1}}, p_{k+1}) + d(x_{n_{k+1}}, p_k) < \frac{1}{2^{k+1}} + \frac{1}{2^k} < \frac{1}{2^{k-1}}.$$

This show that the sequence p_k is cauchy in D and then is convergence to $x^* \in D$. On the other hand,

$$d(p_k, T_ix^*) \leq H(T_ip_k, T_ix^*) \leq d(p_k, x^*) \ for \ i \in \{1, 2\}$$

then $d(x^*, T_ix^*) = 0$ for $i \in \{1, 2\}$ and so $x^* \in F(T_1) \bigcap F(T_2)$, i.e., $\{x_{n_k}\}$ convergence strongly to x^*. Theorem 3.1 implies that the sequence $\{x_n\}$ convergence strongly to x^* common fixed point T_1 and T_2. □

Theorem 3.3. *Let X be a complete CAT(0) space and D a nonempty closed convex subset X. Let $T_1, T_2 : D \to CB(D)$ be a quasi nonexpansive multi-valued map with $F(T_1) \bigcap F(T_2) \neq \emptyset$ and $T_1p = \{p\}$ and $T_2p = \{p\}$ for each $p \in F(T_1) \bigcap F(T_2)$. Assume that T_1, T_2 be hemicompact continuous maps. Suppose $x_1 \in D$ and $\{x_n\}$ is defined by (1.1). Then $\lim\limits_{n\to\infty} d(x_n, p)$ exists.*

Proof. By Theorem 3.1 along with the proof and by equation (3.4) and (3.6), we obtain $\lim\limits_{n\to\infty} d(x_n, T_ix_n) = 0$ for $i = 1, 2$. Since T_1, T_2 are hemicompact, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to p$ for some $p \in D$.

Also, T_1, T_2 are continuous, we obtain

$$d(x_{n_k}, T_i x_{n_k}) \to d(p, T_i p).$$

Since $\lim_{n\to\infty} d(x_{n_k}, T_i x_{n_k}) = 0$, hence $\lim_{n\to\infty} d(p, T_i p) = 0$ and so $p \in F(T_1) \bigcap F(T_2)$.
Theorem 3.1 implies that $\{x_n\}$ strongly convergence to p. □

Theorem 3.4. *Let X be a complete CAT(0) space and D a nonempty closed convex subset X and $T_1, T_2 : D \to CB(D)$ be multi-valued map and $\{P_{T_1}, P_{T_2}\}$ be a quasi nonexpansive and P_{T_2} be $L-$Lipchitzian. Also, $F(T_1) \bigcap F(T_2) \neq \emptyset$ and $T_1 p = \{p\}$ and $T_2 p = \{p\}$ for each $p \in F(T_1) \bigcap F(T_2)$. If $\{T_1, T_2\}$ satisfies condition (II), then the sequence $\{x_n\}$ generated by (1.1) convergence strongly to common fixed point T_1 and T_2.*

Proof. Let $p \in F(T_1) \bigcap F(T_2)$, we have $p \in P_{T_i} = \{p\}$,

$$d(z_n, p) \leq d(z_n, P_{T_1} p) \leq H(P_{T_1} x_n, P_{T_1} p) \leq d(x_n, p), \tag{3.8}$$

$$d(\acute{z_n}, p) \leq d(\acute{z_n}, P_{T_2} p) \leq H(P_{T_2} y_n, P_{T_2} p) \leq d(y_n, p) \tag{3.9}$$

By assumption, we obtain

$$d(y_n, p) = d(\alpha_n z_n \oplus (1-\alpha_n) x_n, p) \leq \alpha_n d(z_n, p) + (1-\alpha_n) d(x_n, p).$$

It follows that

$$\begin{aligned} d(x_{n+1}, p) &= d(\beta_n \acute{z_n} \oplus (1-\beta_n) x_n, p) \\ &\leq \beta_n d(\acute{z_n}, p) + (1-\beta_n) d(x_n, p). \end{aligned}$$

(3.8) together (3.9) implies that

$$d(x_{n+1}, p) \leq d(x_n, p). \tag{3.10}$$

By similar proof argument in Theorem 3.1, one can easy obtain that

$$\lim_{n\to\infty} d(x_n, \acute{z_n}) = 0, \quad \lim_{n\to\infty} d(x_n, z_n) = 0.$$

It follows that

$$d(x_n, T_1 x_n) \leq d(x_n, z_n) \to 0 \quad as \quad n \to \infty$$

then

$$\lim_{n\to\infty} d(x_n, T_1 x_n) = 0.$$

Hausdruff metric space definition implies that for each positive $n \geq 1$ there exist $\widetilde{x_n} \in P_{T_2}$ such that

$$d(\widetilde{x_n}, \acute{z_n}) \leq H(P_{T_2} x_n,\ P_{T_2} y_n) + \frac{1}{n}$$

hence

$$\begin{aligned} d(\widetilde{x_n}, x_n) &\leq d(\widetilde{x_n}, \acute{z_n}) + d(\acute{z_n}, x_n) \\ &\leq H(P_{T_2}x_n,\ P_{T_2}y_n) + \frac{1}{n} + d(\acute{z_n}, x_n). \end{aligned}$$

Since P_{T_2} is $L-$Lipchitzian, we have

$$\begin{aligned} d(\widetilde{x_n}, x_n) &\leq L\, d(x_n, y_n) + d(\acute{z_n}, x_n) + \frac{1}{n} \\ &\leq L\, d(x_n, z_n) + L\, d(y_n, z_n) + d(\acute{z_n}, x_n) + \frac{1}{n} \\ &= L\, d(x_n, z_n) + L\, d(\alpha_n z_n \oplus (1-\alpha_n)x_n, z_n) + d(\acute{z_n}, x_n) + \frac{1}{n} \\ &\leq L\,(1+\alpha_n) d(x_n, z_n) + d(\acute{z_n}, x_n) + \frac{1}{n}. \end{aligned}$$

Then

$$\lim_{n\to\infty} d(\widetilde{x_n}, x_n) = 0.$$

From inequality

$$d(x_n, T_2x_n) \leq d(\widetilde{x_n}, x_n) \to 0$$

we conclude

$$\lim_{n\to\infty} d(x_n, T_2x_n) = 0.$$

Condition (II) implies that $\lim_{n\to\infty} d(x_n, F(T_1)\bigcap F(T_2)) = 0$, then there exist sequence $\{x_{n_k}\}$ of $\{x_n\}$ and sequence $\{p_k\}$ in $F(T_1)\bigcap F(T_2)$ such that

$$d(x_{n_k}, p_k) \leq \frac{1}{2^k} \;\; for \;\; all \;\; k.$$

On the other hand

$$\begin{aligned} d(p_{k+1}, p_k) &\leq d(x_{n_{k+1}}, p_{k+1}) + d(x_{n_{k+1}}, p_k) \\ &< \frac{1}{2^{k+1}} + \frac{1}{2^k} < \frac{1}{2^{k-1}}. \end{aligned}$$

This show that the sequence p_k is cauchy in D and then is convergence to $x^* \in D$.
Also,

$$d(p_k, T_ix^*) \leq H(P_{T_i}p_k, P_{T_i}x^*) \leq d(p_k, x^*) \; for \; i \in \{1,2\}$$

then $d(x^*, T_ix^*) = 0$ for $i \in \{1,2\}$ and so $x^* \in F(T_1)\bigcap F(T_2)$.
It implies that $\{x_{n_k}\}$ convergence strongly to x^*. Theorem 3.1 implies that $\{x_n\}$ strongly convergence to x^* common fixed point T_1 and T_2.

□

4 Acknowledgment

The authors very grateful to the referee for their careful reading, comments and suggestions, which improve the presentation of this article.

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