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Solving Fuzzy Linear System of Equations

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ABSTRACT

Many real problems may often be reduced to solving a system of linear equations. However, in such real-world problems, the existence of uncertainties, like a fuzzy environment, is inevitable. Thus, the main aim of this paper is to propose a solving approach for a fuzzy linear system of equalities. The presented approach is based on the nearest weighted approximation of fuzzy numbers. Two numerical examples illustrate the capabilities of the proposed approach.

1 Introduction

Practical problems in many fields of study—such as biology, business, chemistry, computer science, economics, electronics, engineering, physics, and the social sciences—can often be reduced to solving a system of linear equations. Hence, systems of simultaneous linear equations play a major role in such areas ((1), (2), (3)). However, in many situations, such as in a manufacturing system, a production process, or a service system, inventories or demands are volatile and complex so it is difficult to measure them in an accurate way. Instead, the data can be given as a fuzzy variable ((4), (5)). Many approaches for solving fuzzy linear systems have been introduced in the related literature.
Among them, Horčík (6) presented a method for finding a solution to a fuzzy interval system of linear equations. Vroman et al. (7) provided an approach for solving systems of linear fuzzy equations based on a practical algorithm using parametric functions in which the variables are given by the fuzzy coefficients of the system. Behera and Chakraverty (8) proposed a framework for finding a solution of a general fuzzy complex system of linear equations. Piegat and

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Pluciński (9) proposed a multidimensional fuzzy RDM-arithmetic and its application for solving linear fuzzy equation systems.
The main aim of this paper is to propose an approach for solving a Fuzzy Linear System of Equations in which its right-hand sides are fuzzy numbers. Saeidifar (10) introduced a weighted mean to rank fuzzy numbers. For this purpose, he proposed a new ranking method for fuzzy numbers, which used a defuzzification of fuzzy numbers and a weighting function. Thus, using his method, in this paper, first, we convert each fuzzy right-hand side of the fussy system into an interval one. In this manner, the fuzzy linear system of equations is converted to an interval system of equations. Therefore, we can solve the obtained system by employing the proposed method of Asady (11) for solving an interval linear system of equalities.
The rest of the paper is organized as follows: In section two, the proposed approach is discussed. Section three provides two illustrative examples to demonstrate the proposed approach. Conclusions and directions for future research are given in section four.

2 Proposed Approach

Our proposed approach consists of two stages. In the first stage, we recall the nearest weighted interval approximation method for fuzzy numbers. In the second stage, with the help of this approach, we convert a fuzzy system of equations into an interval one. Finally, we solve the obtained interval system using the embedding approach.

2.1 Nearest Weighted Interval Approximation of Fuzzy Numbers

Let's $a, b,$ and c are real numbers such that $a \leq b \leq c$. Then, a fuzzy number $A = (a, b, c)$ is called a triangular fuzzy number if its membership function, $\mu_A(x)$, has the following form:

$$\mu_A(x) = \begin{cases} \dfrac{x-a}{b-a}, & a \leq x \leq b \\ \dfrac{c-x}{c-b}, & b \leq x \leq c \\ 0, & \text{otherwise} \end{cases} \tag{1}$$

We denote the family of fuzzy numbers by ξ.The membership function of a triangular fuzzy number is depicted as Fig. 1.

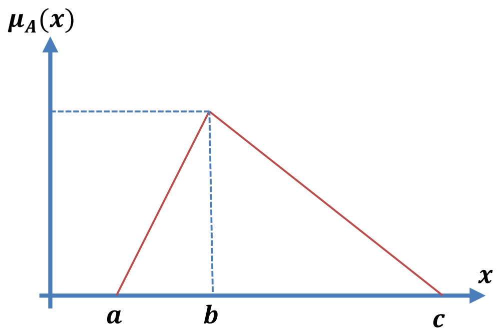

Fig. 1: The membership function of a triangular fuzzy number

To manage the fuzzy numbers, here we employ the concept of the nearest weighted interval approximation to a fuzzy number. The following definition was defined in (10) and applied in (12) and (13) to deal with fuzzy situations.

Definition 1: A weighting function is a function as $f = \left(\underline{f}, \overline{f}\right): ([0,1],[0,1]) \to (\mathbb{R}, \mathbb{R})$ such that the functions $\underline{f}$ and $\overline{f}$ are nonnegative, monotone increasing and satisfies the following normalization condition:

$$\int_0^1 \underline{f}(\alpha)d\alpha = \int_0^1 \overline{f}(\alpha)d\alpha = 1 \tag{2}$$

It must be noted that the function $f(\alpha)$ can be understood as the weight of our interval approximation, the property of monotone increasing of function $f(\alpha)$ means that the higher the cut level is, the more important its weight is in determining the interval approximation of fuzzy numbers. In applications, the function $f(\alpha)$ can be chosen according to the actual situation.

Corollary: Let $\mathrm{A} = (\mathrm{a}, \mathrm{b}, \mathrm{c})$ be a triangular fuzzy number, and $\mathrm{f}(\alpha) = (\mathrm{n}\alpha^{\mathrm{n}-1}, \mathrm{n}\alpha^{\mathrm{n}-1})$ be a weighting function. Then, the nearest weighted interval approximation of the fuzzy number A is obtained as

$$NWIA_f(A) = \left[\frac{a+nb}{n+1}, \frac{nb+c}{n+1}\right] \tag{3}$$

Thus, each fuzzy number can be considered as an interval number.

2.2 Fuzzy Linear System and Solving Methodology

A system of linear equations (or linear system) is a collection of one or more linear equations involving the same set of variables. Generally, the system of linear equations including n equations with n variables with definite data is presented as follows.

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases} \tag{4}$$

In this system, the matrix of coefficients is denoted by A and the right-hand side values of this system are definite. Now, if the right-hand side values of these equations are indefinite, typically fuzzy, we are dealing with a fuzzy system of equations in the following form:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \approx \tilde{b}_1 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \approx \tilde{b}_m \end{cases} \tag{5}$$

Where, the right-hand side values, i.e. $\tilde{b}_i$ $(i = 1, \dots, n)$, are triangular fuzzy numbers as $\tilde{b}_i = (b_{1i}, b_{2i}, b_{3i})$, and thus system (5) is an uncertain system. In order to solve this system, we need to decrease the amount of uncertainties. To solve this fuzzy system, we first use the fuzzy number approximation method and convert it to the form of an interval equation system. For this purpose, suppose that we use the weight function $f\,(\alpha) = (n\alpha^{n-1}, n\alpha^{n-1})$, in which case the right-hand side values are converted to the interval form as follows:

$$\tilde{b}_i = \left[\frac{b_{1i} + nb_{2i}}{n+1}, \frac{nb_{2i} + b_{3i}}{n+1}\right] \tag{6}$$

In this way, it is converted into the following interval linear system

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = [b_1^L, b_1^U] \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = [b_m^L, b_m^U] \end{cases} \tag{7}$$

Where $b_i^L = \frac{b_{1i}+nb_{2i}}{n+1}$ and $b_1^U = \frac{nb_{2i}+b_{3i}}{n+1}$ $(i = 1, \dots, n)$. So, we have an interval linear system. In order to solve an interval linear system like (7), Asady (11) applied the embedding approach and suggested the following $2m \times 2n$ crisp linear system (8).

$$\begin{cases} s_{11}x_1^L + \cdots + s_{1n}x_n^L - s_{1,n+1}x_1^U - \cdots - s_{1,2n}x_n^U = b_1^L \\ \vdots \\ s_{m1}x_1^L + \cdots + s_{mn}x_n^L - s_{m,n+1}x_1^U - \cdots - s_{m,2n}x_n^U = b_m^L \\ s_{m+1,1}x_1^L + \cdots + s_{m+1,n}x_n^L - s_{m+1,n+1}x_1^U - \cdots - s_{m+1,2n}x_n^U = -b_m^U \\ \vdots \\ s_{2m,1}x_1^L + \cdots + s_{2m,n}x_n^L - s_{2m,n+1}x_1^U - \cdots - s_{2m,2n}x_n^U = -b_m^U \end{cases} \tag{8}$$

where s_{ij} are determined as follows:

$$\begin{aligned} &a_{ij} \geq 0 \Rightarrow s_{ij} = a_{ij};\;\; s_{i+m,j+n} = a_{ij}, \\ &a_{ij} < 0 \Rightarrow s_{i,j+n} = -a_{ij};\;\; s_{i+m,j} = -a_{ij}, \end{aligned} \tag{9}$$

And any s_{ij} which is not determined by the above relation is zero. Using the matrix notation (9), we get the following general system.

$$S\tilde{X} = \tilde{b} \tag{10}$$

Where $S = (s_{ij})_{2m,2n} \geq 0, (1 \leq i \leq 2m; 1 \leq j \leq 2n)$, and also we have

$$\tilde{X} = (x_1^L, \dots, x_n^L, -x_1^U, \dots, x_n^U)^t; \text{ and } \tilde{b} = (b_1^L, \dots, b_n^L, -b_1^U, \dots, -b_n^U)^t \tag{11}$$

Asady (11) applied the least square approach and proved that the solution of the interval system (8) can be computed as follows:

$$\tilde{X} = S^t(SS^t)^{-1}\tilde{b} \tag{12}$$

The following Lemma and Theorem from (11), are useful to detect the solution of the proposed fuzzy linear system.

Lemma: The solution $\tilde{X}$ of (8) is an interval vector for an arbitrary $\tilde{b}$ if and only if $S^t(SS^t)^{-1}$ is nonnegative, i.e. $(S^t(SS^t)^{-1})_{ij} \geq 0,\; 1 \leq i \leq 2m, 1 \leq j \leq 2n$.

Theorem: The solution $\tilde{X}$ of (8) is an interval vector for arbitrary Y if the non-diagonal elements of $(SS^t)^{-1}$are non-negative, i.e.

$$((SS^t)^{-1})_{ij} \geq 0, i \neq j,\;\; 1 \leq i, j \leq 2m$$

Accordingly, the following definition describes the property of the interval solution for the system (5).

Definition 2 (Interval Solution): Let the set vector $\{x_i^L, x_i^U, 1 \leq i \leq n\}$ *is* obtained by solving the system (8), then the interval vector $\tilde{X} = \{[\tilde{x}_i^L, \tilde{x}_i^U], 1 \leq i \leq n\}$, *that is defined as (13)* is called the interval solution of (5).

$$\tilde{x}_i^L = \min\{x_i^L, x_i^U\};\ 1 \leq i \leq n\}\ \textit{and}\ \tilde{x}_i^U = \max\{x_i^L, x_i^U\};\ 1 \leq i \leq n\} \tag{13}$$

Remark: If $\tilde{x}_i^L = x_i^L$ and $\tilde{x}_i^U = x_i^U$ $(1 \leq i \leq n\})$, *the solution* $\tilde{X}$ is called a strong interval solution. Otherwise, $\tilde{X}$ is called a weak interval solution.

We should be noted that, the obtained interval solution is completely dependent to the function $f(\alpha)$. We know that, according to the form of the function $f(\alpha)$, the higher the cut level is, the more important its weight is in determining the interval approximation of fuzzy numbers. Obviously, the weighted interval approximation synthetically reflects the information on every membership degree. Its advantage is that different α-cut set plays different roles.

3 Numerical Examples

In this section, we discuss the problems of solving fuzzy linear equation systems and calculate their solution with the proposed method. In this regard, we consider the following two examples.

Example 1: Consider the following fuzzy linear system of equations with two equations and two variables:

$$\begin{cases} 7x_1 - 4x_2 \approx \tilde{5} \\ 3x_1 + 5x_2 \approx \tilde{7} \end{cases}$$

Where $\tilde{5} = (3,5,6)$ and $\tilde{7} = (5,7,11)$ are two triangular fuzzy numbers. We employ the weighting function $f(\alpha) = (3\alpha^2, 3\alpha^2)$. So, $n = 3$ and hence we have

$$\tilde{5} = \left[\frac{3+15}{4}, \frac{15+6}{4}\right] = [4.5,5.25]$$

$$\tilde{7} = \left[\frac{5+21}{4}, \frac{21+11}{4}\right] = [6.5,8]$$

Then, we obtain the following interval system

$$\begin{cases} 7x_1 - 4x_2 = [4.5,5.25] \\ 3x_1 + 5x_2 = [6.5,8] \end{cases}$$

In this regard, the extended matrix (8), is stated as follows:

$$\begin{cases} 7x_1^L - 4x_2^U = 4.5 \\ 3x_1^L + 5x_2^L = 6.5 \\ 4x_2^L - 7x_1^U = -5.25 \\ -3x_1^U - 5x_2^U = -8 \end{cases}$$

Where the matrix S and $\tilde{b}$ are expressed as follows:

$$S = \begin{bmatrix} 7 & 0 & 0 & 4 \\ 3 & 5 & 0 & 0 \\ 0 & 4 & 7 & 0 \\ 0 & 0 & 3 & 5 \end{bmatrix} \text{ and } \tilde{b} = \begin{bmatrix} 4.5 \\ 6.5 \\ -5.25 \\ -8 \end{bmatrix}$$

The solution of the system is computed as follows:

$$\tilde{X} = \begin{bmatrix} x_1^L \\ x_2^L \\ -x_1^U \\ -x_2^U \end{bmatrix} = \begin{bmatrix} 1.185 \\ 0.589 \\ -1.087 \\ -0.948 \end{bmatrix}$$

This system has a weak interval solution as follows:

$$\begin{cases} \tilde{x}_1^L = \min\{x_1^L, x_1^U\} = 1.087 \\ \tilde{x}_1^U = \max\{x_1^L, x_1^U\} = 1.185 \end{cases} \Rightarrow \tilde{x}_1 = [1.087, 1.185]$$

$$\begin{cases} \tilde{x}_2^L = \min\{x_2^L, x_2^U\} = 0.589 \\ \tilde{x}_2^U = \max\{x_2^L, x_2^U\} = 0.948 \end{cases} \Rightarrow \tilde{x}_2 = [0.589, 0.948]$$

Obviously, since $\tilde{x}_i^L \neq x_i^L$ and $\tilde{x}_i^U \neq x_i^U$ $(1 \leq i \leq 2\})$, this solution is a weak solution.

Example 2: Consider the following 3×2 fuzzy linear system of equations:

$$\begin{cases} 2x_1 - 3x_2 + x_3 \approx \tilde{9} \\ -x_1 + 2x_2 - 3x_3 \approx \tilde{8} \end{cases}$$

Where $\tilde{9} = (7,9,12)$ and $\tilde{8} = (6,8,10)$ are two triangular fuzzy numbers. We employ the weighting function $f(\alpha) = (4\alpha^3, 4\alpha^3)$. So, $n = 4$ and hence we have

$$\tilde{9} = \left[\frac{7+36}{5}, \frac{36+12}{5}\right] = [8.6, 9.6]$$

$$\tilde{8} = \left[\frac{6+32}{5}, \frac{32+10}{5}\right] = [7.6, 8.4]$$

Then, we obtain the following interval system

$$\begin{cases} 2x_1 - 3x_2 + x_3 = [8.6,9.6] \\ -x_1 + 2x_2 - 3x_3 = [7.6,8.4] \end{cases}$$

In this regard, the matrix S and $\tilde{b}$ are expressed as follows:

$$S = \begin{bmatrix} 2 & 0 & 1 & 0 & 3 & 0 \\ 0 & 2 & 0 & 1 & 0 & 3 \\ 0 & 3 & 0 & 2 & 0 & 1 \\ 1 & 0 & 3 & 0 & 2 & 0 \end{bmatrix} \text{ and } \tilde{b} = \begin{bmatrix} 8.6 \\ 7.6 \\ -9.6 \\ -8.4 \end{bmatrix}$$

The solution of the system is computed as follows:

$$\tilde{X} = \begin{bmatrix} x_1^L \\ x_2^L \\ x_3^L \\ -x_1^U \\ -x_2^U \\ -x_3^U \end{bmatrix} = \begin{bmatrix} 2.845 \\ -3.067 \\ -5.651 \\ -2.987 \\ 2.853 \\ 5.573 \end{bmatrix}$$

This system has a strong interval solution as follows:

$$\begin{cases} \tilde{x}_1^L = \min\{x_1^L, x_1^U\} = 2.845 \\ \tilde{x}_1^U = \max\{x_1^L, x_1^U\} = 2.987 \end{cases} \Rightarrow \tilde{x}_1 = [2.845,2.987]$$

$$\begin{cases} \tilde{x}_2^L = \min\{x_2^L, x_2^U\} = -3.067 \\ \tilde{x}_2^U = \max\{x_2^L, x_2^U\} = -2.853 \end{cases} \Rightarrow \tilde{x}_2 = [-3.067, -2.853]$$

$$\begin{cases} \tilde{x}_3^L = \min\{x_3^L, x_3^U\} = -5.651 \\ \tilde{x}_3^U = \max\{x_3^L, x_3^U\} = -5.573 \end{cases} \Rightarrow \tilde{x}_3 = [-5.651, -5.573]$$

Obviously, since $\tilde{x}_i^L = x_i^L$ and $\tilde{x}_i^U = x_i^U$ $(1 \leq i \leq 3\})$, this solution is a strong solution.

4 Conclusions

In this study, we developed a systematic approach for solving a fuzzy linear system of equations consisting of m linear equations with n unknown variables and fuzzy right-hand sides. Our proposed approach employed the nearest weighted approximation method of fuzzy numbers to convert the considered fuzzy system into an interval linear system. Then, the obtained interval system was solved using the least square method. Finally, two numerical examples illustrated the capabilities of the proposed approach. In this paper, the uncertainties were considered as triangular fuzzy numbers. For future study, one can use our method for other types of fuzzy numbers such as trapezoidal, intuitionistic, type 2 of fuzzy numbers, etc.

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