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A Strong Convergence for The General Equilibrium System

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Abstract

We suggest a new viscosity iterative scheme for finding a common element of the set of solutions of the general equilibrium problem system (GEPS) and the set of all common fixed points of two noncommutative nonexpansive self mappings in the framework of a real Hilbert space. The results in the paper generalize and improve some well-known results in Jankaew et al.[2] and many others. Moreover, we present a numerical example (by using Maple software) to grantee the main result of this paper.

1 Introduction

Yao and Chen [?] introduced a new iteration for two average self mappings S and T on a closed convex subset C as follows

$$\begin{cases} x_0 = x \in C; \\ x_{n+1} = \alpha_n x + (1-\alpha_n)\frac{2}{(n+1)(n+2)}\sum_{i=0}^{n}\sum_{j=0}^{n-i}((ST)^j S^{i-j} \vee (ST)^i T^{j-i})x_n, \forall n \geq 0. \end{cases}$$

where

$$(ST)^j S^{i-j} \vee (ST)^i T^{j-i} = \begin{cases} (ST)^j S^{i-j} & \text{if } i \geq j \\ (ST)^i T^{j-i} & \text{if } i < j. \end{cases} \quad (1.1)$$

By improving this idea, Jankaew et al. [2] considered the following iteration:

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n \frac{2}{(n+1)(n+2)}\sum_{i=0}^{n}\sum_{j=0}^{n-i}((ST)^j S^{i-j} \vee (ST)^i T^{j-i})x_n, \quad (1.2)$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0,1)$, $\alpha_n + \beta_n + \gamma_n = 1$, f is a contraction mapping on C. They proved that the iteration

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process (1.2) converges strongly to common fixed point of the mapping S and T which solves some variational inequality.

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In this paper, we consider and analyzed an iterative scheme for finding a common element of the set of solutions of the general equilibrium problem system (GEPS) and the set of all common fixed points of two noncommutative nonexpansive self mapping in the framework of a real Hilbert space. The results in the paper generalize and improve some well-known results in Jankaew et al.[2] and many others.

We consider H be a real Hilbert space, I an identity mapping on H and C a nonempty closed convex subset of H. The strong (weak) convergence of $\{x_n\}$ to x is written by $x_n \to x(x_n \rightharpoonup x)$ as $n \to \infty$. Moreover, H satisfies the Opial's condition [4], if for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty}\|x_n - x\| < \liminf_{n\to\infty}\|x_n - y\|,$$

holds for every $y \in H$ with $x \neq y$.

A mapping T of H into itself is said to be nonexpansive if for all $x, y \in D(T)$,

$$\|Tx - Ty\| \leq \|x - y\|.$$

$F(T)$ denotes the set of fixed points of T.

Recall that f said to be weakly contractive **[?]** iff for all $x, y \in D(T)$,

$$\|f(x) - f(y)\| \leq \|x - y\| - \phi(\|x - y\|),$$

Recall that a self mapping $f : H \to H$ is a contraction if there exists $\rho \in (0,1)$ such that $\|f(x) - f(y)\| \leq \rho\|x - y\|$ for each $x, y \in H$.

The mapping A is a strongly positive linear bounded operator on H if there exists a constant $\bar{\gamma} > 0$ such that

$$\langle Ax, x\rangle \geq \bar{\gamma}\|x\|^2, \text{ for all } x \in H.$$

Moreover, $B : C \to H$ is called $\alpha-$inverse strongly monotone if there exists a positive real number $\alpha > 0$ such that for all $x, y \in C$

$$\langle Bx - By, x - y\rangle \geq \alpha\|Bx - By\|^2.$$

2 GEP system

A typical problem is to minimize a quadratic function over the set of the fixed points of a nonexpansive mappings on a real Hilbert space H:

$$\min \ \frac{1}{2}\langle Ax, x\rangle - h(x)$$

where A is strongly positive linear bounded operator and h is a potential function for γf, i. e., $h'(x) = \gamma f$ for all $x \in H$.

Let $A : H \to H$ be an inverse strongly monoton mapping and $F : C \times C \to \mathbb{R}$ be a bifunction. Then we consider the following GEP:

Find $\tilde{x} \in C$ such that

$$F(\tilde{x}, y) + \langle Ax, y - x\rangle \geq 0, forally \in C.$$

The set of such $x \in C$ is denoted by $GEP(F; A)$, i.e.,

$$GEP(F, A) = \{x \in C : F(x, y) + \langle Ax, y - x\rangle \geq 0, \forall y \in\}$$

To study the generalized equilibrium problem, we assume that the bifunction F satisfies the following conditions:

(A1) $F(x, x) = 0$, for all $x \in C$;

(A2) F is monotone, i.e., $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;

(A3) for each $x, y, z \in C$, $\limsup_{t\to 0^-} F(tz + (1 - t)x, y) \leq F(x, y)$;

(A4) for each $x \in C$ $y \mapsto F(x, y)$ is convex and weakly lower semi-continuous.

Lemma 2.1. *([1]) Let C be a nonempty closed convex subset of H and $F : C \times C \to \mathbb{R}$ be a bifunction satisfying $(A1) - (A4)$. Then, for any $r > 0$ and $x \in H$ there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C.$$

Further, define

$$T_r x = \{z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C\}$$

for all $r > 0$ and $x \in H$. Then

(a) T_r is single-valued;

(b) T_r is firmly nonexpansive, i.e., for any $x, y \in H$

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle;$$

(c) $F(T_r) = GEP(F)$;

(d) $\|T_s x - T_r x\| \leq \frac{s-r}{s}\|T_s x - x\|$;

(e) $GEP(F)$ is closed and convex.

Remark 2.1. *It is clear that for any $x \in H$ and $r > 0$, by Lemma 2.1(a), there exists $z \in H$ such that*

$$F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in H. \tag{2.1}$$

Replacing x with $x - r\psi x$ in (2.1), we obtain

$$F(z, y) + \langle \psi x, y - z\rangle + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in H.$$

Lemma 2.2. *([7]) Assume $\{a_n\}$ is a sequence of nonnegative numbers such that*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \delta_n,$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence in real number such that *(i)* $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty,$

(ii) $\limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty,$

Then $\lim_{n\to\infty} a_n = 0.$

Lemma 2.3. *([2]) Let C be a nonempty bounded closed convex subset of a Hilbert space H, and let S, T be two nonexpansive mappings of C into itself such that $F(ST) = F(S) \bigcap F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence defined as follows:*

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n \frac{2}{(n+1)(n+2)} \sum_{i=0}^{n} \sum_{j=0}^{n-i} ((ST)^j S^{i-j} \vee (ST)^i T^{j-i}) x_n,$$

and put

$$\Lambda_n = \frac{2}{(n+1)(n+2)} \sum_{i=0}^{n} \sum_{j=0}^{n-i} ((ST)^j S^{i-j} \vee (ST)^i T^{j-i}) x_n.$$

Then,

$$\limsup_{n\to\infty} \sup_{x\in C} \|\Lambda_n(x) - ST\Lambda_n(x)\| = 0.$$

3 GEPS for two noncommutative nonexpansive

In this section, we introduce an explicit viscosity iterative algorithm for finding a common element of the set of solution for an equilibrium problem system involving a bifunction defined on a closed convex subset and the set of fixed points for two noncommutative nonexpansive mappings.

Theorem 3.1. *Let $x_0 \in C$, $\{u_{n,i}\} \subset C$ and C be a nonempty closed convex subset of a real Hilbert space H, $F_1, F_2, \ldots, F_k$ be bifunctions from $C \times C$ to $\mathbb{R}$ satisfying $(A1) - (A4)$, $\Psi_1, \Psi_2, \ldots, \Psi_k$ be μ_i-inverse strongly monotone mapping on C, f be a weakly contractive mapping with a function ϕ on H, A be a strongly positive linear bounded operator with coefficient $\bar{\gamma}$ such that $\bar{\gamma} \leq \|A\| \leq 1$, B be strongly positive linear bounded operator on H with coefficient $\bar{\beta} \in (0,1]$ such that $\|B\| = \bar{\beta}$, S, T be nonexpansive mappings on C, such that $F(ST) = F(TS) = F(T) \bigcap F(S) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated in the following manner:*

$$\begin{cases} F_1(u_{n,1}, y) + \langle \Psi_1 x_n, y - u_{n,1} \rangle + \frac{1}{r_n} \langle y - u_{n,1}, u_{n,1} - x_n \rangle \geq 0, \text{ for all } y \in C \\ F_2(u_{n,2}, y) + \langle \Psi_2 x_n, y - u_{n,2} \rangle + \frac{1}{r_n} \langle y - u_{n,2}, u_{n,2} - x_n \rangle \geq 0, \text{ for all } y \in C \\ \vdots \\ F_k(u_{n,k}, y) + \langle \Psi_k x_n, y - u_{n,k} \rangle + \frac{1}{r_n} \langle y - u_{n,k}, u_{n,k} - x_n \rangle \geq 0, \text{ for all } y \in C \\ \\ \omega_n = \frac{1}{k} \sum_{i=1}^{k} u_{n,i}, \\ \\ \Lambda_n = \frac{2}{(n+1)(n+2)} \sum_{i=0}^{n} \sum_{j=0}^{n-i} ((ST)^j S^{i-j} \vee (ST)^i T^{j-i}) \omega_n, \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n B x_n + ((1-\varepsilon_n)I - \beta_n B - \alpha_n A)\Lambda_n. \end{cases}$$

where $\{\alpha_n\} \subset (0,1)$, $\{\beta_n\}$ and $\{\varepsilon_n\}$ are the sequences in $[0,1)$ such that $\varepsilon_n \leq \alpha_n$ and $\{r_n\} \subset (0,\infty)$ is a real sequence satisfying the following conditions:

(C1) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$,

(C2) $\lim_{n\to\infty} \beta_n = 0,\ \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$,

(C3) $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$ *and* $\liminf_{n\to\infty} r_n > 0$ *and* $0 < b < r_n < a < 2\mu_i$ *for* $1 \leq i \leq k$,

(C4) $\sum_{n=1}^{\infty} |\varepsilon_{n+1} - \varepsilon_n| < \infty$,

(C5) $\lim_{n\to\infty} \frac{\varepsilon_n}{\alpha_n} = 0.$

Then

(i) the sequence $\{x_n\}$ is bounded.

(ii) $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0.$

(iii) $\lim_{n\to\infty} \|\Psi_i x_n - \Psi_i x^*\| = 0.$ *for* $i \in \{1, 2, \ldots, k\}$.

(iv) $\lim_{n\to\infty} \|x_n - \Lambda_n\| = 0.$

Proof. **(i)** Without loss of generality, we assume that $\alpha_n < (1 - \varepsilon_n - \beta_n\|B\|)\|A\|^{-1}$. Since A, B are two strongly positive bounded linear operator on H, we have

$$\|A\| = \sup\{|\langle Ax, x\rangle| : x \in H, \|x\| = 1\},$$

$$\|B\| = \sup\{|\langle Bx, x\rangle| : x \in H, \|x\| = 1\}.$$

Also, $(1 - \varepsilon_n)I - \beta_n B - \alpha_n A$ is positive. Indeed,

$$\begin{aligned} \langle((1 - \varepsilon_n)I - \beta_n B - \alpha_n A)x, x\rangle &= (1 - \varepsilon_n)\langle x, x\rangle - \beta_n\langle Bx, x\rangle - \alpha_n\langle Ax, x\rangle \\ &\geq 1 - \varepsilon_n - \beta_n\|B\| - \alpha_n\|A\| > 0. \end{aligned}$$

Notic that

$$\begin{aligned} \|(1 - \varepsilon_n)I - \beta_n B - \alpha_n A\| &= \sup\{\langle((1 - \varepsilon_n)I - \beta_n B - \alpha_n A)x, x\rangle : x \in H, \|x\| = 1\} \\ &= \sup\{(1 - \varepsilon_n)\langle x, x\rangle - \beta_n\langle Bx, x\rangle - \alpha_n\langle Ax, x\rangle : x \in H, \|x\| = 1\} \\ &\leq 1 - \varepsilon_n - \beta_n\bar{\beta} - \alpha_n\bar{\gamma} \\ &\leq 1 - \beta_n\bar{\beta} - \alpha_n\bar{\gamma}. \end{aligned}$$

Let $Q = P_{F(ST)\bigcap GEP(F_i,\Psi_i)}$. It is clear that $Q(I - A + \gamma f)$ is a contraction. Hence, there exists a unique element $z \in H$ such that $z = Q(I - A + \gamma f)z$.

Let $x^* \in \bigcap_{i=1}^k F(ST) \bigcap GEP(F_i, \Psi_i)$. For any $i = 1, 2, \ldots, k$, $I - r_n\Psi_i$ is a nonexpansive mapping and $\|u_{n,i} - x^*\| \leq \|x_n - x^*\|$. Also $\|\omega_n - x^*\| \leq \|x_n - x^*\|$. Thus

$$\begin{aligned}
\|x_{n+1} - x^*\| &= \|\alpha_n \gamma f(x_n) + \beta_n B x_n + ((1-\varepsilon_n)I - \beta_n B - \alpha_n A)\Lambda_n - x^*\| \\
&\leq \alpha_n \|\gamma f(x_n) - Ax^*\| + \beta_n \|B\| \|x_n - x^*\| \\
&\quad + \|((1-\varepsilon_n)I - \beta_n B - \alpha_n A)\| \|\Lambda_n - x^*\| + \varepsilon_n \|x^*\| \\
&\leq \alpha_n \gamma \|f(x_n) - f(x^*)\| + \alpha_n \|\gamma f(x^*) - Ax^*\| + \beta_n \bar{\beta} \|x_n - x^*\| \\
&\quad + (1 - \beta_n \bar{\beta} - \alpha_n \bar{\gamma}) \|x_n - x^*\| + \alpha_n \|x^*\| \\
&\leq \alpha_n \gamma \|x_n - x^*\| - \phi(\|x_n - x^*) + \alpha_n \|\gamma f(x^*) - Ax^*\| + \beta_n \bar{\beta} \|x_n - x^*\| \\
&\quad + (1 - \beta_n \bar{\beta} - \alpha_n \bar{\gamma}) \|x_n - x^*\| + \alpha_n \|x^*\| \\
&\leq (1 - (\bar{\gamma} - \gamma)\alpha_n) \|x_n - x^*\| + \alpha_n (\|\gamma f(x^*) - Ax^*\| + \|x^*\|) \\
&\leq \max\{\|x_n - x^*\|, \frac{\|\gamma f(x^*) - Ax^*\|}{\bar{\gamma} - \gamma}\}.
\end{aligned}$$

By induction

$$\|x_n - x^*\| \leq \max\{\|x_1 - x^*\|, \frac{\|\gamma f(x^*) - Ax^*\|}{\bar{\gamma} - \gamma}\}.$$

and the sequence $\{x_n\}$ is bounded and also $\{f(x_n)\}, \{\omega_n\}$ and $\{\Lambda_n\}$ are bounded.

(ii) Note that $u_{n,i}$ can be written as $u_{n,i} = T_{r_n,i}(x_n - r_n \psi_i x_n)$. It follows from Lemma 2.1 that

$$\|u_{n+1,i} - u_{n,i}\| \leq \|x_{n+1} - x_n\| + 2M_i |r_{n+1} - r_n|, \tag{3.1}$$

where
$$M_i = \max\{\sup\{\frac{\|T_{r_{n+1},i}(I - r_n\Psi_i)x_n - T_{r_n,i}(I - r_n\Psi_i)x_n\|}{r_{n+1}}, \sup\{\|\Psi_i x_n\|\}\}\}.$$

Let $M = \frac{1}{k}\sum_{i=1}^{k} 2M_i < \infty$. Next, we estimate $\|\omega_{n+1} - \omega_n\|$,

$$\|\omega_{n+1} - \omega_n\| \leq \frac{1}{k}\sum_{i=1}^{k} \|u_{n+1,i} - u_{n,i}\| \leq \|x_{n+1} - x_n\| + M|r_{n+1} - r_n|. \tag{3.2}$$

Now, we prove that $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$. We observe that

$$\begin{aligned}
\|x_{n+2}-x_{n+1}\| &= \|\alpha_{n+1}\gamma f(x_{n+1})+\beta_{n+1}Bx_{n+1}+((1-\varepsilon_{n+1})I-\beta_{n+1}B-\alpha_{n+1}A)\Lambda_{n+1}\\
&\quad -\alpha_{n+1}\gamma f(x_n)-\beta_n Bx_n-((1-\varepsilon_n)I-\beta_n B-\alpha_n A)\Lambda_n\|\\
&= \|((1-\varepsilon_{n+1})I-\beta_{n+1}B-\alpha_{n+1}A)(\Lambda_{n+1}-\Lambda_n)\\
&\quad +\{(\varepsilon_n-\varepsilon_{n+1})\Lambda_n+(\beta_n-\beta_{n+1})B\Lambda_n+(\alpha_n-\alpha_{n+1})A\Lambda_n\}\\
&\quad +\alpha_{n+1}\gamma(f(x_{n+1})-f(x_n))+(\alpha_{n+1}-\alpha_n)\gamma f(x_n)+\beta_{n+1}B(x_{n+1}-x_n)\\
&\quad +(\beta_{n+1}-\beta_n)Bx_n\|\\
&\leq \|(1-\varepsilon_{n+1})I-\beta_{n+1}B-\alpha_{n+1}A\|\|\Lambda_{n+1}-\Lambda_n\|+\|\varepsilon_n-\varepsilon_{n+1}\|\|\Lambda_n\|\\
&\quad +|\beta_n-\beta_{n+1}|\|B\|\Lambda_n\|+|\alpha_n-\alpha_{n+1}|\|A\Lambda_n\|+\alpha_{n+1}\gamma\|f(x_{n+1})-f(x_n)\|\\
&\quad +|\alpha_{n+1}-\alpha_n|\gamma\|f(x_n)\|+\beta_{n+1}\|B\|\|x_{n+1}-x_n\|+|\beta_{n+1}-\beta_n|\|B\|\|x_n\|\\
&\leq (1-\beta_{n+1}\bar{\beta}-\alpha_{n+1}\bar{\gamma})\|\Lambda_{n+1}-\Lambda_n\|+|\varepsilon_n-\varepsilon_{n+1}|\|\Lambda_n\|+|\beta_n-\beta_{n+1}|\bar{\beta}\|\Lambda_n\|\\
&\quad +|\alpha_n-\alpha_{n+1}|\|A\Lambda_n\|+\alpha_{n+1}\gamma\|x_{n+1}-x_n\|-\alpha_{n+1}\gamma\phi(\|x_{n+1}-x_n\|)\\
&\quad +|\alpha_{n+1}-\alpha_n|\gamma\|f(x_n)\|+\beta_{n+1}\bar{\beta}\|x_{n+1}-x_n\|+|\beta_{n+1}-\beta_n|\bar{\beta}\|x_n\|\\
&\leq (1-\beta_{n+1}\bar{\beta}-\alpha_{n+1}\bar{\gamma})\|\Lambda_{n+1}-\Lambda_n\|+|\varepsilon_n-\varepsilon_{n+1}|\|\Lambda_n\|+|\beta_n-\beta_{n+1}|\bar{\beta}K\\
&\quad +K|\alpha_n-\alpha_{n+1}|+(\alpha_{n+1}\gamma+\beta_{n+1}\bar{\beta})\|x_{n+1}-x_n\|-\alpha_{n+1}\gamma\phi(\|x_{n+1}-x_n\|)
\end{aligned}$$

where $K=\sup\{max\{\|\Lambda_n\|+\|x_n\|,\gamma\|f(x_n)\|+\|A\Lambda_n\|\},\forall n\geq 0\}<\infty$.
Let $\Delta_n=\|\beta_n-\beta_{n+1}\|\bar{\beta}K+K|\alpha_n-\alpha_{n+1}|+|\varepsilon_n-\varepsilon_{n+1}|\|\Lambda_n\|$, then

$$\tag{3.3}$$

$$\begin{aligned}
\|x_{n+2}-x_{n+1}\| &\leq (1-\beta_{n+1}\bar{\beta}-\alpha_{n+1}\bar{\gamma})\|\Lambda_{n+1}-\Lambda_n\|+(\alpha_{n+1}\gamma+\beta_{n+1}\bar{\beta})\|x_{n+1}-x_n\|\\
&\quad -\alpha_{n+1}\gamma\phi(\|x_{n+1}-x_n\|)+\Delta_n,
\end{aligned}$$

From [2], we conclude

$$\|\Lambda_{n+1}-\Lambda_n\|\leq\|\omega_{n+1}-\omega_n\|+\frac{4}{n+3}\|\omega_{n+1}-x^*\|+\frac{4}{n+3}\|x^*\|. \tag{3.4}$$

Substituing (3.2) and (3.4) into (3.3), we arrive at

$$\begin{aligned}
\|x_{n+2}-x_{n+1}\| &\leq (1-\beta_{n+1}\bar{\beta}-\alpha_{n+1}\bar{\gamma})\{\|x_{n+1}-x_n\|+M|r_{n+1}-r_n|+\frac{4}{n+3}\|\omega_{n+1}-x^*\|\\
&\quad +\frac{4}{n+3}\|x^*\|\}+(\alpha_{n+1}\gamma+\beta_{n+1}\bar{\beta})\|x_{n+1}-x_n\|-\alpha_{n+1}\gamma\phi(\|x_{n+1}-x_n\|)\\
&\quad +\Delta_n,
\end{aligned}$$

for some positive constant M. It follows that

$$\begin{aligned}\|x_{n+2}-x_{n+1}\| &\leq (1-\alpha_{n+1}(\bar{\gamma}-\gamma\rho))\|x_{n+1}-x_n\|+M(1-\beta_{n+1}\bar{\beta}-\alpha_{n+1}\bar{\gamma})|r_{n+1}-r_n| \\ &\quad +(1-\beta_{n+1}\bar{\beta}-\alpha_{n+1}\bar{\gamma})\frac{4}{n+3}\|\omega_{n+1}-x^*\|+ \\ &\quad (1-\beta_{n+1}\bar{\beta}-\alpha_{n+1}\bar{\gamma})\frac{4}{n+3}\|x^*\|+\Delta_n.\end{aligned}$$

By Lemma 2.2,

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{3.5}$$

(iii) For any $i\in\{1,2,\ldots,k\}$,

$$\begin{aligned}\|u_{n,i}-x^*\|^2 &\leq \|(x_n-x^*)-r_n(\Psi_i x_n-\Psi_i x^*)\|^2 \\ &= \|x_n-x^*\|^2-2r_n\langle x_n-x^*,\Psi_i x_n-\Psi_i x^*\rangle+r_n^2\|\Psi_i x_n-\Psi_i x^*\|^2 \\ &\leq \|x_n-x^*\|^2-r_n(2\mu_i-r_n)\|\Psi_i x_n-\Psi_i x^*\|^2,\end{aligned}$$

from which it follows that

$$\begin{aligned}\|\omega_n-x^*\|^2 = &\ \|\sum_{i=1}^{k}\frac{1}{k}(u_{n,i}-x^*)\|^2\leq\frac{1}{k}\sum_{i=1}^{k}\|u_{n,i}-x^*\|^2 \\ \leq &\ \|x_n-x^*\|^2-\tfrac{1}{k}\sum_{i=1}^{k}r_n(2\mu_i-r_n)\|\Psi_i x_n-\Psi_i x^*)\|^2.\end{aligned} \tag{3.6}$$

From (3.6), we see that

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &= \|\alpha_n(\gamma f(x_n)-Ax^*)+\beta_n B(x_n-x^*) \\ &\quad +((1-\varepsilon_n)I-\beta_n B-\alpha_n A)(\Lambda_n-x^*)-\varepsilon_n x^*\|^2 \\ &\leq \|\alpha_n(\gamma f(x_n)-Ax^*)+\beta_n B(x_n-x^*) \\ &\quad +((1-\varepsilon_n)I-\beta_n B-\alpha_n A)(\omega_n-x^*)+\varepsilon_n x^*\|^2 \\ &\leq \alpha_n\|\gamma f(x_n)-Ax^*\|^2+\beta_n\|B\|^2\|x_n-x^*\|^2 \\ &\quad +(1-\beta_n\bar{\beta}-\alpha_n\bar{\gamma})\|\Lambda_n-x^*\|^2+\varepsilon_n^2\|x^*\|^2 \\ &\leq \alpha_n\|\gamma f(x_n)-Ax^*\|^2+\beta_n\bar{\beta}^2\|x_n-x^*\|^2+(1-\beta_n\bar{\beta}-\alpha_n\bar{\gamma})\|\omega_n-x^*\|^2 \\ &\quad +\varepsilon_n^2\|x^*\|^2 \\ &\leq \alpha_n\|\gamma f(x_n)-Ax^*\|^2+\beta_n\bar{\beta}\|x_n-x^*\|^2+(1-\beta_n\bar{\beta}-\alpha_n\bar{\gamma})\{\|x_n-x^*\|^2 \\ &\quad -\frac{1}{k}\sum_{i=1}^{k}r_n(2\mu_i-r_n)\|\Psi_i x_n-\Psi_i x^*)\|^2\}+\varepsilon_n^2\|x^*\|^2 \\ &\leq \alpha_n\|\gamma f(x_n)-Ax^*\|^2+\|x_n-x^*\|^2 \\ &\quad -(1-\beta_n\bar{\beta}-\alpha_n\bar{\gamma})\frac{1}{k}\sum_{i=1}^{k}r_n(2\mu_i-r_n)\|\Psi_i x_n-\Psi_i x^*)\|^2++\varepsilon_n^2\|x^*\|^2\end{aligned}$$

and hence

$$
\begin{aligned}
(1-\beta_n\bar{\beta}-\alpha_n\bar{\gamma})\tfrac{1}{k} & \sum_{i=1}^{k} b(2\mu_i-a)\|\Psi_i x_n-\Psi_i x^*\|^2 \\
\leq & \ \alpha_n\|\gamma f(x_n)-Ax^*\|^2+\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2+\varepsilon_n^2\|x^*\|^2 \\
\leq & \ \alpha_n\|\gamma f(x_n)-Ax^*\|^2+\|x_{n+1}-x_n\|(\|x_{n+1}-x^*\|-\|x_n-x^*\|) \\
+ & \ \varepsilon_n^2\|x^*\|^2.
\end{aligned}
$$

Since $\alpha_n\to 0$ and $\varepsilon_n\leq\alpha_n$ then $\varepsilon_n\to 0$ as $n\to\infty$. The inequality (3.5) implies that

$$
\lim_{n\to\infty}\|\Psi_i x_n-\Psi_i x^*\|=0, \forall i=1,2,\ldots,k. \tag{3.7}
$$

(iv) It follows from Lemma 2.1 that

$$
\|u_{n,i}-x^*\|^2\leq\|x_n-x^*\|^2-\|x_n-u_{n,i}\|^2+2r_n\|x_n-u_{n,i}\|\|\Psi_i x_n-\Psi_i x^*\| \tag{3.8}
$$

and hence

$$
\begin{aligned}
\|\omega_n-x^*\|^2= & \ \|\textstyle\sum_{i=1}^{k}\frac{1}{k}(u_{n,i}-x^*)\|^2 \\
\leq & \ \tfrac{1}{k}\textstyle\sum_{i=1}^{k}\|u_{n,i}-x^*\|^2 \\
\leq & \ \|x_n-x^*\|^2-\tfrac{1}{k}\textstyle\sum_{i=1}^{k}\|u_{n,i}-x_n\|^2+\tfrac{1}{k}\sum_{i=1}^{k}2r_n\|x_n-u_{n,i}\|\|\Psi_i x_n-\Psi_i x^*\|.
\end{aligned} \tag{3.9}
$$

From (3.9), we see that

$$
\begin{aligned}
\|x_{n+1}-x^*\|^2 \leq & \ \alpha_n\|\gamma f(x_n)-Ax^*\|^2+\beta_n\bar{\beta}\|x_n-x^*\|^2+(1-\beta_n\bar{\beta}-\alpha_n\bar{\gamma})\|\omega_n-x^*\|^2 \\
& +\varepsilon_n^2\|x^*\|^2 \\
\leq & \ \alpha_n\|\gamma f(x_n)-Ax^*\|^2+\beta_n\bar{\beta}\|x_n-x^*\|^2+(1-\beta_n\bar{\beta}-\alpha_n\bar{\gamma})\{\|x_n-x^*\|^2 \\
& -\frac{1}{k}\sum_{i=1}^{k}\|u_{n,i}-x_n\|^2+\frac{1}{k}\sum_{i=1}^{k}2r_n\|x_n-u_{n,i}\|\|\Psi_i x_n-\Psi_i x^*\|\}+\varepsilon_n^2\|x^*\|^2
\end{aligned}
$$

from which it follows that

$$
(1-\beta_n\bar{\beta}-\alpha_n\bar{\gamma})\frac{1}{k}\sum_{i=1}^{k}\|u_{n,i}-x_n\|^2
$$

$$
\begin{aligned}
\leq & \ \alpha_n\|\gamma f(x_n)-Ax^*\|^2+\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2 \\
& +(1-\beta_n\bar{\beta}-\alpha_n\bar{\gamma})\frac{1}{k}\sum_{i=1}^{k}2r_n\|x_n-u_{n,i}\|\|\Psi_i x_n-\Psi_i x^*\|+\varepsilon_n^2\|x^*\|^2 \\
\leq & \ \alpha_n\|\gamma f(x_n)-Ax^*\|^2+\|x_{n+1}-x_n\|(\|x_{n+1}-x^*\|-\|x_n-x^*\|) \\
& +(1-\beta_n\bar{\beta}-\alpha_n\bar{\gamma})\frac{1}{k}\sum_{i=1}^{k}2r_n\|x_n-u_{n,i}\|\|\Psi_i x_n-\Psi_i x^*\|+\varepsilon_n^2\|x^*\|^2
\end{aligned}
$$

From the condition $(C1)$, (3.5) and (3.7), we get that

$$\lim_{n\to\infty} \|u_{n,i} - x_n\| = 0. \tag{3.10}$$

It is easy to prove

$$\lim_{n\to\infty} \|\omega_n - x_n\| = 0. \tag{3.11}$$

By definition of the sequence $\{x_n\}$, we obtain

$$\begin{aligned} \|x_n - \Lambda_n\| &\leq \|x_{n+1} - x_n\| + \|x_{n+1} - \Lambda_n\| \\ &\leq \|x_{n+1} - x_n\| + \|\alpha_n \gamma f(x_n) + \beta_n B x_n + ((1-\varepsilon_n)I - \beta_n B - \alpha_n A)\Lambda_n - \Lambda_n\| \\ &\leq \|x_{n+1} - x_n\| + \alpha_n \|\gamma f(x_n) - A\Lambda_n\| + \beta_n \bar{\beta} \|x_n - \Lambda_n\| + \varepsilon_n \|\Lambda_n\|. \end{aligned}$$

Then

$$\|x_n - \Lambda_n\| \leq \frac{1}{1-\beta_n\bar{\beta}} \|x_{n+1} - x_n\| + \frac{\alpha_n}{1-\beta_n\bar{\beta}_n} \|\gamma f(x_n) - A\Lambda_n\| + \frac{\varepsilon_n}{1-\beta_n\bar{\beta}} \|\Lambda_n\|.$$

Thanks to the conditions$(C1) - (C2)$ and (3.5), we conclude that

$$\lim_{n\to\infty} \|x_n - \Lambda_n\| = 0. \tag{3.12}$$

Also

$$\|\omega_n - \Lambda_n\| \leq \|\omega_n - x_n\| + \|x_n - \Lambda_n\|,$$

hence

$$\lim_{n\to\infty} \|\omega_n - \Lambda_n\| = 0. \tag{3.13}$$

□

Theorem 3.2. *Suppose all assumptions of Theorem 3.1 are holds. Then the sequence $\{x_n\}$ converges strongly to a point $\bar{x}$, where $\bar{x} \in \bigcap_{i=1}^k F(ST) \bigcap GEP(F_i, \Psi_i)$ solves the variational inequality*

$$\langle (A - \gamma f)\bar{x}, \bar{x} - x \rangle \leq 0.$$

Equivalently, $\bar{x} = P_{\bigcap_{i=1}^k F(ST) \bigcap GEP(F_i, \Psi_i)}(I - A + \gamma f)(\bar{x})$.

Proof. Observe that $P_{\bigcap_{i=1}^k F(ST) \bigcap GEP(F_i, \Psi_i)}(I - A + \gamma f)$ is a contraction of H into itself. Since H is complete, there exists a unique element $\bar{x} \in H$ such that

$$\bar{x} = P_{\bigcap_{i=1}^k F(ST) \bigcap GEP(F_i, \Psi_i)}(I - A + \gamma f)(\bar{x}).$$

Next, we prove

$$\limsup_{n\to\infty} \langle (A - \gamma f)\bar{x}, \bar{x} - \Lambda_n \rangle \leq 0$$

Let $\tilde{x} = P_{\bigcap_{i=1}^{k} F(ST)\bigcap GEP(F_i,\Psi_i)} x_1$, set

$$E = \{\bar{y} \in H : \|\bar{y} - \tilde{x}\| \leq \|x_1 - \tilde{x}\| + \frac{\|\gamma f(\tilde{x}) - A\tilde{x}\|}{\bar{\gamma} - \gamma\rho}\} \bigcap C.$$

It is clear, E is nonempty closed bounded convex subset of C and $S(E) \subset E, T(E) \subset E$. Without loss of generality, we may assume S and T are mappings of E into itself. Since $\{\Lambda_n\} \subset E$ is bounded, there is a subsequence $\{\Lambda_{n_j}\}$ of $\{\Lambda_n\}$ such that

$$\limsup_{n\to\infty} \langle (A - \gamma f)\bar{x}, \bar{x} - \Lambda_n\rangle = \lim_{j\to\infty} \langle (A - \gamma f)\bar{x}, \bar{x} - \Lambda_{n_j}\rangle. \tag{3.14}$$

As $\{\Lambda_{n_j}\}$ is also bounded, there exists a subsequence $\{\Lambda_{n_{j_l}}\}$ of $\{\Lambda_{n_j}\}$ such that $\Lambda_{n_{j_l}} \rightharpoonup \xi$. Without loss of generality, let $\Lambda_{n_j} \rightharpoonup \xi$.

Now, we prove the following items:

(i) $\xi \in F(ST) = F(T)\bigcap F(S)$. Assume $\xi \neq F(ST)$. By Lemma 2.3 and Opial's condition, we have

$$\begin{aligned}
\liminf_{j\to\infty} \|\Lambda_{n_j} - \xi\| &< \liminf_{j\to\infty} \|\Lambda_{n_j} - ST(\xi)\| \\
&\leq \liminf_{j\to\infty} (\|\Lambda_{n_j} - ST(\Lambda_{n_j})\| + \|ST(\Lambda_{n_j}) - ST(\xi)\|) \\
&\leq \liminf_{j\to\infty} \|\Lambda_{n_j} - \xi\|.
\end{aligned}$$

That is a contradiction. Hence $\xi = F(ST)\xi$.

(ii) By the same argument as in the proof of Theorem 3.2 [5], we conclude that $\xi \in GEP(F_i, \Psi_i)$, for all $i = 1, 2, \ldots, k$. Then $\xi \in \bigcap_{i=1}^{k} GEP(F_i, \Psi_i)$. Now, in view of (3.14), we see

$$\limsup_{n\to\infty} \langle (A - \gamma f)\bar{x}, \bar{x} - \Lambda_n\rangle = \langle (A - \gamma f)\bar{x}, \bar{x} - \xi\rangle \leq 0$$

Finally, we show taht $\{x_n\}$ converges strongly to $\bar{x}$. As a matter of fact,

$$\begin{aligned}
\|x_{n+1}-\bar{x}\|^2 &= \|\alpha_n\gamma f(x_n)+\beta_n Bx_n+((1-\varepsilon_n)I-\beta_n B-\alpha_n A)\Lambda_n-\bar{x}\|^2\\
&= \|\alpha_n(\gamma f(x_n)-A\bar{x})+\beta_n B(x_n-\bar{x})+((1-\varepsilon_n)I-\beta_n B-\alpha_n A)(\Lambda_n-\bar{x})\\
&\quad -\varepsilon_n\bar{x}\|^2\\
&\leq \|\alpha_n(\gamma f(x_n)-A\bar{x})+\beta_n B(x_n-\bar{x})+((1-\varepsilon_n)I-\beta_n B-\alpha_n A)(\Lambda_n-\bar{x})\\
&\quad +\varepsilon_n\bar{x}\|^2\\
&= \|\alpha_n(\gamma f(x_n)-A\bar{x})+\beta_n B(x_n-\bar{x})+((1-\varepsilon_n)I-\beta_n B-\alpha_n A)(\Lambda_n-\bar{x})\|^2\\
&\quad +2\varepsilon_n\|\alpha_n(\gamma f(x_n)-A\bar{x})+\beta_n B(x_n-\bar{x})\\
&\quad +((1-\varepsilon_n)I-\beta_n B-\alpha_n A)(\Lambda_n-\bar{x})\|\|\bar{x}\|+\varepsilon_n^2\|\bar{x}\|^2\\
&= \|\beta_n B(x_n-\bar{x})+((1-\varepsilon_n)I-\beta_n B-\alpha_n A)(\Lambda_n-\bar{x})\|^2\\
&\quad +2\alpha_n\langle((1-\varepsilon_n)I-\beta_n B-\alpha_n A)(\Lambda_n-\bar{x}),\gamma f(x_n)-A\bar{x}\rangle\\
&\quad +2\varepsilon_n\|\alpha_n(\gamma f(x_n)-A\bar{x})+\beta_n B(x_n-\bar{x})\\
&\quad +((1-\varepsilon_n)I-\beta_n B-\alpha_n A)(\Lambda_n-\bar{x})\|\|\bar{x}\|\\
&\quad +\varepsilon_n^2\|\bar{x}\|^2+\alpha_n^2\|\gamma f(x_n)-A\bar{x}\|^2+2\alpha_n\beta_n\langle B(x_n-\bar{x}),\gamma f(x_n)-A\bar{x}\rangle\\
&\leq (\beta_n\|B\|\|x_n-\bar{x}\|+\|(1-\varepsilon_n)I-\beta_n B-\alpha_n A\|\|\Lambda_n-\bar{x}\|)^2\\
&\quad +2\alpha_n\gamma\langle\Lambda_n-\bar{x},f(x_n)-f(\bar{x})\rangle+2\varepsilon_n\|\alpha_n(\gamma f(x_n)-A\bar{x})+\beta_n B(x_n-\bar{x})\\
&\quad +((1-\varepsilon_n)I-\beta_n B-\alpha_n A)(\Lambda_n-\bar{x})\|\|\bar{x}\|+\varepsilon_n^2\|\bar{x}\|^2+\alpha_n^2\|\gamma f(x_n)-A\bar{x}\|^2\\
&\quad +2\alpha_n\beta_n\langle B(x_n-\bar{x}),\gamma f(x_n)-A\bar{x}\rangle+2\alpha_n\langle\Lambda_n-\bar{x},\gamma f(x_n)-A\bar{x}\rangle\\
&\quad -2\alpha_n\langle(\varepsilon_n I+\beta_n B+\alpha_n A)(\Lambda_n-\bar{x}),\gamma f(x_n)-A\bar{x}\rangle\\
&\leq (\beta_n\bar{\beta}\|x_n-\bar{x}\|+(1-\beta_n\bar{\beta}-\alpha_n\bar{\gamma})\|x_n-\bar{x}\|)^2+2\alpha_n\gamma\|x_n-\bar{x}\|^2\\
&\quad +2\varepsilon_n\|\alpha_n(\gamma f(x_n)-A\bar{x})+\beta_n B(x_n-\bar{x})\\
&\quad +((1-\varepsilon_n)I-\beta_n B-\alpha_n A)(\Lambda_n-\bar{x})\|\|\bar{x}\|+\varepsilon_n^2\|\bar{x}\|^2\\
&\quad +\alpha_n^2\|\gamma f(x_n)-A\bar{x}\|^2+2\alpha_n\beta_n\langle B(x_n-\bar{x}),\gamma f(x_n)-A\bar{x}\rangle\\
&\quad +2\alpha_n\langle\Lambda_n-\bar{x},\gamma f(x_n)-A\bar{x}\rangle-2\alpha_n\langle(\varepsilon_n I+\beta_n B+\alpha_n A)(\Lambda_n-\bar{x}),\gamma f(x_n)-A\bar{x}\rangle\\
&= (1-2(\bar{\gamma}-\gamma)\alpha_n)\|x_n-\bar{x}\|^2+\alpha_n^2\bar{\gamma}^2\|x_n-\bar{x}\|^2+2\varepsilon_n\|\alpha_n(\gamma f(x_n)-A\bar{x})\\
&\quad +\beta_n B(x_n-\bar{x})+((1-\varepsilon_n)I-\beta_n B-\alpha_n A)(\Lambda_n-\bar{x})\|\|\bar{x}\|+\varepsilon_n^2\|\bar{x}\|^2\\
&\quad +\alpha_n^2\|\gamma f(x_n)-A\bar{x}\|^2+2\alpha_n\beta_n\langle B(x_n-\bar{x}),\gamma f(x_n)-A\bar{x}\rangle\\
&\quad +2\alpha_n\langle\Lambda_n-\bar{x},\gamma f(x_n)-A\bar{x}\rangle-2\alpha_n\langle(\varepsilon_n I+\beta_n B+\alpha_n A)(\Lambda_n-\bar{x}),\gamma f(x_n)-A\bar{x}\rangle\\
&\leq (1-2(\bar{\gamma}-\gamma)\alpha_n)\|x_n-\bar{x}\|^2+\alpha_n\vartheta_n,
\end{aligned}$$

where

$$\begin{aligned}\zeta_n &= (\bar{\gamma}-\gamma\rho)\alpha_n,\\ \vartheta_n &= \alpha_n\bar{\gamma}^2\|x_n-\bar{x}\|^2+2\|\alpha_n(\gamma f(x_n)-A\bar{x})\\ &\quad+\beta_n B(x_n-\bar{x})+((1-\varepsilon_n)I-\beta_n B-\alpha_n A)(\Lambda_n-\bar{x})\|\|\bar{x}\|+\varepsilon_n\|\bar{x}\|^2\\ &\quad+\alpha_n\|\gamma f(x_n)-A\bar{x}\|^2+2\beta_n\langle B(x_n-\bar{x}),\gamma f(x_n)-A\bar{x}\rangle\\ &\quad+2\alpha_n\langle\Lambda_n-\bar{x},\gamma f(x_n)-A\bar{x}\rangle-2\alpha_n\langle(\varepsilon_n I+\beta_n B+\alpha_n A)(\Lambda_n-\bar{x}),\gamma f(x_n)-A\bar{x}\rangle.\end{aligned}$$

With respect to $(C1),(C2)$ and $(C5)$ we have

$$lim_{n\to\infty}\zeta_n=0,\sum_{n=1}^{\infty}\zeta_n=\infty$$

and

$$limsup\frac{\theta_n}{\zeta_n}\leq 0.$$

The lemma 2.2implies that the sequence $\{x_n\}$ strongly convergence to $\bar{x}$. □

Remark 3.1. *If we put $F_i=\Psi_i\equiv 0,\forall i\in\{1,2,\ldots,k\},A=B\equiv I,\gamma=\bar{\gamma}=1$, $\varepsilon_n=0$, and $\omega_n=x_n,\phi(t)=(1-\rho)t$ in theorem 3.1 and 3.2, we get [2], Theorem 3.1*

Remark 3.2. *Let $ST=TS$, $F_i=\Psi_i\equiv 0,\forall i\in\{1,2,\ldots,k\},A=B\equiv I,\gamma=\bar{\gamma}=1$, $f(y)=x,\forall y\in C$, $\varepsilon_n=\beta_n=0$ and $\omega_n=x_n,\phi(t)=(1-\rho)t$ in theorem 3.1 and 3.2, we get the desired [6], Theorem 1.*

Remark 3.3. *Taking S and T be average map of C into itself and $F_i=\Psi_i\equiv 0,\forall i\in\{1,2,\ldots,k\},A=B\equiv I,\gamma=\bar{\gamma}=1$, $\varepsilon_n=0$ and $\omega_n=x_n,\phi(t)=(1-\rho)t$ in theorem 3.1 and 3.2, we get [2], Corollary 3.4 easily.*

4 Numerical Example

In this section, we give some examples and numerical results for supporting our main theorem. All the numerical results have been produced in Matlab 2017 on a Linux workstation with a 3.8 GHZ Intel annex processor and 8 Gb of memory.

Example 4.1. Let $H=\mathbb{R}$, the set of all real numbers, with the inner product defined by $\langle x,y\rangle=xy,\ \forall x,y\in\mathbb{R}$, and induced usual norm $|\ .\ |$. Let $C=[0,2]$; let $F:C\times C\to\mathbb{R}$ be defined by $F(x,y)=(x-4)(y-x),\ \forall x,y\in C$; let $M,N:C\to H$ be defined by $M(x)=x$ and $N(x)=2x,\ \forall x\in C$, such that $\bar{\alpha}=\frac{1}{2}$ and $\bar{\beta}=\frac{1}{3}$ respectively, and let for each $x\in\mathbb{R}$, we define $f(x)=\frac{1}{8}x,\ A(x)=2x,\ B(x)=\frac{1}{3}x$ and

$$Tx=\begin{cases}\{x\}, & 0\leq x\leq 1\\ \{\frac{1}{2}\}, & 1<x\leq 2\end{cases}$$

Then there exist unique sequences $\{x_n\}\subset\mathbb{R}$ and $\{u_n\}\subset C$ generated by the iterative schemes

$$u_n=T^F_{r_n}(x_n-r_n(M+N)x_n); \tag{4.1}$$

$$x_{n+1} = (\frac{1}{8n} + \frac{1}{3n^2})x_n + ((1 - \frac{1}{2n^2-3})I - \frac{1}{n^2}B - \frac{1}{n}A)z_n \tag{4.2}$$

where $\alpha_n = \frac{1}{n}$, $\beta_n = \frac{1}{n^2}$, $\epsilon_n = \frac{1}{2n^2-3}$ and $r_n = 1$. Then $\{x_n\}$ converges to $\{1\} \in \text{Fix}(T) \cap \text{GEPP}$. *It is easy to prove that the bifunction F satisfy the Assumption* **??**. *Further, f is contraction mapping with constant $\alpha = \frac{1}{5}$ and A is a strongly positive bounded linear operator with constant $\bar{\gamma}_1 = 1$ on $\mathbb{R}$. Therefore, we can choose $\gamma = 1$ which satisfies $0 < \gamma < \frac{\bar{\gamma}_1}{\alpha} < \gamma + \frac{1}{\alpha}$. Furthermore, it is easy to observe that Fix$(T) = [0,1]$ and GEPP $= \{1\}$. Hence Fix$(T) \cap$ GEPP $= \{1\} \neq \emptyset$. After simplification, schemes (4.3) and (4.4) reduce to*

$$u_n = 2 - x_n$$

$$Tu_n = \begin{cases} \{2 - x_n\}, & 0 \le u_n \le 1 \quad or(1 \le x_n \le 2) \\ \{\frac{1}{2}\}, & 1 < u_n \le 2 \quad or(0 \le x_n < 1) \end{cases}$$

If $z_n = 2 - x_n$ for $x_n \in [1,2]$, we have

$$x_{n+1} = (-1 + \frac{17}{8n} + \frac{2}{3n^2} + \frac{1}{2n^2-3})x_n + 2(1 - \frac{1}{2n^2-3} - \frac{1}{3n^2} - \frac{2}{n}).$$

If $z_n = \frac{1}{2}$ for $x_n \in [0,1)$, we have

$$x_{n+1} = (\frac{1}{8n} + \frac{1}{3n^2})x_n + \frac{1}{2}(1 - \frac{1}{2n^2-3} - \frac{1}{3n^2} - \frac{2}{n}).$$

Following the proof of Theorem **??**, *we obtain that $\{x_n\}$, $\{u_n\}$ converges strongly to $w = \{1\} \in$ Fix$(T) \cap$ GEPP.*

Figure 1 indicates the behavior of x_n with initial point $x_1 = 0.5$, which converges to the same solution, that is, $w = \{1\} \in$ Fix$(S) \cap$ GEPP as a solution of this example.

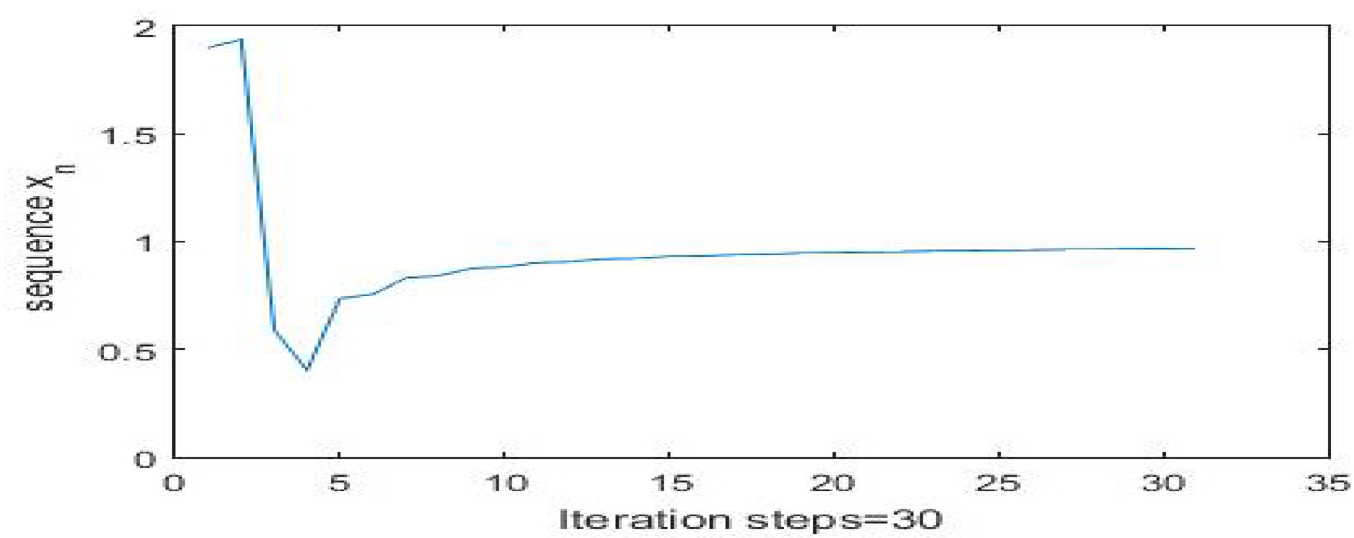

Figure 1: The graph of $\{x_n\}$ with initial value $x_1 = 0.5$.

Example 4.2. Let $H = \mathbb{R}$, the set of all real numbers, with the inner product defined by $\langle x, y \rangle = xy, \ \forall x, y \in \mathbb{R}$, and induced usual norm $|\ .\ |$. Let $C = [-1, 3]$; let $F : C \times C \to \mathbb{R}$ be defined by $F(x, y) = x(y - x),\ \ \forall x, y \in C$; let $M, N : C \to H$ be defined by $M(x) = 2x$ and $N(x) = 3x,\ \ \forall x \in C$, such that $\bar{\alpha} = \frac{1}{3}$ and $\bar{\beta} = \frac{1}{4}$ respectively, and let for each $x \in \mathbb{R}$, we define $f(x) = \frac{1}{6}x,\ A(x) = \frac{x}{2},\ B(x) = \frac{1}{10}x$ and

$$Tx = \begin{cases} \{\frac{x}{2}\}, & 0 < x \leq 3 \\ \{0\}, & -1 \leq x \leq 0 \end{cases}$$

Then there exist unique sequences $\{x_n\} \subset \mathbb{R}$ and $\{u_n\} \subset C$ generated by the iterative schemes

$$u_n = T^F_{r_n}(x_n - r_n(M + N)x_n); \tag{4.3}$$

$$x_{n+1} = (\frac{1}{3\sqrt{n}} + \frac{1}{10(n+1)^2})x_n + ((1 - \frac{2}{n^2})I - \frac{1}{(n+1)^2}B - \frac{1}{\sqrt{n}}A)z_n \tag{4.4}$$

where $\alpha_n = \frac{1}{\sqrt{n}},\ \beta_n = \frac{1}{(n+1)^2},\ \epsilon_n = \frac{2}{n^2}$ and $r_n = 1 + \frac{1}{n}$. Then $\{x_n\}$ converges to $\{0\} \in \text{Fix}(T) \cap \text{GEPP}$. *It is easy to prove that the bifunction F satisfy the Assumption* **??**. *Further, f is contraction mapping with constant $\alpha = \frac{1}{5}$ and A is a strongly positive bounded linear operator with constant $\bar{\gamma}_1 = 1$ on $\mathbb{R}$. Therefore, we can choose $\gamma = 2$ which satisfies $0 < \gamma < \frac{\bar{\gamma}_1}{\alpha} < \gamma + \frac{1}{\alpha}$. Furthermore, it is easy to observe that Fix$(T) = \{0\}$ and GEPP $= \{0\}$. Hence Fix$(T) \cap$ GEPP $= \{0\} \neq \emptyset$. After simplification, schemes (4.3) and (4.4) reduce to*

$$u_n = (\frac{-4n-5}{2n+1})x_n$$

$$Tu_n = \begin{cases} \{0\}, & -15 \leq u_n < 0 \quad or(0 < x_n \leq 3) \\ \{(\frac{-4n-5}{4n+2})x_n\}, & 0 \leq u_n \leq 2 \quad or(-1 \leq x_n \leq 0) \end{cases}$$

If $z_n = \frac{-4n-5}{4n+2}x_n$ for $x_n \in [-1, 0]$, we have

$$x_{n+1} = (\frac{1}{3\sqrt{n}} + \frac{1}{10(n+1)^2})x_n + (1 - \frac{2}{n^2} - \frac{1}{10(n+1)^2} - \frac{1}{2\sqrt{n}})(\frac{-4n-5}{4n+2})x_n.$$

If $z_n = 0$ for $x_n \in (0, 3]$, we have

$$x_{n+1} = (\frac{1}{3\sqrt{n}} + \frac{1}{10(n+1)^2})x_n.$$

Following the proof of Theorem **??**, *we obtain that $\{x_n\},\ \{u_n\}$ converges strongly to $w = \{0\} \in$ Fix$(T) \cap$ GEPP . Figure 2 indicates the behavior of x_n with initial point $x_1 = 0.5$, which converges to the same solution, that is, $w = \{1\} \in$ Fix$(S) \cap$ GEPP as a solution of this example.*

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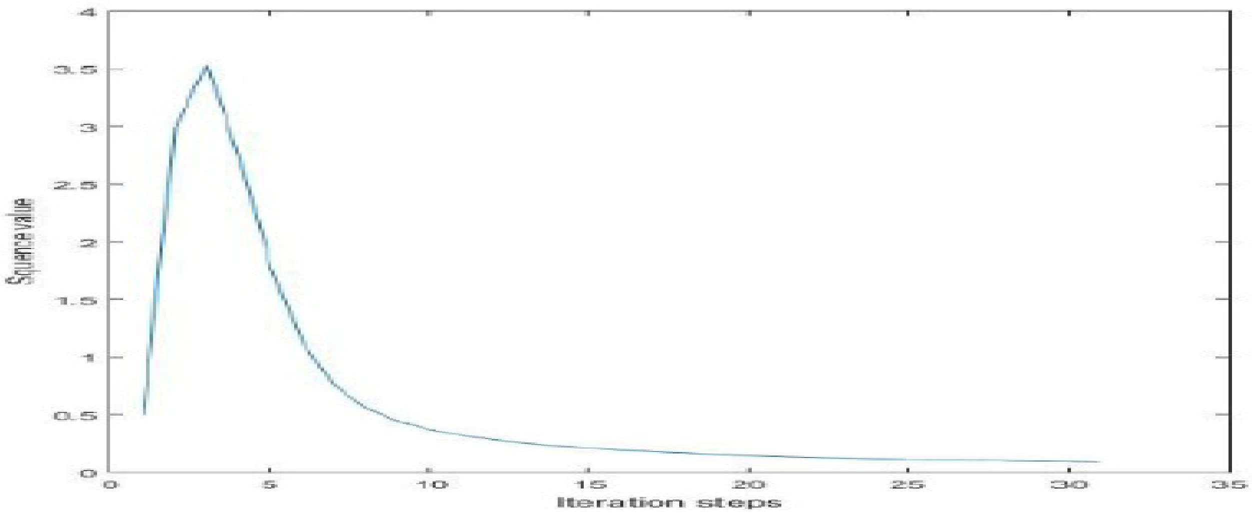

Figure 2: The graph of $\{x_n\}$ with initial value $x_1 = 0.5$.

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