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Midpoint Characterization of Symmetric Hadamard Spaces and its Applications

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Abstract

In this paper, we distinguish symmetric Hadamard spaces by introducing the midpoint properties in these spaces. We prove that any symmetric Hadamard spaces are flat. As an application of the new midpoint property, we characterize the affine mapping in these spaces.

1 Introduction

Let (X, d) be a metric space. A geodesic from x to y is a map γ from the closed interval $[0, d(x, y)] \subset \mathbb{R}$ to X such that $\gamma(0) = x,\ \gamma(d(x, y)) = y$ and $d(\gamma(t), \gamma(t')) = |t - t'|$ for all $t, t' \in [0, d(x, y)]$. The space (X, d) is said to be a geodesic space if every two points of X are joined by a geodesic. The metric segment $[x, y]$ contains the images of all geodesics, which connect x to y. X is called unique geodesic iff $[x, y]$ contains only one geodesic.
Let X be a unique geodesic metric space. For each $x, y \in X$ and for each $t \in [0, 1]$, there exists a unique point $z \in [x, y]$ such that $d(x, z) = td(x, y)$ and $d(y, z) = (1 - t)d(x, y)$. We will use the notation $(1 - t)x \oplus ty$ for the unique point z satisfying in the above statement.

In a unique geodesic metric space X, a set $A \subset X$ is called convex iff for each $x, y \in A$, $[x, y] \subset A$. A unique geodesic space X is called CAT(0) space if for all $x, y, z \in X$ and for each $t \in [0, 1]$, we have the following inequality

$$d^2((1 - t)x \oplus ty, z) \le (1 - t)d^2(x, z) + td^2(y, z) - t(1 - t)d^2(x, y).$$

A complete CAT(0) space is called a Hadamard space.

Berg and Nikolaev in [8,9] have introduced the concept of quasi-linearization along these lines. Let us formally denote a pair $(a, b) \in X \times X$ by $\overrightarrow{ab}$ and call it a vector. Then quasi-linearization is defined as a map $\langle \cdot, \cdot \rangle : (X \times X) \times (X \times X) \to \mathbb{R}$ defined by

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \frac{1}{2}\left(d^2(a, d) + d^2(b, c) - d^2(a, c) - d^2(b, d)\right) \quad (a, b, c, d \in X).$$

It is easily seen that

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$\langle \vec{ab}, \vec{ab} \rangle = d^2(a,b)$, $\langle \vec{ab}, \vec{cd} \rangle = \langle \vec{cd}, \vec{ab} \rangle$, $\langle \vec{ab}, \vec{cd} \rangle = -\langle \vec{ba}, \vec{cd} \rangle$ and $\langle \vec{ax}, \vec{cd} \rangle + \langle \vec{xb}, \vec{cd} \rangle = \langle \vec{ab}, \vec{cd} \rangle$ for all $a,b,c,d,x \in X$. We say that X satisfies the Cauchy-Schwartz inequality if $\langle \vec{ab}, \vec{cd} \rangle \leq d(a,b)d(c,d)$ for all $a,b,c,d \in X$. It is known (Corollary 3 of [9]) that a geodesically connected metric space is a CAT(0) space if and only if it satisfies the Cauchy-Schwartz inequality.

2 Main Result

Let (X,d) be a semi metric space. Define the relation $\sim$ on $X \times X$ as follows:

$$\overrightarrow{xy} \sim \overrightarrow{zt} \quad \Leftrightarrow \quad \langle \overrightarrow{ab}, \overrightarrow{xy} \rangle = \langle \overrightarrow{ab}, \overrightarrow{zt} \rangle \ \ (\forall a,b \in X). \tag{2.1}$$

The equivalent class of $\overrightarrow{xy}$ will be denoted by $[\overrightarrow{xy}]$. If $[\overrightarrow{xy}] = [\overrightarrow{zt}]$, then

$$d^2(x,y) = \langle \overrightarrow{xy}, \overrightarrow{xy} \rangle = \langle \overrightarrow{xy}, \overrightarrow{zt} \rangle = \langle \overrightarrow{zt}, \overrightarrow{xy} \rangle = \langle \overrightarrow{zt}, \overrightarrow{zt} \rangle = d^2(z,t). \tag{2.2}$$

Definition 2.1. *We say that the semi metric space (X,d) satisfies the ($\mathcal{S}$) property if for any $(x,y) \in X \times X$ there exist $y_x \in X$ such that $[\overrightarrow{xy}] = [\overrightarrow{y_x x}]$ (cf. [1, Definition 2.7]); or equivalently if for any $(x,y) \in X \times X$ there exist $x_y \in X$ such that $[\overrightarrow{xy}] = [\overrightarrow{y x_y}]$.*

Any Hadamard space with property $\mathcal{S}$ is called symmetric Hadamard space. It is obvious that, any Hilbert space enjoys the $\mathcal{S}$ and property (let $y_x := 2x - y$, then $[\overrightarrow{xy}] = [\overrightarrow{y_x x}] = [y-x]$ and also $x_y := 2y - x$ and then $[\overrightarrow{xy}] = [\overrightarrow{y x_y}] = [y-x]$).

Remark 2.1. *In the above definition, y_x and x_y are unique. Note that if $[\overrightarrow{xy}] = [\overrightarrow{y_x x}] = [\overrightarrow{ux}]$, then*

$$\begin{aligned} d^2(y_x,u) &= \langle \overrightarrow{y_x u}, \overrightarrow{y_x u} \rangle = \langle \overrightarrow{y_x x}, \overrightarrow{y_x u} \rangle + \langle \overrightarrow{xu}, \overrightarrow{y_x u} \rangle \\ &= \langle \overrightarrow{xy}, \overrightarrow{y_x u} \rangle - \langle \overrightarrow{xy}, \overrightarrow{y_x u} \rangle = 0. \end{aligned}$$

The uniqueness of x_y is similar.

Lemma 2.1. *Let X be a uniquely geodesic metric space satisfying ($\mathcal{S}$) property, and $x,y,z \in X$. Then $[\overrightarrow{xy}] = [\overrightarrow{zx}]$, if and only if, x is the midpoint of the geodesic $[z,y]$.*

Proof. Let $[\overrightarrow{xy}] = [\overrightarrow{zx}]$. Using (2.2) we have $d(x,y) = d(z,x)$. Moreover,

$$\begin{aligned} d^2(z,y) &= \langle \overrightarrow{zy}, \overrightarrow{zy} \rangle = \langle \overrightarrow{zx}, \overrightarrow{zy} \rangle + \langle \overrightarrow{xy}, \overrightarrow{zy} \rangle \\ &= \langle \overrightarrow{xy}, \overrightarrow{zy} \rangle + \langle \overrightarrow{xy}, \overrightarrow{zy} \rangle = 2\langle \overrightarrow{xy}, \overrightarrow{zy} \rangle \\ &= 2(\langle \overrightarrow{xy}, \overrightarrow{zx} \rangle + \langle \overrightarrow{xy}, \overrightarrow{xy} \rangle) \\ &= 2(\langle \overrightarrow{xy}, \overrightarrow{xy} \rangle + \langle \overrightarrow{xy}, \overrightarrow{xy} \rangle) \\ &= 4\langle \overrightarrow{xy}, \overrightarrow{xy} \rangle = 4d^2(x,y). \end{aligned}$$

It means that $d(x,y) = d(x,z) = 1/2d(z,y)$ and so x is the midpoint of $[z,y]$.
Conversely, let m be the midpoint of $[x,y]$. Then we have $d(x,m) = d(m,y) = 1/2d(x,y)$. By ($\mathcal{S}$) there exists $x_m \in X$ such that $[\overrightarrow{xm}] = [\overrightarrow{m x_m}]$. It follows from (2.1) and (2.2) that

$$\langle \overrightarrow{ym}, \overrightarrow{xm} \rangle = \langle \overrightarrow{ym}, \overrightarrow{m x_m} \rangle$$

and $d(m, x_m) = d(m, x)$. It means that

$$d^2(y, m) + d^2(x, m) - d^2(x, y) = d^2(y, x_m) - d^2(y, m) - d^2(m, x_m).$$

Since $d(m, x_m) = d(m, x) = d(m, y) = 1/2d(x, y)$, we have $d^2(y, x_m) = 0$. It means that $x_m = y$ and thus $[\overrightarrow{xm}] = [\overrightarrow{mx_m}] = [\overrightarrow{my}]$. □

Lemma 2.2. *Let C be a nonempty convex subset in the symmetric Hadamard space X, $x \in X$ and $u \in C$. Then, $u = P_C(x)$ if and only if*

$$\langle \overrightarrow{xu}, \overrightarrow{yu} \rangle = 0,$$

for all $y \in C$

Proof. By [36, Theorem 2.2] we have $u = P_C(x)$ if and only if

$$\langle \overrightarrow{xu}, \overrightarrow{yu} \rangle \leq 0,$$

for all $y \in C$. Also, there exists $y_u \in C$ such that $[\overrightarrow{uy}] = [\overrightarrow{y_u u}]$. Thus we have

$$\langle \overrightarrow{xu}, \overrightarrow{yu} \rangle = -\langle \overrightarrow{xu}, \overrightarrow{uy} \rangle = -\langle \overrightarrow{xu}, \overrightarrow{y_u u} \rangle \geq 0$$

It means that $\langle \overrightarrow{xu}, \overrightarrow{yu} \rangle = 0$. □

Theorem 2.1. *Let X be a symmetric Hadamard space and $p, q, r, m \in X$. Then m is the midpoint of $[q, r]$ if and only if the following equality holds*

$$2d^2(p, m) + \frac{1}{2}d^2(q, r) = d^2(p, r) + d^2(p, q).$$

Proof. ($\Rightarrow$) Let m be the midpoint of $[q, r]$. Then $[\overrightarrow{qm}] = [\overrightarrow{mr}]$. Thus,

$$\langle \overrightarrow{ab}, \overrightarrow{qm} \rangle = \langle \overrightarrow{ab}, \overrightarrow{mr} \rangle \quad \forall\, a, b \in X.$$

Now let $a = p$ and $b = m$ we have

$$\langle \overrightarrow{pm}, \overrightarrow{qm} \rangle = \langle \overrightarrow{pm}, \overrightarrow{mr} \rangle.$$

Therefore,

$$d^2(p, m) + d^2(q, m) - d^2(p, q) = d^2(p, r) - d^2(p, m) - d^2(m, r).$$

It means that

$$2d^2(p, m) + \frac{1}{4}d^2(q, r) = d^2(p, q) + d^2(p, r) - \frac{1}{4}d^2(q, r).$$

Hence,

$$2d^2(p, m) + \frac{1}{2}d^2(q, r) = d^2(p, q) + d^2(p, r)$$

and the proof is completed.

($\Leftarrow$) Suppose that $p, q, r \in X$ and $m \in [q, r]$ and

$$2d^2(p, m) + \frac{1}{2}d^2(q, r) = d^2(p, r) + d^2(p, q). \tag{2.3}$$

Since $m \in [q, r]$, we have

$$d(m, q) = td(q, r) \quad and \quad d(m, r) = (1 - t)d(q, r). \tag{2.4}$$

If we take $p = m$ in (2.3), we obtain

$$\frac{1}{2}d^2(q, r) = d^2(p, q) + d^2(p, r). \tag{2.5}$$

Combining (2.4) and (2.5), one can conclude that

$$\frac{1}{2}d^2(q, r) = (1 - t)^2 d^2(q, r) + t^2 d^2(q, r)$$

and so $t^2 + (1 - t)^2 = \frac{1}{2}$ and this holds if and only if $t = \frac{1}{2}$. It means that

$$d(m, q) = d(m, r) = \frac{1}{2}d(q, r).$$

Hence, m is the midpoint of $[q, r]$. □

Definition 2.2. *Let X be an Hardamard space (with or without $(\mathcal{S})$ property). We say that X is flat if for $p, q, r, m \in X$ in which m is the midpoint of $[q, r]$ then the following equality holds*

$$d^2(p, m) = \frac{d^2(p, r) + d^2(p, q)}{2} - \frac{1}{4}d^2(q, r).$$

Corollary 2.1. *Let X be an Hardamard space. Then, if X be symmetric Hadamard space then it is flat.*

Proof. Based on Theorem 2.1 since the CN inequality of Bruhat and Tits turns into equality in symmetric Hadamard space, therefore, it is flat. □

3 Some Applications of Midpoint Characterization

In the current section, as an application of the midpoint property in symmetric Hadamard spaces, we characterize the affine mapping in such spaces.

Theorem 3.1. *Suppose that X is a symmetric Hadamard space. Then*

(a_1) *for each $a, b, c \in X$ the function $f : X \to \mathbb{R}$ defined by $f(x) = \langle \overrightarrow{ab}, \overrightarrow{cx} \rangle + f(b)$ is affine and for each $x_0 \in dom(f)$, $[\overrightarrow{ab}] \in \partial f(x_0)$, i.e., $dom(\partial f) = int(dom(f))$.*

(a_2) *for each affine mapping $f : X \to \mathbb{R}$ and for every $b \in cl(dom(f))$ and $t \in \mathbb{R}$ with $t < f(b)$, there exists a point $a \in X$, such that*

$$f(x) = \frac{1}{f(b) - t}\langle \overrightarrow{ab}, \overrightarrow{ax} \rangle + f(b)$$

for each $x \in X$. Moreover, $d(a, b) \leq \frac{1}{2}(f(b) - t)$.

Proof. (a_1) : Define $g(x) = d^2(x, a) - d^2(x, b)$, for each $x \in X$. Let $m = \frac{1}{2}x \oplus \frac{1}{2}y$. Since m is the midpoint of $[x, y]$ by Theorem we have $[\overrightarrow{xm}] = [\overrightarrow{my}]$. It means that,

$$\langle \overrightarrow{xm}, \overrightarrow{ab} \rangle = \langle \overrightarrow{my}, \overrightarrow{ab} \rangle.$$

we have

$$d^2(x,b)+d^2(m,a)-d^2(x,a)-d^2(m,b)=d^2(m,b)+d^2(y,a)-d^2(m,a)-d^2(y,b).$$

It means that,

$$d^2(\frac{1}{2}x\oplus\frac{1}{2}y,a)-d^2(\frac{1}{2}x\oplus\frac{1}{2}y,a)=\frac{1}{2}(d^2(x,a)-d^2(x,b))+\frac{1}{2}(d^2(y,a)-d^2(y,b)).$$

Thus,

$$g(\frac{1}{2}x\oplus\frac{1}{2}y)=\frac{1}{2}g(x)+\frac{1}{2}g(y).$$

Since f is continuous so for each $\lambda\in[0,1]$, we have

$$g(\lambda x\oplus(1-\lambda)y)=\lambda g(x)+(1-\lambda)g(y).$$

Since

$$\langle\overrightarrow{ab},\overrightarrow{cx}\rangle=\frac{1}{2}(d^2(a,x)+d^2(b,x)-d^2(a,c)-d^2(b,c))$$

It shows that f is affine and the proof is completed.

Also, suppose that $a,b,c\in X$. We have

$$\begin{aligned}f(x)-f(x_0)&=\langle\overrightarrow{ab},\overrightarrow{cx}\rangle-\langle\overrightarrow{ab},\overrightarrow{cx_0}\rangle\\&=\tfrac{1}{2}[d^2(a,x)+d^2(b,c)-d^2(a,c)-d^2(b,x)\\&\quad-d^2(a,x_0)+d^2(b,c)-d^2(a,c)-d^2(b,x_0)]\\&=\tfrac{1}{2}[d^2(a,x)-d^2(b,x)-d^2(a,x_0)+d^2(b,x_0)]\\&=\langle\overrightarrow{ab},\overrightarrow{x_0x}\rangle\end{aligned}$$

If we take $x^*=[\overrightarrow{ab}]$, we have $x^*\in\partial f(x_0)$.

(a_2) : Note that f is affine, thus the set $epi(f)=\{(x,t)\,:\,t\geq f(x)\}$ is nonempty, closed and affine set. Applying the same argument in [37, Lemma 3.2] and the fact that X is symmetric Hadamard space, taking $(a,s)=P_{epi(f)}(b,t)$, one can apply Lemma 2.2 and conclude that

$$\langle(\overrightarrow{ax},f(x)-s),(\overrightarrow{ab},t-s)\rangle=0.$$

for each $x\in dom(f)$. It means that

$$\langle\overrightarrow{ax},\overrightarrow{ab}\rangle+(f(x)-s)(t-s)=0. \tag{3.1}$$

Considering $x=b$, we have

$$d^2(a,b)+(f(b)-s)(t-s)=0. \tag{3.2}$$

It means that, $s=f(b)$ and from (3.1) we have

$$f(x)=\frac{1}{f(b)-t}\langle\overrightarrow{ab},\overrightarrow{ax}\rangle+f(b).$$

Moreover, (3.2) yields that

$$d^2(a,b)+(f(x)-t)(t-s)+(t-s)^2=0.$$

Taking $A = t - s$, we have $A^2 + A(f(x) - t) + d^2(a, b) = 0$. It means that $\Delta = (f(x) - t)^2 - 4d^2(a, b) \geq 0$. Thus,

$$|f(x) - t| \geq 2d(a, b).$$

and so $d(a, b) \leq \frac{1}{2}|f(x) - t|$. □

Remark 3.1. *In [1, Definition 2.7], the author introduced $(\mathcal{S})$ property and said that : "it is not hard to check that any symmetric Hadamard manifold satisfies the $(\mathcal{S})$ property". Based on this sentence and Corollary* **??**, *any symmetric Hadamard spaces is flat! But anyone ia able to exemplify many Hadamard manifolds (see [1] for more details) which are not flat and so the sentence is not true as the author in [1] pointed out.*

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