Theory of Approximation and Applications, 2020, Volume 14, No. 1: 75-88, DOR: 20.1001.1.25382217.2020.14.1.3.5

# Sequential Optimality Conditions and Variational Inequalities

Sanjeev Kumar Singh, J.K. Maurya, S.K. Mishra*

*Department of Mathematics, Institute of Science, Banaras Hindu University, Varanasi, India*

**ARTICLE INFO**

**KEYWORDS**
Optimality conditions
Variational inequalities
Vector variational inequalities
Convex analysis.

**ARTICLE HISTORY**
**RECEIVED:** 2020 JANUARY 9
**ACCEPTED:** 2020 JUNE 12

**ABSTRACT**

In recent years, sequential optimality conditions are frequently used for convergence of iterative methods to solve nonlinear constrained optimization problems. The sequential optimality conditions do not require any of the constraint qualifications. In this paper, We present the necessary sequential complementary approximate Karush Kuhn Tucker (CAKKT) condition for a point to be a solution of a nonlinear optimization problem. The nonlinear optimization problem is associated with the variational inequality problem. We also extend the complementary approximate Karush Kuhn Tucker condition from scalar optimization problem to multiobjective optimization problem and associated with the vector variational inequality problem. Further, we prove that with some extra conditions of convexity and affinity, complementary approximate Karush Kuhn Tucker conditions are sufficient for the variational inequality problem and vector variational inequality problem. Finally, we verify our results via illustrative examples. An example shows that a point which is a solution of variational inequality problem is also a CAKKT point.

## 1 Introduction

Karush Kuhn Tucker conditions [6] play a vital role to solve nonlinear optimization problems, both for scalar optimization and for multiobjective optimization problems. Practically, optimality conditions based on the sequence of iterands, which is known as sequential optimality conditions, do not require any constraint qualification [8].

The sequential optimality conditions, for example, approximate Karush Kuhn Tucker condition [11] needs the existence of a sequence $\{x^k\}$, which is converging to some $x^*$ with the condition that $x^k$ is a Karush Kuhn Tucker point for every natural number $k,$ also there should be an appropriate sequence of Lagrange multipliers with the property that gradient of the Lagrangian function at $x^k$ converges to zero. Another type of sequential optimality conditions, that do not require any of the constraint qualifications, are called complementary approximate Karush Kuhn Tucker condition (CAKKT) [10]. In CAKKT test we need some extra condition to the usual AKKT test, that the product of each multiplier with the corresponding constraint value should be small [11]. Some other sequential optimality conditions are discussed in [8, 12]. The concept of variational inequality problem is introduced by Lions and Stampacchia [7]. Further, Giannessi [1] introduced the concept of vector variational inequalities. Variational inequalities and vector variational inequalities play an important role in deriving necessary and sufficient optimality conditions for scalar and vector optimization problems. Recently, Laha and Mishra [13] established some results in vector optimization problems and vector variational inequalities involving locally Lipschitz functions.

* Corresponding Author's E-mail: bhu.skmishra@gmail.com

Motivated by the work of Andreani *et al.* [10], Haeser and Schuverdt [4], Mastroeni [5] and Giorgi *et al.* [3], we discuss the sufficiency of CAKKT conditions in the nonlinear programming problems and generalize its definition to the structure of variational inequality problems and vector variational inequality problems. This paper is organized as follows. In Section 2, we collected some basic definitions and results. In Section 3, we develop sequential optimality conditions as CAKKT conditions for variational inequality and proved sufficiency with convex and affine conditions. In Section 4, we define CAKKT conditions for vector variational inequality and also proved sufficiency with convex and affine conditions.

# 2 Preliminaries

We recall some basic and essential definitions. The open (closed) ball centered at $y^* \in \mathbb{R}^n$ with radius $\delta > 0$ is denoted by $B(y^*, \delta)$ ( $\bar{B}(y^*, \delta)$). We denote $\mathbb{R}^n_+$ as the nonnegative orthant of $\mathbb{R}^n$. We also denote $c_+ = \max\{0, c\}$, $c^2_+ = (c_+)^2$, where $c \in \mathbb{R}$. The notation $\|\cdot\|$ is the Euclidean norm of $\mathbb{R}^n$ except otherwise specified. For $y, z \in \mathbb{R}^n$, $y \leq z$ iff $y_i \leq z_i$, for $i = 1, ..., n$; $y < z$, $y_i < z_i$, for $i = 1, ..., n$.

Let $X$ be real Banach Space with a norm $\|.\|$ and $X^*$ be its dual space with a norm $\|\cdot\|^*$. Let $K$ be a non-empty open convex subset of $X$, $F : X \to 2^{X^*}$ be a set-valued mapping from real Banach space to the family of non-empty subsets of $X^*$. The following definitions and results are extracted from [2] to resolve difficulties during the derivation of upcoming results.

**Definition 2.1.** *(Generalized directional derivative) Let $f$ be locally Lipschitz at a given point $x \in X$ and $v$ be any other vector in X. The generalized directional derivative of $f$ at $x$ in the direction of $v$, denoted by $f^0(x; v)$, is defined by*

$$f^0(x; v) = \limsup_{y \to x,\ t \downarrow 0} \frac{f(y + tv) - f(y)}{t}.$$

**Definition 2.2.** *(Clarke subdifferential) Let $f$ be locally Lipschitz at a given point $x \in X$ and $v$ be any other vector in X. The Clarke subdifferential of $f$ at $x$, denoted by $\partial^c f(x)$, is defined by*

$$\partial^c f(x) = \{\xi \in X^* : f^0(x; v) \geq \langle \xi, v \rangle, \forall v \in X\}.$$

Next, we gather some properties related to Clarke's generalized subdifferential which can be found in [2].

**Proposition 2.1.** *Let $f : \mathbb{R}^n \to \mathbb{R}$ be locally Lipschitz at $x$ with constant $K$. Then*

1. *$\partial^c f(x)$ is a nonempty, convex, compact set such that $\partial^c f(x) \subset B(0; K)$,*
2. *$f^0(x, v) = \max\{\langle v, \xi \rangle | \xi \in \partial^c f(x)\}\ \forall v \in \mathbb{R}^n$,*
3. *the map $\partial^c f(\cdot) : \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ is upper semicontinuous, where $\mathcal{P}(\mathbb{R}^n)$ denotes the power set of $\mathbb{R}^n$,*
4. *if $f$ is differentiable at $x$, then $\nabla f(x) \in \partial^c f(x)$,*
5. *if $f$ attains its extremum at $x$, then $0 \in \partial^c f(x)$.*

**Proposition 2.2.** *Let the functions $f_i : \mathbb{R}^n \to \mathbb{R}$ be locally Lipschitz at $x$ for $i = 1, 2, ..., k$, then for $\lambda_i \in \mathbb{R}$*

$$\partial^c \Big( \sum_{i=1}^{k} \lambda_i f_i \Big)(x) \subset \sum_{i=1}^{k} \lambda_i \partial^c f_i(x).$$

**Proposition 2.3.** *If $f_1$ and $f_2$ are locally Lipschitz at $x \in \mathbb{R}^n$, then the function $f_1 f_2$ is locally Lipschitz at $x$ and*

$$\partial^c(f_1f_2)(x) \subset \partial^c f_1(x)f_2(x) + f_1(x)\partial^c f_2(x).$$

**Approximate Karush Kuhn Tucker conditions (AKKT)** [11]

We consider the nonlinear constrained optimization problems (OP).

$$\text{(OP) Minimize } f(x) \text{ s.t. } x \in \Omega = \{x \in \mathbb{R}^n : g(x) \leq 0,\ h(x) = 0\}, \tag{1}$$

where $f : \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}^m$, $h : \mathbb{R}^n \to \mathbb{R}^r$ are smooth functions. We say a feasible point $x^*$ satisfies (AKKT) conditions, if there exists a sequences $(\mu^k, \tau^k) \subset \mathbb{R}^m_+ \times \mathbb{R}^r$, $\{x^k\} \subset \mathbb{R}^n$ converging to $x^*$ and satisfies the following:

$$\lim_{k\to\infty} \|\nabla f(x^k) + \sum_{j=1}^{m} \mu_j^k \nabla g_j(x^k) + \sum_{l=1}^{r} \tau_l^k \nabla h_l(x^k)\| = 0,$$

$$g_j(x^*) < 0 \implies \mu_j^k = 0 \text{ for sufficiently large } k,\ j = 1, ..., m. \tag{2}$$

**Complementary Approximate Karush Kuhn Tucker conditions for constrained optimization problems (CAKKT-OP)**

**Definition 2.3.** *(CAKKT-OP conditions)[10] We say that a feasible point $x^*$ satisfies (CAKKT-OP) if there exists sequences $(x^k) \subset \mathbb{R}^n$ and $(\mu^k, \tau^k) \subset \mathbb{R}^m_+ \times \mathbb{R}^r$ such that*

*(C1)* $x^k \to x^*$,

*(C2)* $\nabla f(x^k) + \sum\limits_{j=1}^{m} \mu_j^k \nabla g_j(x^k) + \sum\limits_{i=1}^{r} \tau_i^r \nabla h_i(x^k) \to 0,$

*(C3)* $\lim\limits_{k\to\infty} \mu_j^k g_j(x^k) = 0, \lim\limits_{k\to\infty} \tau_i^k h_i(x^k) = 0, j = 1, ..., m,\ i = 1, ..., r.$

**Remark 2.1.** *It is the direct implication from CAKKT and AKKT conditions that every CAKKT point is also an AKKT point, but the converse need not be true in general (see, for instance [10]).*

Now, we generalize the concept of CAKKT-OP conditions for multiobjective optimization problems (MOP) motivated by the work of [3], say CAKKT-MOP. Consider the problem

$$\text{(MOP) Minimize } (f_1(x), ..., f_p(x)),$$

$$\text{subject to } x \in \Omega = \{x \in \mathbb{R}^n : g(x) \leq 0,\ h(x) = 0\},$$

where $f_i : \mathbb{R}^n \mapsto \mathbb{R}, i = 1, ..., p.$ and g, h are defined earlier.

**Definition 2.4.** *(CAKKT-MOP conditions) We say that a feasible point $x^*$ satisfies CAKKT-MOP conditions for multiobjective optimization problems if there exists sequences $(x^k) \subset \mathbb{R}^n$ and $(\lambda^k, \mu^k, \tau^k) \subset \mathbb{R}^p_+ \times \mathbb{R}^m_+ \times \mathbb{R}^r$ such that*

*(C1)* $x^k \to x^*$,

*(C2)* $\sum_{s=1}^{p} \lambda_s^k \nabla f_s(x^k) + \sum_{j=1}^{m} \mu_j^k \nabla g_j(x^k) + \sum_{l=1}^{r} \tau_{l=1}^{r} \nabla h_l(x^k) \to 0,\ \sum_{s=1}^{p} \lambda_s^k = 1,$

*(C3)* $\lim_{k\to\infty} \mu_j^k g_j(x^k) = 0, \lim_{k\to\infty} \tau_l^k h_l(x^k) = 0, j = 1, ..., m,\ l = 1, ..., r.$

## 3 Necessary and sufficient optimality conditions

Let $\Omega$ be non-empty, convex subset of $\mathbb{R}^n$ and $F : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous map, then variational inequality (VI) problem [7] is stated as follows:

$$VI(F, \Omega) \quad \text{find } y^* \in \Omega, \quad \text{such that} \quad \langle F(y^*), y - y^* \rangle \geq 0,\ \forall y \in \Omega.$$

Consider the following constrained optimization variational inequality (OP-VI) problem

$$\text{(OP-VI) Minimize}\langle F(y^*), y\rangle,\ \text{such that} y \in \Omega,$$
$$\Omega = \{y \in \mathbb{R}^n : g(x) \leq 0,\ h(x) = 0\}.$$

We define the following definition of CAKKT-VI points motivated by [10].

**Definition 3.1.** *(CAKKT-VI conditions) We say that a feasible point $x^*$ satisfies CAKKT-VI conditions for OP-VI problems if there exists sequences $(x^k) \subset \mathbb{R}^n$ and $(\mu^k, \tau^k) \subset \mathbb{R}_+^m \times \mathbb{R}^r$, such that*

*(C1)* $x^k \to x^*,$

*(C2)* $F(x^k) + \sum_{j=1}^{m} \mu_j^k \nabla g_j(x^k) + \sum_{l=1}^{r} \tau_{l=1}^{r} \nabla h_l(x^k) \to 0,$

*(C3)* $\lim_{k\to\infty} \sum_{j=1}^{m} \mu_j^k g_j(x^k) = 0, \lim_{k\to\infty} \sum_{l=1}^{r} \tau_l^k h_l(x^k) = 0,$
$j = 1, ..., m,\ l = 1, ..., r.$

We present a result which states how the solutions of the VI problem are related to the CAKKT-VI conditions.

**Theorem 3.1.** *If $y^*$ is a solution to $VI(F, \Omega)$, then $y^*$ satisfies the $CAKKT - VI$ conditions.*

***Proof*** *Consider the following problem*

$$\textit{Min } \langle F(y^*), y\rangle,\ \textit{subject to } y \in \Omega = \{y \in \mathbb{R}^n : g(x) \leq 0, h(x) = 0\}. \tag{3}$$

*Let $\delta > 0$ be such that $\langle F(y^*), y^*\rangle \leq \langle F(y^*), y\rangle$ for every $y \in \Omega$ and $\|y - y^*\| \leq \delta$. Then, $y^*$ is the unique global minimizer of the problem*

$$Minimize \left[\langle F(y^*), y\rangle + \frac{1}{2}\|y - y^*\|^2\right],\ \textit{subject to}$$
$$h(y) = 0,\ g(y) \leq 0,\ \|y - y^*\| \leq \delta. \tag{4}$$

*For every $k \in \mathbb{N}$, let $y^k$ be a global minimizer of*

*Minimize $\Phi_k(y)$, subject to $y \in \Omega \cap \bar{B}(y^*, \delta)$,*

$$\text{where } \Phi_k(y) = \langle F(y^*), y\rangle + \frac{1}{2}\|y - y^*\|^2 + \frac{1}{2}\Big[\sum_{j=1}^{m} kg_j(y)_+^2 + \sum_{l=1}^{r} kh_l(y)^2\Big], \tag{5}$$

*$\{y^k\}$ is well defined since the objective function is continuous. Since $y^*$ is feasible to problem (5), we have from convergence of external penalty methods [9].*

$$\Phi_k(y^k) = \langle F(y^*), y^k\rangle + \frac{1}{2}\|y^k - y^*\|^2 + \frac{1}{2}\Big[\sum_{j=1}^{m} kg_j(y^k)_+^2 + \sum_{l=1}^{r} kh_l(y^k)^2\Big] \leq \langle F(y^*), y^*\rangle. \tag{6}$$

*As $\lim_{k\to\infty} y^k = y^*$. Then,*

$$\lim_{k\to\infty} \frac{1}{2}\|y^k - y^*\|^2 + \frac{1}{2}\Big[\sum_{j=1}^{m} kg_j(y^k)_+^2 + \sum_{l=1}^{r} kh_l(y^k)^2\Big] = 0.$$

*Therefore,*

$$\sum_{j=1}^{m} kg_j(y^k)_+^2 + \sum_{l=1}^{r} kh_l(y^k)^2 \to 0 \text{ as } k \to \infty.$$

*Let us define $\lambda^k = kh_l(y^k)$, $\mu^k = kg_j(y^k)_+$. Then,*

$$\lim_{k\to\infty} \Big[\sum_{j=1}^{m} |\mu^k g_j(y^k)_+| + \sum_{l=1}^{r} |\lambda^k h_l(y^k)|\Big] = 0.$$

*That is,*

$$\lim_{k\to\infty} \sum_{l=1}^{r} \lambda^k h_l(y^k) = 0, \ \lim_{k\to\infty} \sum_{j=1}^{m} \mu^k g_j(y^k) = \lim_{k\to\infty} \sum_{j=1}^{m} \mu^k g_j(y^k)_+ = 0. \tag{7}$$

*as $y^k \to y^*$, the conditions (C3) of CAKKT-VI are satisfied. In $(6)$, we observe that $y^k$ exists because $\Phi_k(y^k)$ is continuous and $\bar{B}(y^*, \delta)$ is compact. Let $z$ be a limit point of $y^k$. We can assume that $y^k \to z$. From the problem (5), we have*

$$\langle F(y^*), y^k\rangle \leq \Phi_k(y^k),$$

*because of*

$$\Phi_k(y^k) - \langle F(y^*), y^k\rangle = \frac{1}{2}\|y^k - y^*\|^2 + \frac{k}{2}\Big\{\sum_{j=1}^{m}[max(0, g_j(y^k))]^2 + \sum_{l=1}^{r}[h_l(y^k)]^2\Big\} \geq 0.$$

*Since $y^*$ is a feasible solution of the problem (4) and $y^k$ is the solution of problem (5), we have*

$$\langle F(y^*), y^k\rangle \leq \Phi_k(y^k) \leq \Phi_k(y^*) = \langle F(y^*), y^*\rangle. \tag{8}$$

*We claim that $z$ is a feasible solution of the problem (5). Since $\|y^k - y^*\| \leq \delta$, therefore $\|z - y^*\| \leq \delta$, suppose if* *possible*

$$\sum_{j=1}^{m}(g_j(z)_+)^2 + \sum_{l=1}^{r} h_l^2(z) > 0,$$

*for sufficiently large k, then there exists* $c > 0$, *such that*

$$\sum_{j=1}^{m}(g_j(y^k)_+)^2 + \sum_{l=1}^{r} h_l^2(y^k) > c.$$

*Therefore,*

$$\sum_{j=1}^{m}[max(0, g_j(y^k))]^2 + \sum_{l=1}^{r}[h_l(y^k)]^2 > c,$$

*for all* $k$ *large enough. From continuity of* $\langle F(y^*), y\rangle$ *and* $y^k \to z$, *we have*

$$\Phi_k(y^k) = \langle F(y^*), y^k\rangle + \frac{1}{2}\|y - y^*\|^2 + \frac{\rho_k}{2}\Big\{\sum_{j=1}^{m}[max(0, g_j(y^k))]^2\Big\} + \sum_{l=1}^{r}[h_l(y^k)]^2\}$$
$$> \langle F(y^*), y^*\rangle + \frac{\rho_k c}{2}.$$

*Taking the limit, we obtain* $\Phi_k(y^k) \longrightarrow +\infty$, *which contradicts (6). Consequently,* $\sum\limits_{j=1}^{m}(g_j(z)_+)^2 + \sum\limits_{l=1}^{r} h_l^2(z) = 0$, *that is,* $z \in \Omega \cap \bar{B}(x^*, \delta)$, *therefore from (6), we obtain*

$$\Phi_k(y^k) = \langle F(y^*), y^k\rangle + \frac{1}{2}\|y^k - y^*\|^2 + \frac{k}{2}\Big\{\sum_{j=1}^{m}[max(0, g_j(y^k))]^2 + \sum_{l=1}^{r}[h_l(y^k)]^2\Big\} \leq 0, \tag{9}$$

*as* $k \longrightarrow +\infty$.
*Since* $\frac{k}{2}\{\sum_{j=1}^{m}[max(0, g_j(y))]^2 + \sum_{l=1}^{r}[h_l(y)]^2\} \geq 0$, *therefore from (4) and (9), we have* $\langle F(y^*), y^k\rangle + \frac{1}{2}\|y^k - y^*\|^2 \leq 0$. *As* $y^*$ *is a unique solution of the problem (4), we conclude that* $z = y^*$. *Therefore,* $y^k \to y^*$ *and* $\|y^k - y^*\| < \delta$ *for all* $k$ *sufficiently large. As* $y^k$ *is a solution of the smooth problem (5) and it is an interior point of the feasible set, for sufficiently large* $k$, *therefore from Proposition 2.1, it follows that* $\nabla\Phi_k(y^k) = 0$, *then we have*

$$T(y^*) + (y^k - y^*) + \sum_{j=1}^{m} kg_j(y^k)_+\nabla g_j(y^k) + \sum_{l=1}^{r} kh_l(y^k)\nabla h_l(y^k) = 0. \tag{10}$$

*As* $kg_j(y^k)_+ = \mu_j^k$, $\tau_l^k = kh_l(y^k)$, *then from (10), we get*

$$F(y^k) + \sum_{j=1}^{m}\mu_j^k\nabla g_j(y^k) + \sum_{l=1}^{r}\tau_l^k\nabla h_l(y^k) = y^* - y^k \to 0,$$

*as* $y^k \to y^*$ *and* $F(y^k) \to F(y^*)$. *Thus CAKKT-VI conditions are satisfied.*

**Remark 3.1.** *In particular, if* $F$ *is replaced by* $\nabla f$, *CAKKT-VI conditions converges on CAKKT-OP conditions.*

**Remark 3.2.** *We are describing an example in the support of Theorem 3.1.*

**Example 3.1.** *Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by*

$$F({y_1}^k, {y_2}^k) = (2({y_1}^k - 2), {y_2}^k) \text{ subject to}$$
$$g(y_1, y_2) = y_1 + y_2 \leq 0 \text{ and}$$
$$h(y_1, y_2) = y_1 y_2 = 0.$$

*Here $y^k = (2 + \frac{1}{k}, \frac{1}{k}) \to (2, 0)$.*

*Consider the (OP-VI) problem,*

*(OP-VI) Minimize$\langle F(y^*), y\rangle$, such that $y \in \Omega$,*

$$\Omega = \{y \in \mathbb{R}^2 : g(x) \leq 0,\ h(x) = 0\}.$$

*Clearly, $y^k = (2 + \frac{1}{k}, \frac{1}{k}) \to y^* = (2, 0)$ is the solution of (OP-VI) problem. Now, we can easily check that $y^k \to y^*$ satisfies all the CAKKT-VI conditions. Take $\lambda = \frac{1}{k}$ and $\mu = -\frac{1}{k}$*

$$F(y_1, y_2) + \lambda \nabla h(y_1, y_2) + \mu \nabla g(y_1, y_2) \to 0,$$

*and*

$$\lambda h(y^k) = \frac{1}{k}(2 \times \frac{1}{k}) \to 0,$$
$$\mu g(y^k) = -\frac{1}{k} \times (2 + \frac{1}{k} + \frac{1}{k}) \to 0.$$

*Therefore, $y^* = (2, 0)$ is a CAKKT point.*

**Theorem 3.2.** *Let $\langle F(y^*), y\rangle$ and $g$ be convex and $h$ be affine. If $y^* \in \Omega$ satisfies CAKKT-VI, then $y^*$ is solution to $VI(F, \Omega)$.*

***Proof*** *Let $y \in \Omega$ be an arbitrary feasible point. From the convexity assumptions,*

$$\langle F(y^*), y\rangle \geq \langle F(y^*), y^k\rangle + \langle F(y^*), y - y^k\rangle,$$

$$g_j(y) \geq g_j(y^k) + \langle \nabla g_j(y^k), y - y^k\rangle, j = 1, ..., m,$$
$$h_l(y) = h_l(y^k) + \langle \nabla h_l(y^k), y - y^k\rangle, l = 1, ..., r.$$

*Multiplying appropriately by $\mu_j^k,\ \tau_l^k$ and adding, as $h_l(y)=0$ and $g_j(y)\leq 0,$ we have*

$$\begin{aligned}\langle F(y^*),y\rangle &\geq \langle F(y^*),y\rangle+\sum_{j=1}^{m}\mu_j^k g_j(y)+\sum_{l=1}^{r}\tau_l^k h_l(y)\\ &\geq \langle F(y^*),y^k\rangle+\sum_{j=1}^{m}\mu_j^k g_j(y^k)+\sum_{l=1}^{r}\tau_l^k h_l(y^k)\\ &\quad+\left\langle F(y^k)+\sum_{j=1}^{m}\mu_j^k\nabla g_j(y^k)+\sum_{l=1}^{r}\tau_l^k\nabla h_l(y^k),y-y^k\right\rangle\\ &=\langle F(y^*),y^*\rangle,\ \text{as } y^k\to y^*.\end{aligned}$$

*Thus, $y^*$ is the solution to $VI(F,\Omega)$.*

# 4 Complementary Approximate KKT Conditions and Vector Variational inequalities

*Consider the following vector optimization problems (VVI) problem*

$$\text{(VVI-MOP) Minimize } F(x),$$
$$\text{where } F(x)=(\langle F_1(x^*),x\rangle,...,\langle F_p(x^*),x\rangle),\ \text{subject to } x\in\Omega,$$

*A point $x^*\in\Omega$ is an efficient solution of $VVI-MOP$ iff there exists no $x\in\Omega$ such that $F(x)\leq F(x^*),F(x)\neq F(x^*)$. The set of all efficient solution of $VVI-MOP$ is denoted by Min$(F,\Omega)$*

*We establish the complementary approximate Karush Kuhn Tucker necessary and sufficient optimality conditions for vector variational inequality problems, which is natural extension of the results on scalar optimization problems given by Andreani $et\ al.$ [10].*

**Definition 4.1.** *(CAKKT-VVI Conditions) We say that complementary approximate Karush-Kuhn-Tucker (CAKKT-VVI) conditions are satisfied at a feasible point $x^*\in\Omega$ iff there exists sequences $(x^k)\subset\mathbb{R}^n$ and $(\lambda^k,\mu^k,\tau^k)\subset\mathbb{R}^p_+\times\mathbb{R}^m_+\times\mathbb{R}^r$, such that*

*(C1)* $x^k\to x^*,$

*(C2)* $\sum_{i=1}^{p}\lambda_i^k F_i(x^k)+\sum_{j=1}^{m}\mu_j^k\nabla g_j(x^k)+\sum_{l=1}^{r}\tau_l^k\nabla h_l(x^k)\to 0,\ \sum_{i=1}^{p}\lambda_i^k=1,$

*(C3)* $\lim_{k\to\infty}\mu_j^k g_j(x^k)=0,\ \forall\, j=1,\ldots,m,\ \lim_{k\to\infty}\tau_l^k h_l(x^k)=0,\ \forall\, l=1,\ldots,r,$

*Points satisfying the CAKKT-VVI conditions are called CAKKT-VVI points. Note that the sequence $x^k$ is not necessarily in feasible set. In order to establish necessary optimality conditions for the problem VVI-MOP, we need to scalarize through the following nonsmooth function:*

$$\mathcal{F}:\mathbb{R}^p\to\mathbb{R},\ \text{defined by } \mathcal{F}(y)=\max\{y_i\},$$

*clearly $\mathcal{F}(y)\leq 0\Leftrightarrow y\leq 0$ and $\mathcal{F}(y)<0\Leftrightarrow y<0$. Moreover, for the sake of reader's convenience we recall the following well known result from Giorgi $et\ al.$ [3]:*

**Lemma 4.1.** *[3] If $x^*$ is solution of $VVI(F,\Omega)$, then $x^*$ is solution of Min$(\mathcal{F}(F(.)-F(x^*)),\Omega)$.*

*The following necessary optimality conditions for multiobjective optimization problem for local efficient solution of (MOP) to be a complementary approximate Karush Kuhn Tucker point will be helpful to develop the proof in section 4.*

**Theorem 4.1.** *If $x^* \in \Omega$ is solution of $VVI(F,\Omega)$, then $x^*$ satisfies the CAKKT-VVI conditions.*

***Proof*** *Since $x^*$ is local solution of $VVI(F,\Omega)$, so by Lemma 4.1 there exists $\delta > 0$, such that $x^* \in$ Min $\{\mathcal{F}(F(\cdot)-F(x^*)),\ \Omega\cap\bar{B}(x^*,\delta)\}$,*

$$\textit{Min } \mathcal{F}(F(x)-F(x^*)) + \frac{1}{2}\|x-x^*\|^2, \textit{ subject to } x \in \Omega\cap\bar{B}(x^*,\delta). \tag{11}$$

*Therefore, we may suppose that $x^*$ is the unique solution of the problem (11). We define the following function:*

$$\begin{aligned}\varphi_{\rho_k}(x) = \mathcal{F}(F(x)-F(x^*)) + \frac{1}{2}\|x-x^*\|^2 + \frac{\rho_k}{2}\{\sum_{j=1}^{m}[max(0,g_j(x))]^2\} \\ + \sum_{l=1}^{r}[h_l(x)]^2\}, \textit{for all } \rho_k > 0, \textit{ and } \rho_k \to \infty.\end{aligned} \tag{12}$$

*Let $x^k$ be a solution of the problem*

$$\textit{Min } \ \varphi_{\rho_k}(x), \textit{ subject to } \|x-x^*\| \leq \delta. \tag{13}$$

*By the convergence property of penalty methods [9], we have*

$$\begin{aligned}\mathcal{F}(F(x^k)-F(x^*)) + \frac{1}{2}\|x^k-x^*\|^2 + \frac{\rho_k}{2}\Big\{\sum_{j=1}^{m}[max(0,g_j(x^k))]^2 + \sum_{l=1}^{r}[h_l(x^k)]^2\Big\} \\ \leq \mathcal{F}(f(x^*)-f(x^*))\end{aligned}$$

*that is,*

$$\mathcal{F}(F(x^k)-F(x^*)) + \frac{1}{2}\|x^k-x^*\|^2 + \frac{\rho_k}{2}\Big\{\sum_{j=1}^{m}[max(0,g_j(x^k))]^2 + \sum_{l=1}^{r}[h_l(x^k)]^2\Big\} \leq 0.$$

*Suppose that $\mu_j^k = (\rho_k g_j(x^k))_+ \geq 0$ and $\tau_l^k = \rho_k h_l(x^k)$, then we have*

$$\mathcal{F}(F(x^k)-F(x^*)) + \frac{1}{2}\|x^k-x^*\|^2 + \frac{1}{2}\{\sum_{j=1}^{m}|\mu_j^k g_j(x^k)_+| + \sum_{l=1}^{r}|\tau_l^k h_l(x^k)|\} \leq 0. \tag{14}$$

*By the convergence property of exact penalty methods [9], taking the $\lim_{k\to\infty} x^k = x^*$, $\rho_k \to \infty$ and by the continuity of $F$, we have*

$$\lim_{x^k\to x^*} \frac{1}{2}\|x^k-x^*\|^2 + \frac{1}{2}\Big\{\sum_{j=1}^{m}|\mu_j^k g_j(x^k)_+|\sum_{l=1}^{r}|\tau_l^k h_l(x^k)|\Big\} = 0.$$

*Therefore, we have*

$$\lim_{k\to\infty} \mu_j^k g_j(x^k) = \lim_{k\to\infty} \mu_j^k g_j(x^k)_+ = 0 \text{ and } \lim_{k\to\infty} \tau_l^k h_l(x^k) = 0.$$

*as $x^k \to x^*$, hence the conditions (C4).*

*In (13) we observe that $x^k$ exists because $\varphi_{\rho_k}(x)$ is continuous and $\bar{B}(x^*, \delta)$ is compact. Let $z$ be a limit point of $x^k$. We can assume that $x^k \to z$. From the problem (12), we have*

$$\mathcal{F}(F(x^k) - F(x^*)) \leq \varphi_{\rho_k}(x^k),$$

*because of*

$$\varphi_{\rho_k}(x^k) - \mathcal{F}(F(x^k) - F(x^*)) = \frac{1}{2}\|x^k - x^*\|^2 + \frac{\rho_k}{2}\Big\{\sum_{j=1}^{m}[max(0, g_j(x^k))]^2 + \sum_{l=1}^{r}[h_l(x^k)]^2\Big\} \geq 0.$$

*Since $x^*$ is a feasible solution of the problem (11) and $x^k$ is the solution of problem (13), we have*

$$\varphi_{\rho_k}(x^k) \leq \varphi_{\rho_k}(x^*) = 0. \tag{15}$$

*We claim that $z$ is a feasible solution of the Problem (11). Since $\|x^k - x^*\| \leq \delta$, therefore $\|z - x^*\| < \delta$, suppose if possible*

$$\sum_{j=1}^{m}(g_j(z)_+)^2 + \sum_{l=1}^{r} h_l^2(z) > 0,$$

*for sufficiently large k, then there exists $c > 0$, such that*

$$\sum_{j=1}^{m}(g_j(x^k)_+)^2 + \sum_{l=1}^{r} h_l^2(x^k) > c.$$

*Therefore,*

$$\sum_{j=1}^{m}[max(0, g_j(x^k))]^2 + \sum_{l=1}^{r}[h_l(x^k)]^2 > c,$$

*for all $k$ large enough. From continuity of $\mathcal{F}$ and $x^k \to z$, we have*

$$\varphi_{\rho_k}(x) = \mathcal{F}(F(x) - F(x^*)) + \frac{1}{2}\|x - x^*\|^2 + \frac{\rho_k}{2}\Big\{\sum_{j=1}^{m}[max(0, g_j(x))]^2 + \sum_{l=1}^{r}[h_l(x)]^2\} > F(f(x) - f(x^*)) + \frac{\rho_k c}{2}.$$

*Taking the limit $k \to \infty$, we obtain $\varphi_{\rho_k}(x^k) \longrightarrow +\infty$, which contradicts (15). Consequently, $\sum_{j=1}^{m}(g_j(z)_+)^2 +$*

$\sum_{l=1}^{r} h_l^2(z) = 0$, *that is,* $z \in \Omega \cap \bar{B}(x^*, \delta)$, *therefore from (14), we obtain*

$$\begin{aligned} \varphi_{\rho_k}(x^k) = \mathcal{F}(F(x^k) - F(x^*)) + \frac{1}{2}\|x^k - x^*\|^2 \\ +\frac{\rho_k}{2}\Big\{\sum_{j=1}^{m}[max(0, g_j(x^k))]^2 + \sum_{l=1}^{r}[h_l(x^k)]^2\Big\} \leq 0, \text{ as } k \longrightarrow +\infty. \end{aligned} \tag{16}$$

*Since* $\frac{\rho_k}{2}\{\sum_{j=1}^{m}[max(0, g_j(x))]^2 + \sum_{l=1}^{r}[h_l(x)]^2\} \geq 0$, *therefore from (16), we have* $\mathcal{F}(F(x^k) - F(x^*)) + \frac{1}{2}\|x^k - x^*\|^2 \leq 0$. *As* $x^*$ *is a unique solution of the problem (11), we conclude that* $z = x^*$. *Therefore,* $x^k \longrightarrow x^*$ *and* $\|x^k - x^*\| < \delta$ *for all* $k$ *sufficiently large. As* $x^k$ *is a solution of the nonsmooth problem (13) and it is an interior point of the feasible set, for sufficiently large* $k$, *from Proposition 2.1, it follows that* $0 \in \partial^c \varphi_{\rho_k}(x^k)$. *Then, we have*

$$0 \in conv(\bigcup_{i=1}^{p}\{F_i(x^*)\}) + (x^k - x^*) + \sum_{j=1}^{m} \rho_k g_j(x^k)_+ \nabla g_j(x^k) + \sum_{l=1}^{r} \rho_k h_l(x^k) \nabla h_l(x^k). \tag{17}$$

*Hence, there exists* $\lambda_i^k \geq 0,\ i = 1, 2, ..., p$, *such that* $\sum_{i=1}^{p} \lambda_i^k = 1$ *and as* $\rho_k g_j(x^k)_+ = \mu_j^k,\ \rho_k h_l(x^k) = \tau_l^k$, *then from* (17), *we get*

$$\sum_{i=1}^{p} \lambda_i^k F_i(x^k) + \sum_{j=1}^{m} \mu_j^k \nabla g_j(x^k) + \sum_{l=1}^{r} \tau_l^k \nabla h_l(x^k) = x^* - x^k \to 0,$$

*as* $x^k \longrightarrow x^*$ *and* $F_i(x^k) \to F_i(x^*)$.

**Remark 4.1.** *If we redefine* $F_i = \nabla f_i,\ i = 1, ..., p$. *Then, CAKKT-VVI coincide on CAKKT-MOP conditions which is good approach to access the optimality conditions of multiobjective optimization problems.*

**Example 4.1.** *Consider the multiobjective optimization problem:*

$$\begin{aligned} Min\ f(x_1, x_2) = (f_1(x_1, x_2), f_2(x_1, x_2)) \\ subject\ to\ h(x_1, x_2) = x_2 - x_1 = 0, \\ and\ g(x_1, x_2) = x_1^2 - x_2 \leq 0, \\ where\ f_1(x_1, x_2) = x_1 - x_2^2\ and\ f_2(x_1, x_2) = x_1 - x_2. \end{aligned}$$

*The point* $x^0 = (1, 1)$ *is a weak efficient solution of the above problem. In order to find sequences satisfying the conditions (C1), (C2), (C3) and (C4), we solve the equation*

$$\lambda_1 \nabla f_1(x_1, x_2) + \lambda_2 \nabla f_2(x_1, x_2) + \mu_1 \nabla h(x_1, x_2) + \mu_2 \nabla g(x_1, x_2) = (0, 0).$$

*Consider the sequence* $x^k = (1 + \frac{1}{k}, 1 + \frac{1}{k}),\ k \in \mathbb{N}$, *then*
$\lambda_i^k = \{\frac{1}{2} + \frac{1}{k}\},\ i = 1, 2;\ \mu_1^k = \{2 + \frac{1}{k}\},\ \mu_2^k = \{\frac{1}{2} + \frac{1}{k}\}$.
*Then we get*

$$\lim_{k\to\infty} \lambda_1^k \nabla f_1(x_1^k, x_2^k) + \lambda_2^k \nabla f_2(x_1^k, x_2^k) + \mu_1^k \nabla h(x_1^k, x_2^k) + \mu_2^k \nabla g(x_1^k, x_2^k) = (0, 0),$$

$$\sum_{i=1}^{p} \lambda_i^k = 1,$$

$$\mu_1^k h(x^k) = (2 + \frac{1}{k}) \times (1 + \frac{1}{k} - 1 - \frac{1}{k}) \to 0,$$
$$\mu_2^k g(x^k) = (\frac{1}{2} + \frac{1}{k}) \times ((1 + \frac{1}{k})^2 - 1 - \frac{1}{k}) \to 0.$$

*Hence, CAKKT-MOP conditions are satisfied at* $x^0 = (1, 1)$.

*The following result is sufficient optimality conditions for the VVI-MOP problem*

**Theorem 4.2.** *Assume that* $\langle F_i(x^*), x\rangle;\ i = 1, ..., p,\ g_j;\ j = 1, ..., m$ *are convex and* $h_l;\ l = 1, ..., r$ *are affine. If* $x^* \in S$ *satisfies the CAKKT-VVI conditions, then* $x^*$ *is a global weak efficient solution of (VVI-MOP).*

***Proof*** *Suppose that* $x^*$ *is not a weakly efficient solution then, there exists* $\bar{x} \in S$ *such that*

$$\langle F_i(x^*), \bar{x}\rangle < \langle F_i(x^*), x^*\rangle, i = 1, 2, ..., p. \tag{18}$$

*Let* $(x^k)$ *and* $(\lambda^k, \mu^k)$ *be the sequences that satisfies the CAKKT-VVI at* $x^*$*. Therefore, without loss of generality we may assume that* $\lambda^k \to \lambda^*$ *with* $\lambda^* \geq 0$ *and* $\sum_{i=1}^{p} \lambda_i^* = 1$*. As* $f_i, g_j$ *are convex and* $h_l$ *are affine, for all* $k$ *we get*

$$\langle F_i(x^*), \bar{x}\rangle \geq \langle F_i(x^*), x^k\rangle + \langle F_i(x^k), \bar{x} - x^k\rangle, \forall\, i = 1, ..., p, \tag{19}$$

$$g_j(\bar{x}) \geq g_j(x^k) + \langle \nabla g_j(x^k), \bar{x} - x^k\rangle, \forall\, j = 1, ..., m, \tag{20}$$

$$h_l(\bar{x}) = h_l(x^k) + \langle \nabla h_l(x^k),\ \bar{x} - x^k\rangle, \forall\, l = 1, ..., m. \tag{21}$$

*Since* $\bar{x}$ *is feasible point, therefore we can write*

$$\sum_{i=1}^{p} \lambda_i^k \langle F_i(x^*), \bar{x}\rangle \geq \sum_{i=1}^{p} \lambda_i^k \langle F_i(x^*), \bar{x}\rangle + \sum_{j=1}^{m} \mu_j^k g_j(\bar{x}) + \sum_{l=1}^{r} \tau_l^k h_l(\bar{x}) \tag{22}$$

*From (19), (20), (21) and (22), we get*

$$\begin{aligned} \sum_{i=1}^{p} \lambda_i^k \langle F_i(x^*), \bar{x}\rangle &\geq \sum_{i=1}^{p} \lambda_i^k \langle F_i(x^*), x^k\rangle + \sum_{j=1}^{m} \mu_j^k g_j(x^k) + \sum_{l=1}^{r} \tau_l^k h_l(x^k) \\ &+ \left\langle \sum_{i=1}^{p} \lambda_i^k F_i(x^*) + \sum_{j=1}^{m} \mu_j^k \nabla g_j(x^k) + \sum_{l=1}^{r} \tau_l^k \nabla h_l(x^k),\ \bar{x} - x^k \right\rangle. \end{aligned} \tag{23}$$

*Making use of* $(C1) - (C3)$ *in above inequality, we get*

$$\sum_{i=1}^{p} \lambda_i^k \langle F_i(x^*), \bar{x}\rangle \geq \sum_{i=1}^{p} \lambda_i^k \langle F_i(x^*), x^*\rangle, \text{ since } F_i(x^k) \to F_i(x^*)$$

*as $x^k \to x^*$, which contradicts (18).*

# 5 Conclusions

*In this paper, we use variational inequality and vector variational inequality problems as a tool to deduce necessary and sufficient optimality conditions without using any of the constraint qualifications. We proposed new sequential optimality conditions with variational and vector variational inequalities inspired by the work of Andreani et al. [10]. We use practical consequences of variational inequality problems as we can see, the minimization problem can be change into variational inequality problem when we use gradient of the function in place of the function. We utilize this concept to find Complementary Approximate Karush Kuhn Tucker (CAKKT) conditions in the form of our objective function. In general, not all the cases are smooth so the non-smooth analysis of this work is possible in future.*

# Acknowledgment

*The first author is supported by CSIR-UGC JRF, New Delhi, India, through grant no.: 1272/(CSIR-UGC NET DEC.2016). The second author is supported by the Council of Scientific and Industrial Research New Delhi, Ministry of Human Resources Development, Government of India through grant no.: 09/013(0583)/2015-EMR-I.*

# References

[1] *F. Giannessi, Theorems of alternative, quadratic programs and complementarity problems, Wiley, Chichester, 1980.*

[2] *F. H. Clarke, Optimization and nonsmooth analysis, John Wiley & Sons, Inc., New York, 1983.*

[3] *G. Giorgi, B. Jiménez, V. Novo, Approximate Karush-Kuhn-Tucker condition in multiobjective optimization,* Journal of Optimization Theory and Applications 171 *(2016), 70–89.*

[4] *G. Haeser, M. Schuverdt, On approximate KKT condition and its extension to continuous variational inequalities,* Journal of Optimization Theory and Applications 149 *(2011), 528–539.*

[5] *G. Mastroeni, Some remarks on the role of generalized convexity in the theory of variational inequalities, Pitagora, Bologna, 1999.*

[6] *H. W. Kuhn, A. W. Tucker, Nonlinear programming, University of California Press, Berkeley and Los Angeles, 1951.*

[7] *J.-L. Lions, G. Stampacchia, Variational inequalities,* Communications on Pure and Applied Mathematics 20 *(1967), 493–519.*

[8] *J. M. Martínez, B. F. Svaiter, A practical optimality condition without constraint qualifications for nonlinear programming,* Journal of Optimization Theory and Applications 118 *(2003), 117–133.*

[9] *M. R. Hestenes, Optimization theory: the finite dimensional case, John Wiley & Sons, New York, 1975.*

[10] *R. Andreani, J. M. Martínez, B. F. Svaiter, A new sequential optimality condition for constrained optimization and algorithmic consequences,* SIAM Journal on Optimization 20 *(2010), 3533–3554.*

[11] *R. Andreani, G. Haeser, J. M. Martínez, On sequential optimality conditions for smooth constrained optimization,* Optimization 60 *(2011), 627–641.*

[12] *R. Andreani, J. M. Martínez, A. Ramos, P. J. Silva, Strict constraint qualifications and sequential optimality conditions for constrained optimization,* Mathematics of Operations Research 43 *(2018), 693–717.*

[13] *V. Laha, S. K. Mishra, On vector optimization problems and vector variational inequalities using convexificators,* Optimization 66 *(2017), 1837–1850.*