An extension of stochastic differential models by using the Grunwald-Letnikov fractional derivative

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Abstract

Stochastic differential equations (SDEs) have been applied by engineers and economists because it can express the behavior of stochastic processes in compact expressions. In this paper, by using Grunwald-Letnikov fractional derivative, the stochastic differential model is improved. Two numerical examples are presented to show efficiency of the proposed model. A numerical optimization approach based on least square approximation is applied to determine the order of the fractional derivative. Numerical examples show that the proposed model works better than the SDE to model stochastic processes with memory.

1 Introduction

Differential equations are applied to model different processes in science and engineering. These equations with integer orders, cannot describe processes with memory but many processes in finance and economics depend on the history of previous changes. In mathematics, fractional derivatives have memory. Recently, many models based on applications of fractional derivatives and integrals, have been proposed to describe behavior of financial and economical processes. Fractional derivatives are extensions of integer derivatives, in a way that derivatives are defined for arbitrary real order. In many phenomena, fractional derivatives allow to model better than integer derivatives (for further see [7,8] and references therein). There are different definitions for fractional derivatives which do not coincide in general. The most used ones are Caputo, Riemann-Louville, and Grunwald-Letnikov fractional derivatives [6]. This paper regards one of them, namely the Grunwald-Letnikov.

Definition 1.1 *Let $\alpha > 0$ and n be the smallest natural number such that $\alpha \leq n$. If $f \in \mathcal{C}^n[a, x]$ then Grunwald-Letnikov fractional derivative is defined as [6]*

$$d^\alpha f(x) = \lim_{h\to 0} \frac{1}{h^n} \sum_{m=0}^{n} (-1)^m \frac{\Gamma(\alpha+1)}{m!\Gamma(\alpha-m+1)} f(x-mh), \tag{1.1}$$

where $h = \dfrac{x-a}{n}$. It is possible to present the Grunwald-Letnikov derivative in a rather compact notation

$$d^\alpha f(x) = \lim_{h\to 0} \frac{1}{h^n} \sum_{m=0}^{n} w_m^{(\alpha)} f(x-mh), \tag{1.2}$$

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where $w_m^{(\alpha)}$ is defined

$$w_m^{(\alpha)} = (-1)^m \frac{\Gamma(\alpha+1)}{m!\Gamma(\alpha-m+1)} = (-1)^m \binom{\alpha}{m}. \tag{1.3}$$

It is easy to show that

$$w_m^{(\alpha)} = \left(1 - \frac{\alpha+1}{m}\right) w_{m-1}^{(\alpha)}. \tag{1.4}$$

Additionally, let $0 < \alpha < 1$. Then

1. $w_0^{(\alpha)} = 1$ and $w_k^{(\alpha)} \leqslant 0$ for any $k \geqslant 1$.

2. $|w_{k+1}^{(\alpha)}| < |w_k^{(\alpha)}| < w_0^{(\alpha)}$ for any $k \geqslant 1$.

3. $\sum_{k=0}^{\infty} w_k^{(\alpha)} = 0$.

This definition can be easily approximated by taking parameter h small enough. The relation (1) can be rewritten for $-\alpha$ (Grunwald-Letnikov fractional integral) as

Definition 1.2 *Let $\alpha > 0$ and n be the smallest natural number such that $\alpha \leq n$. If $f \in \mathcal{C}^n[a,x]$ then Grunwald-Letnikov fractional integral is defined as [6]*

$$d^{-\alpha} f(x) = \lim_{h\to 0} \frac{1}{h^n} \sum_{m=0}^{n} \frac{\Gamma(\alpha+m)}{m!\Gamma(\alpha)} f(x-mh), \tag{1.5}$$

where $h = \dfrac{x-a}{n}$.

This paper is organized as follows. In section 2, a new fractional stochastic model based on SDEs are presented. In section 3, to show efficiency of the proposed model, two numerical examples are presented. Finally, in section 4, conclusions are expressed.

2 Main results

SDE is a differential equation in which one or more of the terms has a random components. In general, SDE has the following form

$$\begin{cases} dX(t) = \mu(X(t),t)dt + \sigma(X(t),t)dW(t), \\ X(0) = x_0. \end{cases} \tag{2.1}$$

where $\mu(X(t),t)$ is drift term, and $\sigma(X(t),t)$ is diffusion term. Solution of the equation (6), based on Euler-Maruyama method, can be approximated by

$$\begin{cases} X(t+\Delta t) = X(t) + \mu(X(t),t)\Delta t + \sigma(X(t),t)(W(t+\Delta t) - W(t)), \\ X(0) = x_0. \end{cases} \tag{2.2}$$

SDEs are used in biology (for e.g. in the epidemic models, predator-prey models, and population models), physics (for e.g. in the ion transport, nuclear reactor kinetics, chemical reaction, and cotton fiber breakage), and stochastic control to model various phenomena [1-5]. In this paper, the equation (6) is improved as

$$\begin{cases} d^{\alpha}X(t) = \mu(X(t),t)dt + \sigma(X(t),t)dW(t), \\ \\ X(0) = x_0. \end{cases} \tag{2.3}$$

where $\alpha \in (0, 1]$. Therefore, using the relation (1) result in

$$\lim_{h\to 0} \frac{1}{h^n} \sum_{m=0}^{n} (-1)^m \frac{\Gamma(\alpha+1)}{m!\Gamma(\alpha-m+1)} X(t-mh) = \mu(X(t),t)dt + \sigma(X(t),t)dW(t). \tag{2.4}$$

Finally, based on observed data, by truncating an infinite sum and approximating above sum by a finite sum, the order of fractional derivative (i.e. α) is obtained.

In the next section, to show efficiency of the proposed method, two numerical examples are presented.

3 Numerical examples

Example 1 As a first example, consider the well known Black-Scholes SDE

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma dW(t). \tag{3.1}$$

The solution of this SDE is the classical geometric Brownian motion and it is given by

$$S(t) = S(0) \exp\left\{ \left(\mu - \frac{1}{2}\sigma^2 \right) + \sigma W(t) \right\}. \tag{3.2}$$

In this model, it is assumed that the price of the underlying asset (typically a stock) follows a geometric Brownian motion. By discretization (10) along a family $t_0, t_1, t_2, \cdots, t_N$ of observation times, μ and σ are estimated as

$$\begin{cases} \mu = \dfrac{1}{N} \displaystyle\sum_{k=0}^{N-1} \frac{1}{t_{k+1}-t_k} \left(\frac{S_{t_{k+1}} - St_k}{S_{t_k}} \right) \\ \\ \sigma^2 = \dfrac{1}{N-1} \displaystyle\sum_{k=0}^{N-1} \frac{1}{t_{k+1}-t_k} \left(\frac{S_{t_{k+1}} - St_k}{S_{t_k}} - \mu(t_{k+1}-t_k) \right)^2. \end{cases} \tag{3.3}$$

Now, by considering the relation (6) for the following data $x[1] = 44.48$, $x[2] = 45.15$, $x[3] = 45.73$, $x[4] = 45.73$, $x[5] = 46.31$, $x[6] = 46.92$, $x[7] =$47.76, $x[8] = 48.27$, $x[9] = 49.58$, $x[10] = 52.19$, $x[11] = 52.42$, $x[12] = 51.44$, $x[13] = 50.31$, $x[14] = 51.80$, $x[15] = 52.08$, $x[16] = 51.83$, $x[17] = 53.60$, $x[18] = 53.095$, $x[19] = 52.59$, $x[20] = 52.44$, $x[21] = 52.94$, $x[22] = 53.53$, $x[23] = 51.79$, $x[24] = 53.07$, $x[25] = 54.18$, $x[26] = 53.84$, and $x[27] = 55.29$ (crude oil price West Texas Intermediate (WTI) from 2018-12-27 to 2019-02-01) μ and σ^2 are obtained as $\mu = 0.007001$ and $\sigma^2 = 0.000412$. Above data are plotted in figure 1.

In 1000 times simulation by formula (7), the oil price in 2019-02-02 ($x[28] = 54.57$) is approximated by 55.70417463. Therefore absolute error and relative error are 1.13417463 and 0.020783849. Now, by considering the equation (8),

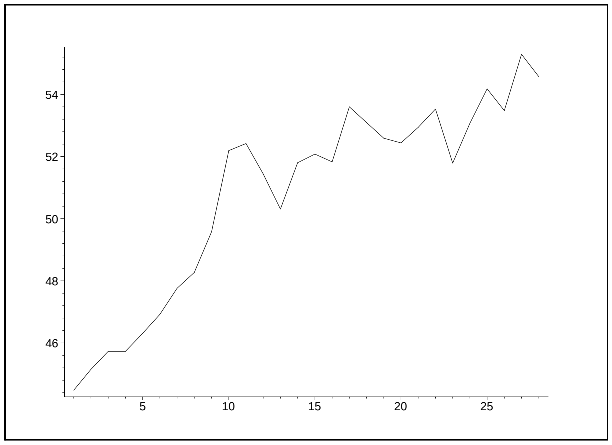

Figure 1: Crude oil price West Texas Intermediate (WTI) from 2018-12-27 to 2019-02-01.

let

$$d^{\alpha}S(t) = \mu S(t)dt + \sigma S(t)dW(t). \tag{3.4}$$

Discretizing the relation (13) along the family $t_0, t_1, \cdots, t_N$, result in

$$d^{\alpha}S(t_{k+1}) = \mu S(t_k)(t_{k+1} - t_k) + \sigma S(t_k)(W(t_{k+1}) - W(t_k)), \;\; i = m \cdots, N-1, \tag{3.5}$$

where α is the order of approximation of fractional derivative. μ and σ can be calculated by equations (12). In addition, by taking expectation from both sides of the relation (14), α is determined to minimize the following function

$$E(\alpha) = \frac{1}{N-m} \sum_{k=m}^{N-1} \left[d^{\alpha}S(t_{k+1}) - \mu S(t_k)(t_{k+1} - t_k) \right]^2. \tag{3.6}$$

Hence, using the relation (9) results in

$$E(\alpha) = \frac{1}{N-m} \sum_{k=m}^{N-1} \left[\frac{1}{h^n} \sum_{i=0}^{m} (-1)^i \frac{\Gamma(\alpha+1)}{i!\Gamma(\alpha-i+1)} S(t_{k+1-i}) - \mu S(t_k)(t_{k+1} - t_k) \right]^2. \tag{3.7}$$

Using the above data, and with 3-term approximation for fractional derivative, $\alpha = 0.99$ minimize above relation. Therefore, by running simulation 1000 times, $x[28]$ is approximated by 55.34945932. The absolute error and relative error are 0.77945932 and 0.01428366. In this example, by using the above mentioned data, the proposed model can predict better. It seems that in stochastic processes with memory (i.e. stochastic processes without jump), using more data from past, result in better prediction.

Example 2 As a second example, consider the following SDE

$$dX(t) = \theta_1 X(t)dt + \sqrt{\theta_2 X(t)}dW(t), \tag{3.8}$$

where $X(t)$ is population size and $\theta = [\theta_1, \theta_2]$ is to be determine. This SDE is applied to model Aransas-Wood Buffalo population of whooping cranes [1]. The population size over the years 1939-1985 are $x[1939] = 18$, $x[1340] = 22$, $x[1941] = 26$, $x[1942] = 16$, $x[1943] = 19$, $x[1944] = 21$, $x[1945] = 18$, $x[1946] = 22$, $x[1947] = 25$,

$x[1348] = 31$, $x[1949] = 30$, $x[1950] = 34$, $x[1951] = 31$, $x[1952] = 25$, $x[1953] = 21$, $x[1954] = 24$, $x[1955] = 21$, $x[1356] = 28$, $x[1957] = 24$, $x[1958] = 26$, $x[1959] = 32$, $x[1960] = 33$, $x[1961] = 36$, $x[1962] = 39$, $x[1963] = 32$, $x[1364] = 33$, $x[1965] = 42$, $x[1966] = 44$, $x[1967] = 43$, $x[1968] = 48$, $x[1969] = 50$, $x[1970] = 56$, $x[1971] = 57$, $x[1372] = 59$, $x[1973] = 51$, $x[1974] = 49$, $x[1975] = 49$, $x[1976] = 57$, $x[1977] = 69$, $x[1978] = 72$, $x[1979] = 75$, $x[1380] = 76$, $x[1981] = 78$, $x[1982] = 73$, $x[1983] = 73$, $x[1984] = 75$, and $x[1985] = 86$. These data are plotted in Figure 2.

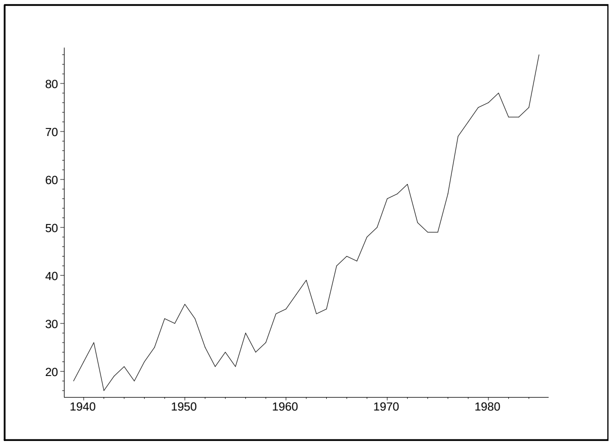

Figure 2: Population data of whooping cranes in Wood Buffalo National Park in Canada

As in the previous example, it is assumed that $x_0, x_1, x_2, \cdots, x_N$ are observed values of $X(t)$. θ_1 and θ_2 are obtained as [1]

$$\begin{cases} \theta_1 = \dfrac{\displaystyle\sum_{i=0}^{N-1}(x_{i+1}-x_i)}{\displaystyle\sum_{i=0}^{N-1} x_i}, \\ \\ \theta_2 = \dfrac{\displaystyle\sum_{i=0}^{N-1}(x_{i+1}-x_i)^2}{\displaystyle\sum_{i=0}^{N-1} x_i}. \end{cases} \tag{3.9}$$

Now, using the above relations for $x[1939], x[1940], \cdots, x[1984]$ result in $\theta_1 = \frac{57}{1808}$ and $\theta_2 = \frac{969}{1808}$. In 1000 times simulations, $x[1985]$ is approximated by 77.23692513. Therefore absolute error and relative error are 2.23692513 and 0.029825668. Now, by considering the equation (8), let

$$d^\alpha X(t) = \theta X(t)dt + \sqrt{\theta X(t)}dW(t). \tag{3.10}$$

Discretizing the relation (19) along the family $t_0, t_1, \cdots, t_N$, result in

$$d^\alpha X(t_{k+1}) = \theta X(t_k)(t_{k+1}-t_k) + \sqrt{\theta_2 X(t_k)}(W(t_{k+1}) - W(t_k)), \ \ i = m\cdots, N-1. \tag{3.11}$$

where m is the order of approximation of fractional derivative. By taking expectation from both sides of the relation (20), α is determined to minimize a function

$$E(\alpha) = \frac{1}{N-m} \sum_{k=m}^{N-1} \left[d^{\alpha} X(t_{k+1}) - \theta_1 X(t_k)(t_{k+1} - t_k)\right]^2. \tag{3.12}$$

Hence, the relation (9) simplifies (21) as

$$E(\alpha) = \frac{1}{N-m} \sum_{k=m}^{N-1} \left[\frac{1}{h^n} \sum_{i=0}^{m} (-1)^i \frac{\Gamma(\alpha+1)}{i!\Gamma(\alpha-i+1)} X(t_{k+1-i}) - \theta_1 X(t_k)(t_{k+1} - t_k)\right]^2. \tag{3.13}$$

Using the above data, and with 3-term approximation for fractional derivative, result in $\alpha = 0.99$. Therefore, in 1000 times simulations, $x[1985]$ is approximated by 77.0807717. The absolute error and relative error are 8.91922829 and 0.103711957. Since the absolute and relative errors of the proposed model is more than the corresponding errors of the relation (8), so it cannot predict better. It seems that the jump in $x[1985]$ results in that the proposed model does not work as well as the (17) model. It seems that the jump leads the past and the future independence.

4 Conclusions

In this paper, for the first time, the classical SDE is improved. For this purpose, the Grunwald-Letnikov fractional derivative is applied. For $\alpha = 1$, the classical stochastic model is obtained (i.e. equation (6)). For $\alpha \in (0, 1)$, fractional SDE is obtained. Fractional derivatives have memory therefore can model stochastic processes with memory in economy and finance better that integer derivative. In the first example, due to memory in the first series, the proposed model predict better. In the second example, because of the jump in the model (i.e. in $x[1985]$), the data are independence and so the proposed model does not work as well as the SDE model.

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