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Strong Algebrability of C^* algebras

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ABSTRACT

In this paper, we introduce the concept strong algebrability of certain C^* algebras generated by finite generators. In fact, using Gelfand theorem, we identify the members of the C^* algebra generated by one element, with the continuous functions on its spectrum, and use some recent result for strong algebrability for functions spaces. Moreover, we introduce the new concept unitable elements in unital C^* algebras, and then we express our main result for this kind of elements. In fact, the C^* subalgebra generated by a non unitable element in a C^* algebra is strongly c algebrable. As the last result in this paper, we show 2^c strong algebrability of direct sums of C^* algebras, using non unitable elements of them.

1 Introduction

The concepts of lineability and algebrability have been investigated for several kind of spaces. Recently, there were published two surveys in this topic, that contain several examples. First, we recall the definition of these concepts, which its origins are in the works of R.M. Aron, V.I. Gurariy, D. Perez-Garcia, J.B. Seoane-Sepulveda.

Definition 1.1 Let κ be a cardinal number.

(1) Let L be a vector space and a set $A \subseteq L$. We say that A is κ-lineable if $A \cup \{0\}$ contains a κ-dimensional vector space;

(2) Let L be a Banach space and a set $A \subseteq L$. We say that A is spaceable if $A \cup \{0\}$ contains an infinite dimensional closed vector space;

(3) Let L be a linear commutative algebra and a set $A \subseteq L$. We say that A is κ-algebrable if A∪{0} contains a κ-generated algebra B (i.e. the minimal system of generators of B has cardinality κ).

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Moreover, A. Bartoszewicz and S. Glab in [8] went further and asked for existence of free structures inside some set $A \cup \{0\}$. They introduced the notion of strong algebrability.

Definition 1.2 Let κ be a cardinal number. Let L be a linear commutative algebra and a set A ⊆ L. We say that A is strongly κ-algebrable if $A \cup \{0\}$ contains a κ-generated algebra B that is isomorphic with a free algebra (denote by $X = \{x_\alpha : \alpha < \kappa\}$ the set of generators of this free algebra).

Note that every free algebra is a structure like a free group, that there is no non trivial relation between its generators. Consequently, every member of a free algebra has the following form

$$\boldsymbol{x_{\alpha_1}^{k_1} \dots x_{\alpha_n}^{k_n}}$$

Which is called a "word". Here, x_α's denotes the generators of X .
It is trivial that for any cardinal number κ, the following implications hold.

$$\kappa \text{ strong algebrability} \Rightarrow \kappa \text{ algebrability} \Rightarrow \kappa \text{ lineability.}$$

Moreover, since every infinite dimensional Banach space has a linear base of cardinality c, then

$$\text{spaceability} \Rightarrow \mathfrak{c} \text{ lineability.}$$

In the rest of this paper, the concept "exponential like" function is a key tool for our aims.

Definition 1.3 We say that a function $f: \mathbb{R} \to \mathbb{R}$ is exponential-like (of rank m) whenever f is given by

$$f(x) = \sum_{1}^{m} \alpha_i e^{\beta_i x}$$

for some distinct nonzero real numbers $\beta_1, \dots, \beta_m$ and some nonzero real numbers $\alpha_1, \dots, \alpha_m$. We will also consider exponential-like functions (of the same form) with the domain [0, 1].

Note that the set of exponential-like functions form an algebra in the space of real functions. In fact, if f, g are two exponential-like functions, then fg is also an exponential-like function.
To see this, let

$$f(x) = \sum_1^m \alpha_i e^{\beta_i x} \qquad , \qquad g(x) = \sum_1^n \alpha_j e^{\beta_j x}$$

Then we have

$$(fg)(x) = \sum_1^m \sum_1^n \alpha_i \alpha_j e^{(\beta_i + \beta_j)x}$$

There is a very useful criterion for characterization strong algebrability of some families of certain functions.

Theorem 1.1 Let $F \subseteq \mathbb{R}^{[0,1]}$ and assume that there exists a function $g \in F$ such that $fog \in F \cup \{0\}$. For every exponential-like function $f : \mathbb{R} \to \mathbb{R}$. Then F is strongly c-algebrable. More exactly, if $H \subseteq \mathbb{R}$ is a set of cardinality c and linearly independent over the rationals Q, then $expo(rf)$, $r \in H$, are free generators of an algebra contained in $F \cup \{0\}$.

In the following examples, we see the application of this theorem for some certain spaces.

Example 1.1 Let C($\mathbb{R}$) denote the set of continuous real functions. It is clear that every exponential-like function is continuous, moreover it is clear that the composition of two continuous functions is also continuous. So, the condition of the theorem1.1 is trivially hold, and consequently, the space C($\mathbb{R}$) is strongly c-algebrable.

Example 1.2 Let $C^n(\mathbb{R})$ denote the set of continuously n-times differentiable real functions. It is clear that every exponential-like function is n-times differentiable, moreover it is clear that the composition of two n-times differentiable functions is also n-times differentiable. So, the condition of the theorem1.1 is trivially hold, and consequently, the space $C^n(\mathbb{R})$ is strongly c algebrable.

2 Strong algebrability of finitely generated C^* algebra

Definition 2.1 A C^* algebra, is a Banach algebra A, which equipped with a mapping

$$*: A \to A$$

With the following properties;

i. $x^{**} = x$

ii. $(x + y)^* = x^* + y^*$

iii. $(xy)^* = y^* x^*$

iv. $(\alpha x + \beta y)^* = \bar{\alpha} x^* + \bar{\beta} y^*$

v. $\|x^* x\| = \|x\|^2$

For every x, y in A and every complex numbers α, β. Note that the last property is the most important property. It is not difficult to see that $\|x^*\| = \|x\|$, i.e. the operation * is an isometric isomorphism.

The set of complex number is the simplest C^* algebra, with the operation $z^* = \bar{z}$.

Example 2.1 Let H be a Hilbert space. Denote the set of bounded linear operators on H by B(H). For any S in B(H), there is a unique T in B(H) such that for any x, y in H,

$$\langle Sx, y \rangle = \langle x, Ty \rangle$$

In this case, the operator T is called the adjoint operator of S and is denoted by S^*. so

$$\langle Sx, y \rangle = \langle x, S^* y \rangle$$

It is well known that B(H) with this involution is a C^* algebra.

Definition 2.2 Let A be a C^* algebra, and assume that $a_1, \dots, a_n$ are elements of A.The C^* algebra generated by $a_1, \dots, a_n$ in A is the smallest C^* algebra contained in A which contains $a_1, \dots, a_n$. We denote this C^* algebra by $C^*(a_1, \dots, a_n)$.

There are some special cases for elements of a C^* algebra; here we see some of them;

i. An element x is called self adjoint if $x^* = x$.

ii. An element x is called normal if $x^* x = x x^*$.

iii. An element x is called unitary if $x^* x = 1 = x x^*$.

iv. An element x is called projection if it is self adjoint and idempotent. i.e. we have $x^* = x\, , x^2 = x$.

v. An element x is called positive if there exists an element y such that $= y^* y$.

For normal elements, we can say that the members of $C^*(a)$ has the following form

$$\overline{span}\,\{a^m(a^*)^n :\ m\,,n\ are\ integer\ numbers\}$$

Definition 2.3 Let A be a C^* algebra, and assume that a is an element of A . The spectrum of a is defined as the following set in complex numbers

$$\sigma(a) = \{\gamma : a - \gamma 1\ is\ not\ invertible\ in\ A\},$$

where 1 denotes the identity element of A .

Now, we express a key theorem in our paper, which is called Gelfand theorem;

Theorem 2.1 (E. Gelfand) Let A be a C^* algebra, and assume that a is a normal element in A. The C^* algebra generated by a in A is corresponding to the set of continuous functions over the spectrum of a. In other words, we have

$$C^*(a) \sim C\big(\sigma(a)\big),$$

where $C(\sigma(a))$ denotes the set of continuous functions with domain $\sigma(a)$.

In fact, this theorem asserts that the element a is corresponding to the identity map, i.e.

$$a \sim f(x) = x,$$

Moreover

$$a^* \sim f(x) = \bar{x},$$

And since the Gelfand map is an isometric isomorphism, we conclude that every polynomial

$$f(x) = \sum_1^N a_i x^{m_i} \bar{x}^{n_i},$$

Is corresponding to the element

$$\sum_1^N a_i a^{m_i} (a^*)^{n_i},$$

Gelfand theorem is a key theorem in the theory of C^* algebras, and has many consequences.

As an useful consequence of Gelfand theorem, we have the following result;

Theorem 2.2 Let A be a C^* algebra, and assume that a is an arbitrary element in A. The set of continuous functions over the spectrum of a can be imbedded in the C^* algebra generated by a in A. in other words, we have an onto map

$$C^*(a) \to C(\sigma(a))$$

This map is one to one if a is a normal. For example, both of two elements

$$a^*a \quad , \quad aa^*$$

Are mapped to the same function $(x) = \bar{x}x$.

Here $C(\sigma(a))$ denotes the set of continuous functions with domain (a) .

Theorem2.3. Let A be a C^* algebra, and assume x is an arbitrary element in A. Let f, g be two analytic functions over (x) . then the elements $f(x), g(x)$ commutes, i.e.

$$f(x)g(x) = g(x)f(x),$$

Proof. Since f, g are analytic, they have suitable Taylor series over (a) . assume that

$$f(x) = \sum_0^\infty \alpha_n x^n \quad , \quad g(x) = \sum_0^\infty \beta_n x^n$$

Are Taylor series corresponding to f, g, respectively. Therefore

$$f(x)g(x) = \left(\sum_0^\infty \alpha_n x^n\right)\left(\sum_0^\infty \beta_n x^n\right) = \sum_0^\infty \left(\sum_{i+j=n} (\alpha_i \beta_j)\right) x^n,$$

A similar calculations show that

$$g(x)f(x) = \left(\sum_0^\infty \beta_n x^n\right)\left(\sum_0^\infty \alpha_n x^n\right) = \sum_0^\infty (\sum_{i+j=n} (\alpha_i \beta_j)) x^n,$$

As claimed.

The following theorem is well known in the theory of C^* algebras.

Theorem 2.4 Let A be a C^* algebra, and assume a, b are arbitrary elements in A. If $exp(a), exp(b)$ commutes, then

$$exp(a)exp(b) = exp(a+b).$$

Corollary 2.1 Since the exponential function is analytic everywhere and entire, we have

$$exp(r_1 a)^{k_{i1}} exp(r_2 a)^{k_{i2}} \dots \exp(r_n a)^{k_{in}} = exp\left(\sum_1^n r_j k_{ij}\, a\right).$$

Theorem 2.5 Let A be a C^* algebra, and assume a is a normal element in A. in this case, the C^* algebra generated by a is strongly c algebrable.

Proof. Consider a set H of cardinality $\mathfrak{c}$, such that it is linearly independent over $\mathbb{Q}$.
By the Gelfand theorem, we have that $\{exp(ra) : r \in H\} \subseteq C^*(a)$. In fact, the element

$$exp(ra),$$

in $C^*(a)$ is corresponded to the function in $\sigma(a)$.
To show that it is a set of free generators, consider n ∈ N and a non zero polynomial P in n variables without a constant term. The function given by

$$a \to P\big(exp(r_1 a), exp(r_2 a), \dots, exp(r_n a)\big),$$

is of the form

$$\sum_1^m \alpha_i exp(r_1 a)^{k_{i1}} exp(r_2 a)^{k_{i2}} \dots exp(r_n a)^{k_{in}} = \sum_1^m \alpha_i exp(\sum_1^n r_j k_{ij}\, a),$$

where $\alpha_1, \dots, \alpha_m$ are nonzero real numbers and the matrix $[k_{ij}]$ of nonnegative integers has distinct nonzero rows.
Since H is linearly independent over $\mathbb{Q}$, we conclude that the set

$$\{exp(ra) : r \in H\},$$

is a set of free generator for an algebra contained in $C^*(a)$. In fact, if there is a non trivial relation between its members, we must have for example

$$exp(r_1 a)^{k_1} exp(r_2 a)^{k_2} \dots exp(r_n a)^{k_n} = 1$$

Which 1 denotes the identity element of A . This implies that

$$exp\left(\sum_1^n r_j k_j\, a\right) = 1.$$

Or equivalently

$$\sum_1^n r_j k_j = 0.$$

Since $r_j's$ are linearly independent, we conclude that $k_j = 0$. This means that there is no non trivial relation between elements

$$exp(r_1 a), \dots, exp(r_n a)$$

And this completes the proof. ■

Corollary 2.2 Every finitely generated C^* algebra is strong c algebrable.

Proof. Denote by $C^*(a_1, \dots, a_n)$ the C^* algebra generated by $a_1, \dots, a_n$. It is trivial that

$$C^*(a_1) \subseteq C^*(a_1, \dots, a_n).$$

So, the strong algebrability of $C^*(a_1)$ implies the same for $C^*(a_1, \dots, a_n)$.

3 Unitable elements in C^* algebras

In this section, we first introduce the new concept unitable elements in unital C^* algebras, and then we express our main result for this kind of elements.

Definition. Let A be a unital C^* algebra and a be an element in A . in this case we say a is unitable if there is a non zero real number α such that $a^\alpha = 1$. The set of unitable elements of A is denoted by unit(A) . Therefore

$$unit(A) = \{a : a \epsilon A \ , \ \exists \alpha \neq 0 \quad a^\alpha = 1\}$$

For more convince, we may assume that a is positive. In fact, we consider positive unitable .

Example. Consider the set of complex number as the simplest C^* algebra. In this C^* algebra, the set of unitable elements is exactly the unit circle. In fact, if z is a unitable complex number, then $z^\alpha = 1$ for some suitable . now, consider the polar decomposition of z ,

$$z = re^{i\theta}$$

So

$$z^\alpha = r^\alpha e^{i\alpha\theta} = 1$$

Hence $r^\alpha = 1$ and $= 1$. Since r is a positive number. Therefore $= e^{i\theta}$.

And so z belongs to the unit circle.

Conversely, it is clear that every members of the unit circle as $e^{i\theta}$ is unitable, since for $= \frac{2\pi}{\theta}$, we have

$$z^\alpha = e^{i\alpha\theta} = e^{2\pi i} = 1$$

Lemma. Let A be a unital C^* algebra and $a_1, \dots, a_n$ be arbitrary elements in A . Then, the element

$$\begin{bmatrix} a_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & a_n \end{bmatrix} \epsilon A \oplus \dots \oplus A,$$

is unitable, if and only all $a_1, \dots, a_n$ are unitable.

Proof. If the matricial element

$$\begin{bmatrix} a_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & a_n \end{bmatrix},$$

is unitable, then for some suitable α we have

$$\begin{bmatrix} a_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & a_n \end{bmatrix}^\alpha = \begin{bmatrix} 1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 1 \end{bmatrix}.$$

So all $a_1, \dots, a_n$ are unitable.

Conversely, assume that all $a_1, \dots, a_n$ are unitable. So, there are $\alpha_1, \dots, \alpha_n$ such that

$$a_1{}^{\alpha_1} = 1, \dots, a_n{}^{\alpha_n} = 1.$$

Therefore, for $= \alpha_1 \dots \alpha_n$, we have

$$\begin{bmatrix} a_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & a_n \end{bmatrix}^{\alpha} = \begin{bmatrix} a_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & a_n \end{bmatrix}^{\alpha_1 \dots \alpha_n} = \begin{bmatrix} a_1{}^{\alpha_1 \dots \alpha_n} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & a_n{}^{\alpha_1 \dots \alpha_n} \end{bmatrix} = \begin{bmatrix} 1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 1 \end{bmatrix}. \qquad \blacksquare$$

A slight modification in the proof of Therom 2.5 will give the following theorem.

Theorem 3.1 Let A be a unital C^* algebra, and assume a is a non unitable element in A. In this case, the C^* algebra generated by a is strongly c algebrable.

Corollary 3.1 Every finitely generated C^* algebra whit at least one non unitable generator is strong c algebrable.

Proof. Denote by $C^*(a_1, \dots, a_n)$ the C^* algebra generated by $a_1, \dots, a_n$. Assume that a_1 is non unitable. It is trivial that $C^*(a_1) \subseteq C^*(a_1, \dots, a_n)$.

So, the strong algebrability of $C^*(a_1)$ implies the same for $C^*(a_1, \dots, a_n)$.

Example 3.2 Let A be a unital C^* algebra and $a_1, \dots, a_n$ be arbitrary elements in A such that at least one of them is non unitable. Then, the C^* algebra generated by the element

$$\begin{bmatrix} a_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & a_n \end{bmatrix}$$

Is strong c algebrable in $\oplus \dots \oplus A$. In fact, in this case, this element is non unitable, so we can apply previous theorem about it.

4 2^c strong algebrability of direct sums of C^* algebras

Theorem. Let A_α be a uncountable collection of unital C^* algebras. Then the C^* algebra

$$\bigoplus_{\alpha \epsilon I} A_\alpha$$

Is strongly 2^c algebrable.

Proof. Let a_α be a non unitable element in A_α . Consider the following collection

$$\{diag(a_\alpha^{r_\alpha}) : r_\alpha\},$$

We claim that this collection is a basis for a free subalgebra of $\oplus_{\alpha\epsilon I} A_\alpha$. To this aim, consider the following members

$$\left\{diag\left(a_\alpha^{r_\alpha^1}\right), diag\left(a_\alpha^{r_\alpha^2}\right), \dots, diag\left(a_\alpha^{r_\alpha^n}\right)\right\},$$

Assume that there is a nontrivial relation between them

$$\left(diag\left(a_\alpha^{r_\alpha^1}\right)\right)^{\beta_1}\left(diag\left(a_\alpha^{r_\alpha^2}\right)\right)^{\beta_2}\dots\left(diag\left(a_\alpha^{r_\alpha^n}\right)\right)^{\beta_n} = 1,$$

Therefore

$$diag\left(a_\alpha^{\beta_1 r_\alpha^1}\right) diag\left(a_\alpha^{\beta_2 r_\alpha^2}\right)\dots diag\left(a_\alpha^{\beta_n r_\alpha^n}\right) = 1,$$

Or

$$diag\left(a_\alpha^{\beta_1 r_\alpha^1} a_\alpha^{\beta_2 r_\alpha^2} \dots a_\alpha^{\beta_n r_\alpha^n}\right) = 1,$$

Or

$$diag\left(a_\alpha^{\beta_1 r_\alpha^1+\beta_2 r_\alpha^2+\cdots+\beta_n r_\alpha^n}\right) = 1,$$

Since the identity element in $\oplus_{\alpha\epsilon I} A_\alpha$ has the form

$$1_{\oplus_{\alpha\epsilon I} A_\alpha} = diag(1_{A_\alpha}),$$

We conclude that

$$diag\left(a_\alpha^{\beta_1 r_\alpha^1+\beta_2 r_\alpha^2+\cdots+\beta_n r_\alpha^n}\right) = diag(1_{A_\alpha}),$$

Therefore

$$a_\alpha^{\beta_1 r_\alpha^1+\beta_2 r_\alpha^2+\cdots+\beta_n r_\alpha^n} = 1_{A_\alpha},$$

Since a_α is a non unitable element of A_α , we conclude that

$$\beta_1 r_\alpha^1 + \beta_2 r_\alpha^2 + \cdots + \beta_n r_\alpha^n = 0$$

Therefore

$$\beta_1 = 0 \quad , \quad \beta_2 = 0 \quad , \quad \ldots \quad , \quad \beta_n = 0$$

Therefore, this collection is a basis for a free subalgebra of $\bigoplus_{\alpha \in I} A_\alpha$. Finally, we assert that the cardinal of

$$\{diag(a_\alpha^{r_\alpha}) : r_\alpha\}$$

is equal to 2^c . ■

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