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Characterization of Regularity of Posemigroups by High-Quasi-Ideals

M. M. Shamivand *

Department of Mathematics, Borujerd Branch, Islamic Azad University, Borujerd, Iran.

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ABSTRACT

A semigroup S is called a posemigroup if S is equipped with an ordering relation " $\leq$ " such that $a \leq b$ in S implies $xa \leq xb$ and $ax \leq bx$, for all $x \in S$. In what follows we study necessary and sufficient conditions that a posemigroup S to be regular, in terms of certain conditions of Q^*_S, the semigroup of high-quasi-ideals of S. This study gives us a characterization method of the regular posemigroups.

1 Introduction

In abstract theory of semigroups, the notion of regular elements was firstly introduced by Thierrin [11] and has been effectively used in the ideals theory of semigroups. The bi-ideals of posemigroups were introduced by Good [1] and Steinfeld [10], respectively. Recall from [2] that in a posemigroup S, the element $a \in S$ is called regular, if $a \leq axa$, for some $x \in S$. Also S is called regular if every element of S is regular. For further information we refer to [2, 3, 5, 7, 9]. Here we generalize the notions of bi-ideal and quasi-ideal to high-bi-ideal and high-quasi-ideal, respectively. Throughout the paper S stands for a posemigroup. Following [4] and [12] we recall the definitions of $(A]$ and AB as:

$$(A] := \{s \in S \mid s \leq a,\ for\ some\ a \in A\}, AB := \{ab \mid a \in A, b \in B\},$$

for subposets A and B of a posemigroup S. By a left ideal of a posemigroup S we mean a non-empty subset L of S satisfying $SL \subseteq L$ and $(L] \subseteq L$. A right ideal may be defined in a similar way. A tow sided ideal of S is a left as well as a right ideal of S. For every non-empty subset A of S, let

$$(A]^* := \{s \in S | s \leq a^n,\ for\ some\ a \in A, n \in \mathbb{N}\}.$$

Let S be a posemigroup. A non-empty subset Q of S is called a *high-quasi-ideal* of S if $(QS]^* \cap (SQ]^* \subseteq Q$ and $(Q]^* \subseteq Q$. The set of all high-quasi-ideals of S will be denoted by Q^*_S. Also a non-empty subset B of S is said to be a *high-bi-ideal* of S if $BSB \subseteq B$ and $(B]^* \subseteq B$.

* Corresponding Author's E-mail: m.shamivand@yahoo.com(M.M. Shamivand)

Clearly, every one-sided ideal of S is a high-quasi-ideal of S. We denote the high-quasi-ideal and high-bi-ideal generated by an element $a \in S$ by $Q^*(a)$ and $B^*(a)$, respectively. One can easily show that $Q^*(a) = (a \cup ((aS]^* \cap (Sa]^*)]^*$ and $B^*(a) = (a \cup a^2 \cup aSa]^*$. Note that every high-quasi-ideal of a posemigroup S is a high-bi-ideal of S, for, if B is a high-quasi-ideal of S, then we get $BSB \subseteq SSB$ and $BSB \subseteq BSS$, so, $BSB \subseteq SSB \cap BSS \subseteq (SSB]^* \cap (BSS]^* \subseteq B$. The second condition is trivial. Hence, B is a high-bi-ideal of S. In general the converse is not true. For an example we may consider the posemigroup $S = \{a, b, c, d\}$ by the multiplication table:

.	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	b
d	a	a	b	a

and the ordering relation:

$$\leq := \{(a, a), (a, b), (b, b), (c, c), (d, d)\}.$$

Clearly, $B = \{a, d\}$ is a high-bi-ideal of S however, it is not a high-quasi-ideal of S. Our notation are merely standard and follow [6, 10]. We prefer to introduce the notations:

$$P_S^* = \{X \mid \varnothing \neq X \subseteq S \; and \; (X]^* \subseteq X\},$$
$$L_S = \{L \mid L \; is \; a \; left \; ideal \; of \; S\},$$
$$R_S = \{R \mid R \; is \; a \; right \; ideal \; of \; S\},$$
$$I_S = \{I \mid I \; is \; a \; two{-}sided \, ideal \, of \, S\}.$$

A multiplication on P_S^* may be defined by $XoY = (XY]^*$, for every $X, Y \in P_S^*$.
Our main results concerning the high-quasi-ideals which are the generalizations of the quasi-ideals of semigroups, are:

Proposition 1.1. *Let S be a posemigroup. Then the following are equivalent:*
(i) *S is regular.*
(ii) *For every right ideal R and left ideal L of S, $(RL]^* = R \cap L$.*
(iii) *For every right ideal R and left ideal L of S, $(R^2]^* = R$, $(L^2]^* = L$ and $(RL]^*$ is a high-quasi-ideal of S.*

Proposition 1.2. *Let S be a posemigroup. Then the following are equivalent:*
(i) *S is regular.*
(ii) *(Q_S^*, o) is the subsemigroup of the semigroup (P_S^*, o) generated by the bands (L_S, o) and (R_S, o).*
(iii) *(Q_S^*, o) is regular.*
(iv) *Every high-quasi-ideal Q of S is in the form $Q = (QSQ]^*$.*

2 The Proofs

To prove our assertions first we have to give certain preliminary results concerning the notions of Section 1. First we give a generalized results of [8] for high-quasi-ideals:

Lemma 2.1. *For a posemigroup S and two non-empty subset A and B of S,*

(i) $A \subseteq (A]^*$.

(ii) *If A is a high-quasi-ideal of S, then $A = (A]^*$ and $(A]^* = ((A]^*]^*$.*

(iii) *If $A \subseteq B$, then $(A]^* \subseteq (B]^*$.*

(iv) *$(A]^*(B]^* \subseteq (AB]^*$ for any two subposemigroups A and B of S.*

(v) *$(A \cap B]^* \neq (A]^* \cap (B]^*$. In particular, if A and B are two high-quasi-ideals of S, then $(A \cap B]^* = (A]^* \cap (B]^*$.*

Proof. (i), (v) and second part of (ii) are evident.

(ii) It suffices to show that $(A]^* \subseteq A$. If $t \in (A]^*$, there exists $a \in A$ and $n \in \mathbb{N}$ such that $t \leq a^n$. Since A is a high-quasi-ideal of S, then $t \in A$ and $A = (A]^*$.

(iii) Let $t \in (A]^*$. Then there exists $a \in A$ and $n \in \mathbb{N}$ such that $t \leq a^n$. Since $A \subseteq B$, there exists $a \in B$ and $n \in \mathbb{N}$ such that $t \leq a^n$. Thus $t \in (B]^*$.

(iv) Take any $x \in (A]^*(B]^*$. This implies that $x = ab$ for some $a \in (A]^*$ and $b \in (B]^*$. Then $a \leq h^m$ and $b \leq k^n$ for some $h \in A$, $k \in B$ and $m, n \in \mathbb{N}$. It follows that $ab \leq h^m k^n$. Since $h^m \in A$ and $k^n \in B$, we obtain $h^m k^n \in AB$. Therefore, $ab \leq h^m k^n \in AB$ showing that $x \in (AB]^*$. □

Lemma 2.2. *Let S be a posemigroup. Then*

(i) *$(P_S^*, o, \subseteq)$ is a posemigroup.*

(ii) *$(L_S, o, \subseteq)$, $(R_S, o, \subseteq)$ and $(I_S, o, \subseteq)$ are subposemigroups of $(P_S^*, o, \subseteq)$.*

Proof. (i) The binary operation "o" is well-defined. Let $A, B, C \in P_S^*$. Then

$(AoB)oC = (AB]^*oC = ((AB]^*C]^* = ((AB)C]^* = (ABC]^*$,

and

$Ao(BoC) = Ao(BC]^* = (A(BC]^*]^* = (A(BC)]^* = (ABC]^*$.

So, $(AoB)oC = Ao(BoC)$ holds. Thus (P_S^*, o) is a semigroup. Let $A, B, C \in P_S^*$ and $A \subseteq B$, then $AoC = (AC]^* \subseteq (BC]^* = BoC$ and $CoA = (CA]^* \subseteq (CB]^* = CoB$. Hence, $(P_S^*, o, \subseteq)$ is a posemigroup.

(ii) Evidently, L_S, R_S and I_S are non-empty subset of P_S^*. Let $J, K \in L_S$. It is obvious that $(JoK]^* = ((JK]^*]^* = (JK]^*$. Further, by $S(JoK) = S(JK]^* \subseteq (S(JK]^*]^* \subseteq (S(JK)]^* = ((SJ)K]^* \subseteq (JK]^* = JoK$, we conclude that JoK is a left ideal of S, i.e.; $JoK \in L_S$. Thus $(L_S, o, \subseteq)$ is a subposemigroup of $(P_S^*, o, \subseteq)$. Similarly, we can show that $(R_S, o, \subseteq)$ is a subposemigroup of $(P_S^*, o, \subseteq)$. By $I_S = L_S \cap R_S$ it follows that $(I_S, o, \subseteq)$ is a subposemigroup of $(P_S^*, o, \subseteq)$. □

Note that each high-quasi-ideal Q of posemigroup S is a subsemigroup of S, for,

$Q^2 \subseteq QS \cap SQ \subseteq (QS]^* \cap (SQ]^* \subseteq Q$.

Lemma 2.3. *For every left ideal L and right ideal R of a posemigroup S, $L \cap R$ is a high-quasi-ideal of S.*

Proof. Proof is easy by considering the relations $RL \subseteq SL \subseteq L$ and $RL \subseteq RS \subseteq R$. □

Lemma 2.4. *For every high-quasi-ideal Q of a posemigroup S,*

$$Q = L(Q) \cap R(Q) = (SQ \cup Q]^* \cap (Q \cup QS]^*.$$

Proof. Clearly that $Q \subseteq (SQ \cup Q]^* \cap (Q \cup QS]^*$. Conversely, let $t \in (SQ \cup Q]^* \cap (Q \cup QS]^*$. Then $t \leq q^m$, or $t \leq (xu)^n$ and $t \leq (vy)^p$, for some $q, u, v \in Q$ and $x, y \in S$ and $m, n, p \in \mathbb{N}$. Since Q is a high-quasi-ideal of S, so $t \in (Q]^* \subseteq Q$ or $t \in (SQ]^* \cap (QS]^* \subseteq Q$. Hence, $(SQ \cup Q]^* \cap (Q \cup QS]^* = Q$. □

Lemma 2.5. *Let I be a two-sided ideal of a posemigroup S and Q be a high-quasi-ideal of I. Then Q is a high-bi-ideal of S.*

Proof. Since Q is a high-quasi-ideal of I then,
$QSQ \subseteq QSI = Q(SI) \subseteq QI \subseteq (QI]^* \subseteq (SI]^* \subseteq (I]^* \subseteq I$,
$QSQ \subseteq ISQ = (IS)Q \subseteq IQ \subseteq (IQ]^* \subseteq (IS]^* \subseteq (I]^* \subseteq I$,
and for $x \in (Q]^*$ we get:
$\exists q \in Q \subseteq I, n \in \mathbb{N}; x \leq q^n$, so,
$x \in (I]^* = I$, hence,
$x \in I \cap (Q]^* = (Q]^* \subseteq Q$.
So, $QSQ \subseteq (I \cap (IQ]^*) \cap (I \cap (QI]^*) = (IQ]^* \cap (QI]^* \subseteq Q$, and $(Q]^* \subseteq Q$.

□

Lemma 2.6. *For every posemigroup S,*

$$\langle L_S \cup R_S \rangle = L_S \cup R_S \cup (R_S o L_S).$$

Proof. Since,
$\langle L_S \cup R_S \rangle = \{X_1 o X_2 o ... o X_n \mid X_i \in L_S \text{ or } X_i \in R_S, 1 \leq i \leq n, n \in \mathbb{N}\}$.
Then by letting $X_i, X_{i+1} \in L_S \cap R_S$ we consider four cases:
(i) $X_i, X_{i+1} \in L_S$. In this case, $X_i o X_{i+1} \in L_S$ (by Lemma 2.2.).
(ii) $X_i, X_{i+1} \in R_S$. In this case, $X_i o X_{i+1} \in R_S$ (by Lemma 2.2.).
(iii) $X_i \in L_S$ and $X_{i+1} \in R_S$. In this case, $X_i o X_{i+1} = (X_i X_{i+1}]^*$ is an ideal of S, so $X_i o X_{i+1} \in I_S = L_S \cap R_S$.
(iv) $X_i \in R_S$ and $X_{i+1} \in L_S$. In this case, $X_i o X_{i+1} \in R_S o L_S$. Thus, for every $n \in \mathbb{N}$ and $X_1, X_2, ..., X_n \in L_S \cup R_S$ there are three casas to recognize:
(a) If $X_1 \in L_S$, then $X_1 o X_2 o ... o X_n \in L_S$.
(b) If $X_n \in R_S$, then $X_1 o X_2 o ... o X_n \in R_S$.
(c) If $X_1 \in R_S$ and $X_n \in L_S$ ($n \geq 2$), then $X_1 o X_2 o ... o X_n \in R_S o L_S$.
Therefore, this gives the result. □

We are now ready to prove the propositions.

Proof of Proposition 1.1. (i) ⇒ (ii) Let R and L be right and left ideals of S respectively, then $(RL]^* \subseteq R \cap L$. Let S be regular and $a \in R \cap L$, so $a \leq axa$ for some $x \in S$, whence $a \in R$ and $xa \in L$. So $axa \in RL$ which implies that $a \in (RL]^*$. Therefore, $R \cap L = (RL]^*$.

(ii) ⇒ (iii) Using Lemma 2.3 and the assumption, $(RL]^*$ is a high-quasi-ideal of S. Since $(R \cup SR]^*$ is a two-sided ideal of S generated by R, it follows that,

$R = R \cap (R \cup SR]^* = (R(R \cup SR]^*]^*$, so, $(R^2]^* \subseteq (R(R \cup SR]^*]^* = R$.

Conversely, let $x \in R = (R(R \cup SR]^*]^*$. Then, $x \leq (r_1 t)^n$ for some $r_1 \in R, t \in (R \cup SR]^*$ and $n \in \mathbb{N}$. From $t \in (R \cup SR]^*$, we obtain $t \leq (u)^m$ where $u = r_2 \in R$ or $u = sr_3$ for some $s \in S, r_3 \in R$ and $m \in \mathbb{N}$. Hence,

$x \leq (r_1 u^m)^n = (r_1 r_2^m)^n$, or
$x \leq (r_1 u^m)^n = (r_1 (sr_3)^m)^n$.
But, $r_1 r_2^m \in R^2$ and $r_1 (sr_3)^m \in R^2$, so $x \in (R^2]^*$ which gives that $R \subseteq (R^2]^*$. Therefore, $(R^2]^* = R$.

(iii) $\Rightarrow$ (i) Let $a \in S$, then
$a \in (R(a)L(a)]^* = (R(a)((L(a))^2]^*]^* \subseteq (R(a)(SL(a)]^*]^*$, hence,
$a \in ((a \cup aS]^*(S(Sa \cup a]^*]^*]^* \subseteq (aSa]^*$, so $a \in (aSa]^*$.
Then there exists $x \in S$ and $n \in \mathbb{N}$ such that,
$a \leq (axa)^n = (axa)(axa)...(axa) = (a(xa^2xa^2...xa^2x)a)$.
Letting $y := xa^2xa^2...xa^2x$ which is an element of S yields $a \leq aya$. Hence S is a regular posemigroup. □

Proof of Proposition 1.2. First we observe that for a posemigroup S and non-empty subset X of S,

$$Q^*(X) = L(X) \cap R(X) = (SX \cup X]^* \cap (X \cup XS]^*. \qquad (*)$$

(i)$\Rightarrow$ (ii) Let S be regular, then by Proposition 1.1 (iii) we get (L_S, o) and (R_S, o) are bands and,
$R_S o L_S \subseteq Q_S^*$, so $\langle L_S \cup R_S \rangle \subseteq Q_S^*$ by Lemma 2.6.

Conversely, let $Q \in Q_S^*$. Then $(Q \cup SQ]^*$ is a left ideal of S generated by Q. Now by the condition (iii) in Proposition 1.1,
$Q \subseteq (Q \cup SQ]^* = (((Q \cup SQ]^*)^2]^* \subseteq (Q^2 \cup SQ^2 \cup QSQ \cup (SQ)^2]^* \subseteq (SQ]^*$.

Similarly, one can show that $Q \subseteq (QS]^*$. These relations and Lemma 2.4 give the following:
$Q \subseteq (SQ]^* \cap (QS]^* \subseteq (SQ \cup Q]^* \cap (Q \cup QS]^* = Q$, that is,

$$(\forall Q \in Q_S^*)\ Q = (SQ]^* \cap (QS]^*. \qquad (2.1)$$

Again by Proposition 1.1 (iii) and (1), we have

$$(\forall R \in R_S)\ (\forall L \in L_S)\ (RL]^* = (S(RL]^*]^* \cap ((RL]^*S]^*. \qquad (2.2)$$

Furthermore, $S = (S^2]^*$ and,
$(SQ]^* = (((SQ]^*)^2]^* = ((SQ]^*(SQ]^*]^* = ((SQ]^*((S^2]^*Q]^*]^*$, hence,
$(SQ]^* \subseteq (SQSSQ]^* \subseteq (S(QS]^*(SQ]^*]^* \subseteq (S((QS]^*(SQ]^*]^*]^*$, so,
$(SQ]^* \subseteq (S(QS^2Q)]^* \subseteq (SQ]^*$. Therefore,
$(SQ]^* = (S((QS]^*(SQ]^*]^*]^*$. Similarly,
$(QS]^* = (((QS]^*(SQ]^*]^*S]^*$. Consequently, in view of (1), (2) and Lemma 2.6 we obtain

$$\begin{aligned} Q = (QS]^* \cap (SQ]^* &= (((QS]^*(SQ]^*]^*S]^* \cap (S((QS]^*(SQ]^*]^*]^* \\ &= ((QS]^*(SQ]^*]^* = (QS]^* o (SQ]^* \in R_S o L_S \subseteq \langle L_S \cup R_S \rangle. \end{aligned} \qquad (2.3)$$

Hence, $Q_S^* \subseteq \langle L_S \cup R_S \rangle$. So, $Q_S^* = \langle L_S \cup R_S \rangle$.
Therefore (Q_S^*, o) is a subsemigroup of (P_S^*, o) generated by the bands (L_S, o) and (R_S, o).

(ii) ⇒ (iii) By assumption, the condition (iii) in Proposition 1.1 is followed immediately from Lemma 2.6. For every $Q \in Q^*_S$, by (3)
$Q = ((QS]^*(SQ]^*]^* \subseteq (QS^2Q]^* \subseteq (QSQ]^* \subseteq Q$,
whence, $Q = (QSQ]^* = QoSoQ$, with $S \in Q^*_S$. Thus, (Q^*_S, o) is regular.

(iii) ⇒ (iv) Let $Q \in Q^*_S$. Then, using Lemma 2.4, there exists a high-quasi-ideal X of S such that,
$Q = QoXoQ = (QXQ]^* \subseteq (QSQ]^* \subseteq (SQ]^* \cap (QS]^* \subseteq (SQ \cup Q]^* \cap (Q \cup QS]^* = Q$. So, $Q = (QSQ]^*$.

Finally, let (iv) holds. We show that S is regular. It is obvious from the assumption that $(Q^*_S, o, \subseteq)$ is a regular subsemigroup of the posemigroup $(P^*_S, o, \subseteq)$. For every $a \in S$, by (*)(with $X = \{a\}$), $R(a) \cap L(a)$ is a high-quasi-ideal of S containing a. Hence, there exists $Q \in Q^*_S$ such that,
$a \in R(a) \cap L(a) \subseteq (R(a) \cap L(a))oQo(R(a) \cap L(a))$, hence,
$a \in ((R(a) \cap L(a))Q(R(a) \cap L(a))]^* \subseteq (R(a)SL(a))]^*$, so,
$a \in ((a \cup aS]^*S(aSa \cup a]^*]^*$.
Then we get $a \in (aSa]^*$. It follows that $a \leq (axa)^n = (axa)(axa)...(axa) = a(xa^2xa^2...xa^2x)a$ for some $x \in S$ and $n \in \mathbb{N}$.
Letting $y := xa^2xa^2...xa^2x$ which is an element of S yields $a \leq aya$. Hence S is a regular posemigroup.

□

3 Examples

Justifying the conditions of Proposition 1.2 in connection with the regularity of posemigroups involves the from of high-quasi-ideals, will be studied in the examples of this section. Indeed, we present two non-commutative posemigroups where, one satisfies the condition of the Proposition 1.2 and the other dosen,t.

Example 3.1. *(i) The posemigroup $S = \{a, b, c, d, e\}$ defined by the multiplication table and the order as below is regular.*

.	a	b	c	d	e
a	d	b	b	d	e
b	b	b	b	b	e
c	b	b	c	b	e
d	d	b	b	d	e
e	b	b	e	b	e

$$\leq := \{(a,a),(a,b),(a,c),(a,e),(b,b),(b,c),(b,e),(c,c),(d,b),(d,c),(d,d),(d,e),(e,e)\}.$$

To see this, the high-quasi-ideals of S are the sets: $\{a, b, d\}$, $\{a, b, d, e\}$ and S. So, $Q^*_S = \{\{a, b, d\}, \{a, b, d, e\}, S\}$. Hence, for every Q in Q^*_S, we have $(QSQ]^* = Q$. So, S is regular.

(ii) The posemigroup $S = \{a, b, c, d, e\}$ defined by the multiplication table and the order as below is regular.

.	a	b	c	d	e
a	a	d	a	d	d
b	a	b	a	d	d
c	a	d	c	d	e
d	a	d	a	d	d
e	a	d	c	d	e

$$\leq := \{(a,a),(a,c),(a,d),(a,e),(b,b),(b,d),(b,e),(c,c),(c,e),(d,d),(d,e),(e,e)\}.$$

Indeed, the high-quasi-ideals of S *are the sets:* $\{a\}$, $\{a,c\}$, $\{a,b,d\}$, $\{a,b,c,d\}$ *and* S. *So,* $Q_S^* = \{\{a\},\{a,c\},\{a,b,d\},\{a,b,c$
For every Q *in* Q_S^*, *we have,* $(QSQ]^* = Q$. *So,* S *is regular.*

Example 3.2. *(i) The posemigroup* $S = \{a,b,c,d,e\}$ *defined by the multiplication table and the order as below is not regular.*

.	a	b	c	d	e
a	a	a	a	d	e
b	a	b	a	d	a
c	a	a	a	d	e
d	d	d	d	d	e
e	d	d	d	d	e

$$\leq := \{(a,a),(a,d),(a,e),(b,b),(b,d),(b,e),(c,c),(c,e),(d,d),(d,e),(e,e)\}.$$

Indeed, the high-quasi-ideals of S *are the sets:* $\{a,b,d\}$, $\{a,b,c,d\}$ *and* S. *So* $Q_S^* = \{\{a,b,d\},\{a,b,c,d\},S\}$. *But for* $Q = \{a,b,c,d\}$ *which is an element of* Q_S^*, *we have* $(QSQ]^* = \{a,b,d\} \neq Q$. *So,* S *is not regular.*

(ii) The posemigroup $S = \{a,b,c,d,e\}$ *defined by the multiplication table and the order as below is not regular.*

.	a	b	c	d	e
a	a	b	c	b	b
b	b	b	b	b	b
c	a	b	c	b	b
d	d	b	d	b	b
e	e	e	e	e	e

$$\leq := \{(a,a),(a,c),(b,b),(b,d),(c,c),(d,d),(e,b),(e,d),(e,e)\}.$$

For this, the high-quasi-ideals of S *are the sets:* $\{e\}$, $\{b,e\}$, $\{a,b,e\}$, $\{b,d,e\}$, $\{a,b,c,e\}$, $\{a,b,d,e\}$ *and* S. *Thus,* $Q_S^* = \{\{e\},\{b,e\},\{a,b,e\},\{b,d,e\},\{a,b,c,e\},\{a,b,d,e\},S\}$.
But for $Q = \{b,d,e\}$ *which is an element of* Q_S^*, *we have* $(QSQ]^* = \{b,e\} \neq Q$. *So,* S *is not regular.*

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