Remotal Centers and Chebyshev Centers in Normed Spaces

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Article Info

Keywords
Chebyshev centers
Remotal centers
Nearest points
Farthest points.

Article History
Received: 2019 August 23
Accepted: 2020 April 17

Abstract

In this paper, we consider **"Nearest points"** and **"Farthest points"** in normed linear spaces. For normed space $(X, \|.\|)$, the set $W \subseteq X$, we define P_g, F_g, R_g where $g \in W$. We obtion results about on P_g, F_g, R_g. We find new results on Chebyshev centers in normed spaces. In finally we define remotal center in normed spaces.

1 Introduction

Approximation theory, which mainly consists of theory of nearest points (best approximation) and theory of farthest points (worst approximation), is an old and rich branch of analysis. The theory is as old as Mathematics itself. Starting in 1853, a Russian mathematician P.L. Chebyshev made significant contributions in the theory of best approximation. The Weierstrass approximation theorem of 1885 by K. Weierstrass is well known.The study was followed in the first half of the 20th Century by L.N.H. Bunt (1934) , T.S . Motzkin (1935) and B. Jessen (1940). B. Jessen was the first to make significant contributions in the theory of farthest points. This theory is less developed as compared to the theory of best approximation .

Let $(X, \|.\|)$ be a normed linear space, W a non-empty subset of X. A point $y_0 \in W$ is said to be a best approximation point (nearest point) for $x \in X$, if

$$\|x - y_0\| \leq \|x - y\|,$$

for each $y \in W$. For each $x \in X$, put

$$P_W(x) = \{y_0 \in W : \ \|x - y_0\| = d(W, x) = \inf_{y \in W} \|x - y\|\}.$$

For each $x \in X$, if $P_W(x)$ is non-empty (a singleton), we say that W is proximinal (Chebyshev). For each $x \in X \backslash W$, if $P_W(x) = \emptyset$, we say that W is anti-proximinal (see [2 , 5 , 10, 17]).

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Let X be a normed linear space and W a bounded non-empty subset of X. A point $q(x) \in W$ is said to be a farthest point for $x \in X$, if

$$\|x - q(x)\| \geq \|x - y\|,$$

for each $y \in W$. For each $x \in X$, put

$$F_W(x) = \{y_0 \in W : \ \|x - y_0\| = \delta(W, x) = \sup_{y \in W} \|x - y\|\}.$$

For each $x \in X$, if $F_W(x)$ is non-empty (a singleton), we say that W is remotal (uniquely remotal). For each $x \in X$, if $F_W(x) = \emptyset$, we say that W is anti-remotal. (see [3 , 4 , 6 , 7 , 9 , 10 , 13 , 14]).
Let $(X, \|.\|)$ be a normed space, $W \subseteq X$ a non-empty subset of X and $g \in W$. We set:

$$\begin{aligned} \delta(W) &= \inf_{x \in X} \delta(W, x), \\ E(W) &= \{x \in X : \delta(W, x) = \delta(W)\}, \\ P_g &= \{x \in X \setminus W : g \in P_W(x)\}, \\ F_g &= \{x \in X \setminus W : g \in F_W(x)\}. \end{aligned}$$

For $x \in X \setminus \overline{W}$, we set :

$$R_W(x) = \{g_0 \in W : ||g_0 - g|| \leq ||x - g|| \quad \forall g \in W\},$$

and

$$R_g = \{x \in X \setminus W : g \in R_W(x)\}.$$

If $c \in E(W)$, we say that c is a Chebyshev center of W, We denote by c_W and $\delta(W)$ is called Chebyshev radius of W.

Definition 1.1. *Let $(X, \|.\|)$ be a normed space, W a Chebyshev subset in X and $x \in X$. W is called sun set if $x \notin W$, $y = P_W(x)$, then*

$$P_W(\lambda x + (1 - \lambda)g) = g, \ for \ every \ \lambda \geq 0.$$

Definition 1.2. *Let $(X, \|.\|)$ be a normed space, W is a uniquely remotal set in X and $x \in X$. W is called sunrise set if $x \notin W$, $g = F_W(x)$, then*

$$F_W(\lambda x + (1 - \lambda)g) = g, \text{ for every } \lambda \geq 1.$$

In the sequel, we will present some known lemmas which are needed in the main results.

Lemma 1.1. *Let $(X, \|.\|)$ be a normed linear space, W a subspace of X, $x \in X \setminus \overline{W}$ and $g_0 \in W$. Then the following statements are equivalent:*

1) $g_0 \in P_W(x)$,

2) there exists a $f \in X^*$ such that $\|f\| = 1, \ f(x - g_0) = \|x - g_0\|, \ f|_W = 0$.

Lemma 1.2. *Let $(X, \|.\|)$ be a normed space, W a bounded subset of X, $x \in X \setminus \overline{W}$ and $g_0 \in W$. Then the following statements are equivalent:*

1) $g_0 \in F_W(x)$,

2) there exists a $f \in X^*$ *such that* $|f(g_0 - x)| = \sup_{g\in W} \|g - x\|$
and

$$|f(g_0 - x)| \geq |f(g - x)|, \qquad g \in W.$$

Lemma 1.3. *Let* $(X, \|.\|)$ *be a normed space,* W *a subset of* X, $x \in X \setminus \overline{W}$ *and* $g \in W$*.Then the following statements are equivalent:*

1) $g_0 \in R_W(x)$,

2) for every $g \in W$ *there exists* $f^g \in X^*$ *such that* $\|f^g\| = 1, f^g(x - g_0) = 0$ *and* $f^g(x - g_0) = \|x - g_0\|$.

2 Chebyshev centers

In this section we obtain some results on Chebyshev centers.

Theorem 2.1. *Let* $(X, \|.\|)$ *be a normed space,* W *a subspace of* X *and* $g \in W$*.Then the following conditions are equivalent:*
1) $x \in P_g$,
2) there exists a $f \in X^*$ *s.t.* $||f|| = 1,\ f(x - g) = ||x - g||,\ f|_W = 0$.

Proof. 1) $\Rightarrow$ 2). Suppose $x \in P_g$, then $g \in P_W(x)$. From Lemma 1.1, there exists a $f \in X^*$ s.t. $||f|| = 1,\ f(x - g) = ||x - g||,\ f|_W = 0$.
2) $\Rightarrow$ 1) If there exists a $f \in X^*$ s.t. $||f|| = 1,\ f(x - g) = ||x - g||,\ f|_W = 0$. From Lemma 1.1, $g \in P_W(x)$, and $x \in P_g$. ☐

Theorem 2.2. *Let* $(X, \|.\|)$ *be a normed space ,* W *a bounded subset of* X *and* $g \in W$ *.Then the following condition are equivant:*
1) $g \in F_g$,
2) there exists a $f \in X^*$ *s.t* $||f|| = 1, |f(g - x)| = \sup_{w\in W} ||w - x||$
and

$$|f(g - x)| \geq |f(w - x)|, \qquad w \in W.$$

Theorem 2.3. *Let* $(X, \|.\|)$ *be a normed space ,* W *a subset of* X *and* $g \in W$ *.Then the following condition are equivent:*
1)$g \in R_g$,
2) for every $w \in W$ *there exists* $f^w \in X^*$*s.t* $||f^*|| = 0$ *and* $f^w|_w = 0$

Lemma 2.1. *Let* $(X, \|.\|)$ *be a normed space,* $W = \{x \in X : \|x\| = 1\}$ *a subset of* X *and* $x \in X \setminus W$*. Then*
1) $d(W, x) = |1 - \|x\||$,
2) nearest point $(x) = \frac{x}{\|x\|}$,
3) farthest point $(x) = \frac{-x}{\|x\|}$,
4) $\delta(W, x) = |1 + \|x\||$,
5) $c_W = 0$,
6) $\delta(W) = 1$.

Proof. It is clear. □

Theorem 2.4. *Let $(X, \|.\|)$ be a normed space and $W = \{x \in X : \|x - x_0\| = 1\}$. Then*
1) $c_W = x_0$ and $\delta(W) = 1$,
2) $\delta(W, x - x_0) = 1 + \|x - c_W\|$.
3) $d(W, x - x_0) = 1 - \|x - c_W\|$,
4) nearest point $(x - x_0) = \frac{x-x_0}{\|x-x_0\|}$,
5) farthest point $(x) = -\frac{x-x_0}{\|x-x_0\|}$.

Example 2.1. *Let $(X, \|.\|)$ be a normed space, $W = \{x \in X : \|x\| = 1\}$ and $x \in X$. We show that W is sun and sunrise set.*

If $g \in W$, put $x = -\lambda g$ for every $\lambda \geq 1$. Therefore $q(x) = g$ and $x \in F_g$.
If $x \in F_g$, since $q(x) = -\frac{x}{\|x\|} = g$. Therefore $x = -\|x\|g$ and $\|x\| \geq 1$. It follows that

$$F_g = \{-\lambda g : \lambda \geq 1\},$$

Put $x = \lambda g$, for every $\lambda \geq 0$. Therefore nearest point $(x) = g$ and $x \in P_g$.
If $x \in P_g$, since nearest point $(x) = \frac{x}{\|x\|} = g$ and $\|x\| \geq 1$. Therefore $x = \|x\|g$ and $\|x\| \geq 1$. .It follows that

$$P_g = \{\lambda g : \lambda \geq 1\}.$$

Suppose $g = F_W(x)$, then $x \in F_g$. Therefore for some $\lambda_0 \geq 1 : x = -\lambda_0 g$.

$$\begin{aligned} \lambda x + (1-\lambda)g &= -\lambda\lambda_0 g + g - \lambda g \\ &= -(-1 + \lambda + \lambda\lambda_0)g, \end{aligned}$$

Note that $-1 + \lambda + \lambda\lambda_0 \geq 1$. Therefore

$$F_W(\lambda x + (1-\lambda)g) = g.$$

and W is a sunrise set.
Suppose $g = P_W(x)$, then $x \in P_g$. Therefore for some $\lambda_0 \geq 1$ we have $x = \lambda_0 g$.
For $\lambda \geq 0$, we have

$$\begin{aligned} \lambda x + (1-\lambda)g &= \lambda\lambda_0 g + g - \lambda g \\ &= (\lambda\lambda_0 - \lambda + 1)g. \end{aligned}$$

Note that $\lambda\lambda_0 - \lambda + 1 \geq 0$. Therefore

$$P_W(\lambda x + (1-\lambda)g) = g.$$

and W is a sun set in X.

Theorem 2.5. *Let $(X, \|.\|)$ be a normed space and W an uniquely remotal convex subset of X. Then W is a sunrise set.*

Proof. Suppose $x \notin W$, $F_W(x) = y$ and $F_W(\lambda x + (1-\lambda)y) = w \neq y$ for every $\lambda \geq 1$. If $z = \lambda x + (1-\lambda)y$, for every $u \in W$. We put

$$w = \frac{1}{\lambda}x + (1 - \frac{1}{\lambda})y$$

since W is convex, $w \in W$ and

$$\begin{aligned}
\|z-y\| &= \|\lambda(x) + (1-\lambda)y - y\| \\
&= \|\lambda x - \lambda y\| \\
&= \lambda\|x-y\| \\
&\geq \lambda\|x-y\| \\
&= \|\lambda x - \lambda w\| \\
&= \|\lambda(x) - w - (\lambda-1)y\| \\
&= \|\lambda(x) + (1-\lambda)y - u\| \\
&= \|z-u\|.
\end{aligned}$$

Therefore

$$\|z-y\| > \|z-u\|,$$

$$F_W(z) = y.$$

□

Theorem 2.6. *Let $(X, \|.\|)$ be a normed space and W a Chebyshev convex set in X.Then W is a sun set.*

Proof. Suppose $x \in X \setminus W, g \in W$ and $g = P_W(x)$. If $z = \lambda x + (1-\lambda)y$ for $0 \leq \lambda < 1$. Then

$$\begin{aligned}
\|x-z\| &= \|\lambda x + (1-\lambda)x - \lambda x - (1-\lambda)g\| \\
&= \|(1-\lambda)(x-g)\| \\
&= (1-\lambda)\|(x-g)\| \\
&= \|(x-g)\| - \lambda\|(x-g)\| \\
&= \|(x-g)\| - \|(\lambda x - \lambda g)\| \\
&= \|(x-g)\| - \|(\lambda x + (1-\lambda)g - \lambda g - (1-\lambda)g)\| \\
&= \|(x-g)\| - \|(z-g)\|.
\end{aligned}$$

Since $g = P_W(x)$, we have $\|x-g\| \leq \|x-y\|$ for every $g \in W$. Therefore

$$\begin{aligned}
\|z-g\| &= \|x-g\| - \|x-z\| \\
&\leq \|x-g\| \\
&\leq \|x-y\| \qquad (g \in W).
\end{aligned}$$

It follows that $g = P_W(z)$. If $\lambda > 1$, for every $u \in W$. We put

$$w = \frac{1}{\lambda}u + (1 - \frac{1}{\lambda})g.$$

Suppose W is convex, then $w \in W$ and

$$\begin{aligned}
\|z - g\| &= \|\lambda(x) + (1-\lambda)g - g\| \\
&= \|\lambda x - \lambda g\| \\
&= \lambda\|x - g\| \\
&\leq \lambda\|x - y\| \\
&= \|\lambda x - \lambda u\| \\
&= \|\lambda x - u - (\lambda - 1)g\| \\
&= \|\lambda x + u - (1-\lambda)g - u\| \\
&= \|z - u\|, \qquad\qquad (u \in W).
\end{aligned}$$

Therefore $P_W(z) = g$. □

3 Remotal centers

In this section we define remotal center and remotal radius a set in normed spaces.

Definition 3.1. *Let $(X, \|.\|)$ be a normed space, W a subset of X. We set*

$$d(W) = \sup_{x \in X} d(x, W),$$

and we call remotal radius of W. If there exists a $x \in X$ such that

$$d(x, W) = d(W),$$

we call x a remotal center of W.

Example 3.1. *Let $(X, \|.\|)$ be a normed space and $W = \{x \in X : \|x\| = 1\}$. Then*

$$\delta(W) = d(W) = 1.$$

Theorem 3.1. *Let $(X, \|.\|)$ be a normed space, W a subset of X, c_W is a Chebyshev center of W, $d(W) \leq d(c_W, W), d(W) = \delta(W)$ and $g \in W$. If $\|g - c_W\| = \delta(W)$, then*

$$g \in P_W(x).$$

Proof. We have

$$\begin{aligned} d(c_W, W) &= \inf_{y \in W} \|c_W - y\|, \\ &\leq \|c_W - g\|, \\ &= \delta(W), \\ &= d(W), \\ &= d(c_W, W). \end{aligned}$$

Therefore

$$d(c_W, W) = \|c_W - g\|,$$

and

$$g \in P_W(x).$$

□

Theorem 3.2. *Let* $(X, \|.\|)$ *be a normed space,* W *a subset of* X, c_W *is a chebyshev center of* W. $\delta(W) \geq \delta(c_W, W), \delta(W) = d(W)$ *and* $g \in W$. *If* $\|g - c_W\| = d(W)$, *then*

$$g \in F_W(x).$$

Proof. We have

$$\begin{aligned} \delta(c_W, W) &= \sup_{y \in W} \|c_W - y\|, \\ &\leq \|c_W - g\|, \\ &= d(W), \\ &= \delta(W), \\ &\geq \delta(c_W, W). \end{aligned}$$

Therefore

$$\delta(c_W, W) = \|c_W - g\|,$$

and

$$g \in P_W(x).$$

□

Theorem 3.3. *Let* $(X, \|.\|)$ *be a normed space and* W *a compact subset of* X. *Then there exists a remotal center of* W.

Proof. We know that $d(W, .)$ is conttinous on X. The proof is trival. □

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