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On co-Farthest Points in Normed Linear Spaces

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Abstract

In this paper, we consider the concepts co-farthest points in normed linear spaces. At first, we defi ne farthest points, farthest orthogonality in normed linear spaces. Then we de fine co-farthest points, co-remotal sets, co-uniquely sets and co-farthest maps. We shall prove some theorems about co-farthest points, co-remotal sets. We obtain a necessary and coeficient conditions about co-farthest points and dual spaces.

Key words: Farthest points, Co-farthest points, Co-farthest map.

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1 Introduction

A kind of approximation, called best co-approximation was introduced by Franchettei and Furi in 1972 [12]. Some results on best co-approximation theory in linear normed spaces have been obtained by P. L. Papini and I. Singer [35]. In this section we consider co-proximinality and co-remotality in normed linear spaces.

Definition 1.1 *Let* $(X, \|.\|)$ *be a normed linear space,* G *a non-empty subset of* X *and* $x \in X$*. We say that* $g_0 \in G$ *is a best co-approximation of* x *whenever* $\|g - g_0\| \leq \|x - g\|$ *for all* $g \in G$*. We denote the set of all best co-approximations of* x *in* G *by* $R_G(x)$*.*
We say that G *is a co-proximinal subset of* X *if* $R_G(x)$ *is a non-empty subset of* G *for all* $x \in X$*. Also, we say that* G *is a co-Chebyshev subset of* X *if* $R_G(x)$ *is a singleton subset of* G *for all* $x \in X$*.*

Definition 1.2 *Let* $(X, \|.\|)$ *be a normed linear space,* A *a subset of* X*,* $x \in X$ *and* $m_0 \in A$*. We say that* m_0 *is co-farthest to* x *if* $\|m_0 - a\| \geq \|x - a\|$ *for every* $a \in A$*. The set of co-farthest points to* x *in* A *is denoted by*

$$C_A(x) = \{a_0 \in A : \ \|a_0 - a\| \geq \|x - a\| \ for \ every \ a \in A \backslash \{a_0\}\}.$$

The set A *is said to be co-remotal if* $C_A(x)$ *has at least one element for every* $x \in X$*. If for each* $x \in X$*,* $C_A(x)$ *has exactly one element in* A*, then the set* A *is called co-uniquely remotal. We define for* $a_0 \in A$*,*

$$C_A{}^{-1}(a_0) = \{x \in X : \ \|a_0 - a\| \geq \|x - a\| \ for \ every \ a \in A\}.$$

$C_A{}^{-1}(a_0)$ *is a closed set and* $a_0 \in C_A{}^{-1}(a_0)$*. Note that if* $x \in A$*, then* $x \in C_A(x)$*.*

Example 1.1 *Suppose* $X = \mathbb{R}$ *and* $A = [1, 2] \bigcup \{3\} \backslash \{1\}$*. We set* $x = 1$ *and* $a_0 = 3$*. Then* $a_0 \in C_A(x)$*.*

2 Co-Proximinality, co-Chebyshevity and co-Remotality

In this section we consider co-proximinality and co-Chebyshevity and co-remotality in normed linear spaces.

Theorem 2.1 *Let $(X, \|.\|)$ be a normed linear space and A a subset of X.*

a) If for every $x \in X$ and for every $a \in A$, $a \in H_{d_x}$, then A is co-proximinal.

b) If for every $x \in X$ and for every $a \in A$, there exists a unique $b \in H^{\bigoplus}_{\|x-a\|}$, then A is co-Chebyshev.

Proof. a) Suppose $x \in X$, for every $a \in A$ there exists $a_0 \in A$ such that $a - a_0 \in B[0, d_x]$. Therefore for every $a \in A$

$$\begin{aligned} \|a - a_0\| &\leq d_x \\ &\leq \|x - a\|. \end{aligned}$$

That is $a_0 \in R_A(x)$ so A is co-proximinal.
b) Suppose $x \in X$, $a \in A$ and there exists an unique $b \in H^{\bigoplus}_{\|x-a\|}$, by part (a), $R_A(x)$ is non-empty. The set A is co-proximinal.

For each $x \in X$ if there exist $a_1, a_2 \in R_A(x)$, then for $a \in A$ we have $\|a_i - a\| \leq \|x - a\|$ for $i = 1, 2$. Therefore for $a \in A$, $a_i - a \in B[0, \|x - a\|]$, and for $a \in A$, we have $a_i \in H^{\bigoplus}_{\|x-a\|}$. This is a contraction. It follows that A is co-Chebyshev.

Theorem 2.2 *Let $(X, \|.\|)$ be a normed linear space and A a subset of X.*

a) If for every $x \in X$ and for every $a \in A$, $a \in K_{\delta_x}$, then A is co-remotal.

b) If for every $x \in X$ and for every $a \in A$, there exists a unique $b \in K^{\bigoplus}_{\|x-a\|}$, then A is co-uniquely remotal.

Proof. a) Suppose $x \in X$ and $a \in A$. Suppose there exists an $a_0 \in A$ such that $a - a_0 \in B^c[0, \delta_x]$. Therefore for every $a \in A$

$$\begin{aligned} \|a - a_0\| &\geq \delta_x \\ &\geq \|x - a\|. \end{aligned}$$

That is $a_0 \in C_A(x)$ so A is co-remotal.
b) If $x \in X$ and $a \in A$ if there exists an unique $b \in K^{\bigoplus}_{\|x-a\|}$, then $C_A(x)$ is non-empty. The set A is co-remotal.

For $x \in X$ if there exist $a_1, a_2 \in C_A(x)$, then for $a \in A$ we have $\|a_i - a\| \leq \|x - a\|$ for $i = 1, 2$. Therefore for $a \in A$, $a_i - a \in B^c[0, \|x - a\|]$, and for $a \in A$, we have $a_i \in K^{\bigoplus}_{\|x-a\|}$. This is a contraction. It follows that A is co-uniquely remotal. Let W be a non-empty bounded subset of a normed linear space $(X, \|.\|)$. If there exists a point $\omega_0 \in W$ such that $\delta(x, W) = \sup\{\|x - \omega\| : \ \omega \in W\} = \|x - \omega_0\|$ for $x \in X$. Then ω_0 is called farthest point in W from x. The set of all such $\omega_0 \in W$ is denoted by $F_W(x)$.

Theorem 2.3 *Let A be a bounded subset of a normed linear space, $A + A = A$, $-A = A$ and $0 \in A$,*

(i) If $a_0 \in A$, then ${C_A}^{-1}(a_0) = -a_0 + {C_A}^{-1}(0)$,

(ii) $C_A(x) = (-x + {C_A}^{-1}(0)) \bigcap A$.

(iii) If $a_0 \in A$, then $x \in C_A(a_0)$ if and only if $x - a_0 \in C_A^{-1}(a_0)$

Proof. (i)

$$
\begin{aligned}
x \in {C_A}^{-1}(a_0) &\Leftrightarrow a_0 \in C_A(x)\\
&\Leftrightarrow \|a_0 - a\| \geq \|x - a\| \ for\ every\ a \in A\backslash\{a_0\}\\
&\Leftrightarrow \|u\| \geq \|x - a_0 - u\| \ for\ every\ u \in A\ since\ A + A = A\\
&\Leftrightarrow x + a_0 \in {C_A}^{-1}(0)\\
&\Leftrightarrow x \in -a_0 + {C_A}^{-1}(0).
\end{aligned}
$$

(ii)

$$
\begin{aligned}
a_0 \in C_A(x) &\Leftrightarrow x \in {C_A}^{-1}(a_0)\\
&\Leftrightarrow x + a_0 \in {C_A}^{-1}(0)\\
&\Leftrightarrow a_0 \in -x - {C_A}^{-1}(0)\ and\ a_0 \in A.
\end{aligned}
$$

(iii) Suppose $x - a_0 \in C_A^{-1}(a_0)$, then

$$\|a\| \geq \|x - a_0 - a\|.$$

Since $A + A = A$ and $-A = A$, then $a - a_0 \in A + A$. Then

$$\|b\| \geq \|x - a_0 - b\| \quad b \in A,$$

Therefore $x - a_0 \in C_A^{-1}(a_0)$.

Theorem 2.4 *Let A be a bounded subset of a normed linear space, then the following statements are equivalent:*

(i) A is co-remotal,

(ii) $X = -A + {C_A}^{-1}(0)$.

Proof. $(i) \to (ii)$. Suppose A is co-remotal and $x \in X$, there exists a $a_0 \in A$ such that $a_0 \in C_A(x)$. Then $u_0 = x + a_0 \in {C_A}^{-1}(0)$, and $x = -a_0 + u_0 \in -A + {C_A}^{-1}(0)$.

$(ii) \to (i)$. if $X = -A + {C_A}^{-1}(0)$ and $x \in X$. Then there exist a $a_0 \in A$ such that $x + a_0 \in {C_A}^{-1}(0)$. Thus $a_0 \in C_A(x)$ and A is co-remotal.

Theorem 2.5 *Let A be a co-remotal subset of a normed linear space,*

$A = A + A$ and $0 \in A$, then there exists an element $z \in X\backslash\{0\}$ such that $0 \in C_A(z)$.

Proof. Suppose $x \in X\backslash A$, since A is co-remotal, there exists $a_0 \in C_A(x)$ and so $z = x + a_0 \in {C_A}^{-1}(0)$. Hence $0 \in C_A(z)$, $z \neq 0$.

Theorem 2.6 *Let $(X, \|.\|)$ be a normed linear space, A a bounded subset of X, $x \in X$, $A = A + A$ and $0 \in A$. If $0 \in C_A(x)$, then $A \perp_F x$.*

Proof. If $0 \in C_A(x)$ and $a \in A$. Then $\|a\| \geq \|x - a\|$, therefore $A \perp_F x$.

Theorem 2.7 *Let $(X, \|.\|)$ be a normed linear space and $x, y \in X$. Then the following statements are equivalent:*

(i) $A \perp_F x$ or $0 \in C_A(x)$,
(ii) For every $m \in A$, there exists an $f \in X^*$ such that f satisfies $\|f\| = 1$ and $|f(m)| \geq \delta(x, A)$.

Proof. $(i) \to (ii)$. Suppose $A \perp_F x$ then for $m \in A$, $m \perp_F x$. That is $\|m\| \geq \delta(x.A)$. By Hahn-Banach Theorem, there exists an $f \in X^*$ such that $\|f\| = 1$ and $|f(m)| = \|m\| \geq \delta(x, A)$.
$(ii) \to (i)$. Suppose there exists an $f \in X^*$ such that f satisfies $\|f\| = 1$ and $|f(m)| \geq \delta(x, A)$. For $m \in A$, we have

$$\begin{aligned}\|m\| &= \|f\|\|m\| \\ &\geq |f(m)| \\ &\leq \|x - m\|.\end{aligned}$$

Therefore $m \perp_F x$ and $A \perp_F x$.

Theorem 2.8 *Let $(X, \|.\|)$ be a normed linear space and $x \in X$.*

(i)If a nonempty bounded set A in X is co-remotal then

$$A \bigcap \Big(\bigcap_{g \in X} C_{\|x-a\|} \Big) \neq \emptyset,$$

where $C_{\|x-a\|} = A \bigcap B^c[g, \delta_x]$.

(ii) For every $x \in X$*, if* $A \bigcap (\bigcap_{g \in X} C_{\|x-a\|}) \neq \emptyset$*. Then* A *is co-remotal.*

Proof. (i) Suppose A is co-remotal and $x \in X$. Then there exists a $a_0 \in A$ such that $\|g - a_0\| \geq \|g - x\|$ for every $g \in A$. Therefore $a_0 \in C_{\|x-a\|}$ for every $g \in A$, it follows that $a_0 \in \bigcap_{g \in X} C_{\|x-g\|}$, and $A \bigcap (\bigcap_{g \in X} C_{\|x-g\|}) \neq \emptyset$.
(ii) Suppose $x \in X$, since $A \bigcap (\bigcap_{g \in X} C_{\|x-g\|}) \neq \emptyset$. There exists a $a_0 \in A$ such that $a_0 \in (\bigcap_{g \in X} C_{\|x-g\|})$. Therefore $\|a_0 - g\| \geq \|x - g\|$ for every $g \in A \backslash \{a_0\}$. Therefore A is co-remotal.

Theorem 2.9 *Let* $(X, \|.\|)$ *be a normed linear space and* A *a co-remotal subset of* X*,* $A = A + A$ *and* $0 \in A$*. If* ${C_A}^{-1}(0)$ *is singleton, then* A *is co-uniquely remotal.*

Proof. Suppose $x \in X$ and $a_1, a_2 \in C_A(x)$. Then $x \in {C_A}^{-1}(a_i)$ for $i = 1, 2$. Therefore $x - a_i \in {C_A}^{-1}(0)$ for $i = 1, 2$. It follow that $x - a_1 = x - a_2$ and $a_1 = a_2$. Thus A is co-uniquely remotal.

Theorem 2.10 *Let* $(X, \|.\|)$ *be a normed linear space, and* A *be a bounded subset. Then* ${C_A}^{-1}(a_0)$ *is convex.*

Proof. If $x_1, x_2 \in {C_A}^{-1}(a_0)$ and $0 < \lambda < 1$. Since $\|a_0 - a\| \geqslant \|x_1 - a_0\|$ and $\|a_0 - a\| \geqslant \|x_2 - a_0\|$, for every $a \in A \backslash \{a_0\}$. Then

$$\begin{aligned}
\|\lambda x_1 + (1-\lambda) x_2 - a\| &= \|\lambda(x_1 - a) + (1-\lambda)(x_2 - a)\| \\
&\leq \lambda \|x_1 - a\| + (1-\lambda)\|x_2 - a\| \\
&\leq \lambda \|a_0 - a\| + (1-\lambda)\|a_0 - a\|,
\end{aligned}$$

for every $a \in A \backslash \{a_0\}$. Therefore $\lambda x_1 + (1-\lambda) x_2 \in {C_A}^{-1}(a_0)$. It follows that ${C_A}^{-1}(a_0)$ is convex.

Theorem 2.11 *Let* $(X, \|.\|)$ *be a normed linear space,* A *a subset of* X*,* $-A = A$*,* $A = A + A$ *and* $0 \in A$*. If* A *is co-remotal, then* A *is co-uniquely remotal.*

Proof. Suppose $x \in X$ and $g_1, g_2 \in C_A(x)$ by $g_1 \neq g_2$. Since $g_1, g_2 \in C_A(x)$, We have $x + g_1, x + g_2 \in {C_A}^{-1}(0)$. Also $-g_2 - x \in {C_A}^{-1}(0)$, therefore $\frac{1}{2}[g_1 - g_2] = \frac{1}{2}[g_1 + x - x - g_2] \in C_A^{-1}(0)$. That is, for every $a \in A \backslash \{0\}$,

$$\|\frac{1}{2}[g_1 - g_2] - a\| \leqslant \|a\|.$$

Since $g_1 - g_2 \in A$ and $a = (g_1 - g_2) \in A$. Then

$$\|\frac{1}{2}[g_1 - g_2] + [g_1 - g_2]\| \leqslant \|g_1 - g_2\|,$$

and

$$\frac{3}{2}\|g_1 - g_2\| \leqslant \|g_1 - g_2\|$$

and

$$\frac{3}{2} \leqslant 1$$

is contraction. That is, A is co-uniquely remotal.

Theorem 2.12 *Let $(X, \|.\|)$ be a normed linear space, A a subset of X and $x \in X$. If A compact(weakly compact) then $C_A(x)$ is compact(weakly c ompact).*

Proof. Suppose $\{x_n\}_{n \geq 1}$ is a sequence in $C_A(x)$. Then for every sequence $\{a_n\}_{n \geq 1}$ in $A \backslash \{x\}$

$$\|x_n - a_n\| \geq \|x - a_n\|.$$

Since A is compact, there exists a convergent subsequence $\{a_{nk}\}$ and $\{x_{nl}\}$ in A, x_0 and $a_0 \in A$ such that $x_{nn \geq 1} \longrightarrow x_0$ and $a_{nk} \longrightarrow a_0$. Then $\|x_{np} - a_{np}\| \geq \|x - a_{np}\|$. Then $\|x_0 - a_0\| \geq \|x - a_0\|$. Therefore $x_0 \in C_A(x)$ and $x_{np} \longrightarrow x_0$. Therefore $\{x_n\}_{n \geq 1}$ has a subsequence in $C_A(x)$ and $C_A(x)$ is compact (weakly compact).

Theorem 2.13 *Let A be a compact subset of a normed linear space $(X, \|.\|)$. Then*

(i) for every $x \in X$, $C_A(x)$,

(ii) C_A is upper semi-continues on $D(C_A)$.

Proof. (i) Suppose $\{a_n\}_{n\geq 1}$ is any sequence in $C_A(x)$. Therefore for every $n \geq 1$, $\|a_n - a\| \geq \|x - a\|$ for every $a \in A\backslash\{a_n\}$. Since A is compact, the sequence $\|a_n\}_{n\geq 1}$ has a subsequence $\{a_{n_i}\}$ such that $a_{n_i} \to a_0 \in A$. Therefore

$$\|a_0 - a\| = lim_{i\to\infty}\|a_{n_i} - a\| \geq \|x - a\|,$$

for every $a \in A\backslash\{a_n\}$, it follows that $a_0 \in C_A(x)$. Thus $C_A(x)$ is compact.
(ii) Suppose N is a closed subset of A and $B = \{x \in D(C_A) : \ C_A(x) \cap N \neq \emptyset\}$. To show that B is closed, if x is a limit point of B. Then there exists a sequence $\{x_n\}_{n\geq 1}$ in B such that $x_n \to x$. Now, $x_n \in B$, implies that there exists a $a_n \in C_A(x_n) \cap N$, and so $\|a_n - a\| \geq \|x_n - a\|$ for every $a \in A\backslash\{x_n\}$. Since A is compact, there exists a subsequence $\{a_{n_i}\}_{i\geq 1}$ of $\{a_n\}_{n\geq 1}$ such that $a_{n_i} \to a_0$, and so $\|a_{n_i} - a\| \geq \|x_{n_i} - a\|$ for every $a \in A\backslash\{a_{n_i}\}$. Implies that $\|a_0 - a\| \geq \|x - a\|$ for every $a \in A\backslash\{a_0\}$. Therefore $a_0 \in C_A(x) \cap N$, i.e., $x \in B$, so that B is closed. Therefore C_A is upper semi-continues.

Theorem 2.14 *Let A be a compact subset of a normed linear space $(X, \|.\|)$. Then for every subset B of $D(C_A)$, the subset $C_A(B)$ is compact in A.*

Proof. Suppose $\{a_n\}_{n\geq 1}$ is a sequence in $C_A(B)$. Then there exists a $x_n \in B$, such that $a_n \in C_A(x_n)$, so that $\|a_n - a\| \geq \|x_n - a\|$ for every $a \in A\backslash\{a_n\}$. Since A is compact, there exists a subsequence $\{a_{n_i}\}_{i\geq 1}$ of $\{a_n\}_{n\geq 1}$ such that $a_{n_i} \to a_0 \in A$. Since $x_{n_i} \in A$, the compactness of B implies that the existence of a subsequence $\{x_{i_m}\}_{m\geq 1}$ such that $x_{i_m} \to x \in B$. Now, $a_{i_m} \in C_A(x_{i_m}$, implies $\|a_{i_m} - a\| \geq \|x_{i_m} - a\|$ for every $a \in A\backslash\{a_{i_m}\}$, in limiting case implies $\|a_0 - a\| \geq \|x - a\|$ for every $a \in A\backslash\{a_0\}$. Therefore $a_0 \in C_A(x) \subseteq C_A(B)$. Hence $C_A(B)$ is compact.

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