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Generalization of Dodgson's "Virtual Center" Method; an Efficient Method for Determinant Calculation

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Abstract

Charles Dodgson (1866) introduced a method to calculate matrices determinant, in a simple way. The method was highly attractive, however if the sub-matrix or the main matrix determination is divided by zero, it would not provide the correct answer. This paper explains the Dodgson method's structure and provides a solution for the problem of "dividing by zero" called "virtual center".

Key words: Determinant, Matrix center, Virtual matrix method, Dodgson method

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1 Introduction

Various methods have been used to calculate matrices determinant, including Laplus generalization, Sarrus method, Gaus method, etc. All methods have both weak points and strengths. The main weak points are:

- Being too complex

- Having too long calculations

- Errors in manual calculation

Dodgson method [1-3] was very simple; it was similar to calculation of 2×2 matrices determinant accompanied by some divisions, for example:

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 3 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$A^{(2)} = \begin{pmatrix} \begin{vmatrix} 1 & 0 \\ 1 & 3 \end{vmatrix} & \begin{vmatrix} 0 & 1 \\ 3 & 1 \end{vmatrix} \\ \begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} 3 & -3 \\ 1 & 2 \end{pmatrix}$$

$$\det(A) = A^{(1)} = \left(\frac{\begin{vmatrix} 3 & -3 \\ 1 & 2 \end{vmatrix}}{3} \right) = 3$$

where the power "(k)" denotes the order of the matrix.

Shivanian).

Definition 1

Each square matrix has a center. The center of each matrix is a $(n-2)\times(n-2)$ sub-matrix with a distance of 1 from other elements of the main matrix.

For instance, the center of a 3×3 matrix is a 1×1 matrix, e. g.

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 3 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$center = 3$$

Dodgson used the center of matrix to calculate $n\times n$ matrix determinant. In the Dodgson method each $n \times n$matrix was changed into $(n-1)^2$ 2×2 matrices, $(n-2)^2$3×3 matrices, $(n-3)^2$4×4 matrices, and so on. In calculation of $n \times n$matrices $n \geq 3$, Dodgson had to determine 3×3 matrix determinant; therefore, the center of the 3×3 matrices was non-zero. In addition, in $n\times n$ matrices, $n \geq 4$, the determinant of the center of the main matrix (A) must be non-zero.

This point may provide difficulties in some cases. To overcome the probable, it is necessary to displace the rows and columns positions, and in some cases, we may not calculate the determinant at all. However, the problem can improved considerably.

2 Virtual Center

Consider 3×3 matrix A, we can use a_{12},a_{21},a_{23} and a_{32} instead of a_{22} as the center of this matrix. Since, the elements are not in the center position, we can apply algorithm in which a_{12},a_{21},a_{23} and a_{32} are virtually center of the 3×3 matrix.

One may easily prove that this displacement is practical:

Assume that $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{pmatrix}$

Case 1) a_2 is the virtual center

Assume that $a_2 \neq 0$, then:

$$A = \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{pmatrix}$$

$$A^{(2)} = \begin{pmatrix} \begin{vmatrix} a_1 & a_2 \\ a_4 & a_5 \end{vmatrix} & \begin{vmatrix} a_2 & a_3 \\ a_5 & a_6 \end{vmatrix} \\ \begin{vmatrix} a_1 & a_2 \\ a_7 & a_8 \end{vmatrix} & \begin{vmatrix} a_2 & a_3 \\ a_8 & a_9 \end{vmatrix} \end{pmatrix}$$

$$A^{(2)} = \begin{pmatrix} a_1a_5 - a_2a_4 & a_2a_6 - a_3a_5 \\ a_1a_8 - a_2a_7 & a_2a_9 - a_3a_8 \end{pmatrix}$$

$$A^{(1)} = \frac{a_1a_5a_2a_9 - a_1a_5a_3a_8 - a_4{a^2}_2a_9 + a_2a_4a_3a_8 - a_2a_6a_8a_1 + a_6{a^2}_2a_7 + a_3a_5a_1a_8 - a_3a_5a_2a_7}{a_2}$$

$$= a_1a_5a_9 - a_4a_2a_9 + a_4a_3a_8 - a_6a_8a_1 + a_6a_2a_7 - a_3a_5a_7$$

Case 2) a_6 is the virtual center

Assume that $a_6 \neq 0$, then:

$$A = \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{pmatrix}$$

$$A^{(2)} = \begin{pmatrix} \begin{vmatrix} a_1 & a_3 \\ a_4 & a_6 \end{vmatrix} & \begin{vmatrix} a_2 & a_3 \\ a_5 & a_6 \end{vmatrix} \\ \begin{vmatrix} a_4 & a_6 \\ a_7 & a_9 \end{vmatrix} & \begin{vmatrix} a_5 & a_6 \\ a_8 & a_9 \end{vmatrix} \end{pmatrix}$$

$$A^{(2)} = \begin{pmatrix} a_1a_6 - a_3a_4 & a_2a_6 - a_3a_5 \\ a_4a_9 - a_6a_7 & a_5a_9 - a_6a_8 \end{pmatrix}$$

$$A^{(1)} = \frac{a_1a_6a_5a_9 - a_1{a^2}_6a_8 - a_3a_4a_5a_9 + a_3a_4a_6a_8 - a_2a_6a_4a_9 + a_2{a^2}_6a_7 + a_3a_5a_4a_9 - a_3a_5a_6a_7}{a_6}$$

$$= a_1a_5a_9 - a_1a_6a_8 + a_3a_4a_8 - a_2a_4a_9 + a_2a_6a_7 - a_3a_5a_7$$

Case 3) a_4 is the virtual center

Assume that $a_4 \neq 0$, then:

$$A = \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{pmatrix}$$

$$A^{(2)} = \begin{pmatrix} \begin{vmatrix} a_1 & a_2 \\ a_4 & a_5 \end{vmatrix} & \begin{vmatrix} a_1 & a_3 \\ a_4 & a_6 \end{vmatrix} \\ \begin{vmatrix} a_4 & a_5 \\ a_7 & a_8 \end{vmatrix} & \begin{vmatrix} a_4 & a_6 \\ a_7 & a_9 \end{vmatrix} \end{pmatrix}$$

$$A^{(2)} = \begin{pmatrix} a_1a_5 - a_2a_4 & a_1a_6 - a_3a_4 \\ a_4a_8 - a_5a_7 & a_4a_9 - a_6a_7 \end{pmatrix}$$

$$A^{(1)} = \frac{a_1a_5a_4a_9 - a_1a_5a_6a_7 - a_2{a^2}_4a_9 + a_2a_4a_6a_7 - a_1a_6a_4a_8 + a_1a_6a_7a_5 + a_3{a^2}_4a_8 - a_3a_4a_5a_7}{a_4}$$

$$= a_1a_5a_9 - a_2a_4a_9 + a_2a_6a_7 - a_1a_6a_8 + a_3a_4a_8 - a_3a_5a_7$$

Case 4) a_8 is the virtual center

Assume that $a_8 \neq 0$, then:

$$A = \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{pmatrix}$$

$$A^{(2)} = \begin{pmatrix} \begin{vmatrix} a_1 & a_2 \\ a_7 & a_8 \end{vmatrix} & \begin{vmatrix} a_2 & a_3 \\ a_8 & a_9 \end{vmatrix} \\ \begin{vmatrix} a_4 & a_5 \\ a_7 & a_8 \end{vmatrix} & \begin{vmatrix} a_5 & a_6 \\ a_8 & a_9 \end{vmatrix} \end{pmatrix}$$

$$A^{(2)} = \begin{pmatrix} a_1a_8 - a_2a_7 & a_2a_9 - a_3a_8 \\ a_4a_8 - a_5a_7 & a_5a_9 - a_6a_8 \end{pmatrix}$$

$$A^{(1)} =$$

$$\frac{a_1a_8a_5a_9 - a_1{a^2}_8a_6 - a_2a_7a_5a_9 + a_2a_7a_6a_8 - a_2a_9a_4a_8 + a_2a_9a_5a_7 + a_3{a^2}_8a_4 - a_3a_8a_5a_7}{a_8}$$

$$= a_1a_5a_9 - a_1a_8a_6 + a_2a_7a_6 - a_2a_9a_4 + a_3a_8a_4 - a_3a_5a_7$$

The above proof methods are called U, R, L, D and the Dodgson method is called M. Consider the following matrix

$$A = \begin{pmatrix} 2 & 3 & 5 \\ 1 & 0 & 6 \\ 7 & 2 & 1 \end{pmatrix}$$

Based on Dodgson method, we have

$$A^{(2)} = \begin{pmatrix} \begin{vmatrix} 2 & 3 \\ 1 & 0 \end{vmatrix} & \begin{vmatrix} 3 & 5 \\ 0 & 6 \end{vmatrix} \\ \begin{vmatrix} 1 & 0 \\ 7 & 2 \end{vmatrix} & \begin{vmatrix} 0 & 6 \\ 2 & 1 \end{vmatrix} \end{pmatrix}$$

$$A^{(2)} = \begin{pmatrix} -3 & 18 \\ 2 & -12 \end{pmatrix}$$

$$A^{(1)} = \left(\frac{\begin{vmatrix} -3 & 18 \\ 2 & -12 \end{vmatrix}}{0} \right) = 0$$

As it is seen, Dodgson method faced difficulty with zero at center and could not calculate determinant, correctly. In this method, we have changed the positions of rows and columns to overcome the problem, so that the center is no more at the center. As a result, the final determination is multiplied by -1 and the number of multiplications increases for calculation of the determination. However, if we apply one of the virtual center methods, we would not face such a problem. Now, we apply method L then

$$A^{(2)} = \begin{pmatrix} \begin{vmatrix} 2 & 3 \\ 1 & 0 \end{vmatrix} & \begin{vmatrix} 2 & 5 \\ 1 & 6 \end{vmatrix} \\ \begin{vmatrix} 1 & 0 \\ 7 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 6 \\ 7 & 1 \end{vmatrix} \end{pmatrix}$$

$$A^{(1)} = \left(\frac{\begin{vmatrix} -3 & 7 \\ 2 & -41 \end{vmatrix}}{1} \right) = 109$$

If we apply method R, then we have

$$A^{(2)} = \begin{pmatrix} \begin{vmatrix} 2 & 5 \\ 1 & 6 \end{vmatrix} & \begin{vmatrix} 3 & 5 \\ 0 & 6 \end{vmatrix} \\ \begin{vmatrix} 1 & 6 \\ 7 & 1 \end{vmatrix} & \begin{vmatrix} 0 & 6 \\ 2 & 1 \end{vmatrix} \end{pmatrix}$$

$$A^{(2)} = \begin{pmatrix} 7 & 18 \\ -41 & -12 \end{pmatrix}$$

$$A^{(1)} = \left(\frac{\begin{vmatrix} 7 & 18 \\ -41 & -12 \end{vmatrix}}{6} \right) = 109$$

Now, consider the following example

$$A = \begin{pmatrix} 1 & 3 & 3 & 4 \\ 6 & 5 & 0 & 9 \\ 1 & 3 & 8 & 10 \\ 2 & 4 & 3 & 1 \end{pmatrix}$$

Matrix $A$is a 4×4 matrix and has a 2×2 center. This center is $\begin{bmatrix} 5 & 0 \\ 3 & 8 \end{bmatrix}$ and the determinant is non-zero; therefore, it has no difficulty, concerning the center; and to calculate determinant of matrix A, we need to calculate determinant 4 of the 3×3 sub-matrix of A_1, A_2, A_3 and A_4:

$$A_1 = \begin{bmatrix} 1 & 3 & 3 \\ 6 & 5 & 0 \\ 1 & 3 & 8 \end{bmatrix}, A_2 = \begin{bmatrix} 3 & 3 & 4 \\ 5 & 0 & 9 \\ 3 & 8 & 10 \end{bmatrix}, A_3 = \begin{bmatrix} 6 & 5 & 0 \\ 1 & 3 & 8 \\ 2 & 4 & 3 \end{bmatrix}, A_4 = \begin{bmatrix} 5 & 0 & 9 \\ 3 & 8 & 10 \\ 4 & 3 & 1 \end{bmatrix}$$

As it is seen, the center of the sub determinant A_2 is zero, because of being in the center. That is why the determinant of the matrix A would be calculated wrongly, based on the Dodgson method. To overcome this problem, we can displace the row 2 by the first or fourth row and the third column by the first or fourth column. The number of necessary multiplication, divisions, addition and subtraction for the Dodgson algorithm is4 $\left[(n-1)^2 + (n-2)^2 + ... + 1\right] \approx 4 \bullet n \bullet n^2$, when we displace a row column, we actually add this value.

Condition are not always so proper for the Dodgson algorithm, and a simple displacement would not solve the problem. Sometimes it may require 2 and even 3 displacements. In some cases, the algorithm is totally inconsistent and unable to calculate determinant. We are going to present a complementary method for Dodgson method and reduce number of displacement to one.

Theorem (Jacobi's Theorem). Let

Abe an $n \times n$ matrix;

M an $m \times m$ minor of$A$, where $m < n$, chosen from rows $i_1, ..., i_m$ and columns $j_1, ..., j_m$,

M' the corresponding $m \times m$ minor ofA', the matrix of cofactors of$A$, and

$M^*$the $(n-m) \times (n-m)$ minor of$A$complementary to MThen $\det M' = \det(A)^{m-1}. \det M^*.(-1)^{\sum_{L=1}^{m} i_L + j_L}$

3 Virtual center algorithm

Assume that A_n is a non-diametric $n \times n$matrix with Has its center. It firstly, the center should not be of determinant zero; otherwise we displace columns and rows. Hwould be as follow and since there is a 3×3 matrix in $A_n (n \geq 3)$, the centers of the 3×3 matrices form the components of $H$from left to night, respectively, i.e.

$$H = \begin{pmatrix} a_{2,2} & \cdots & a_{2,n} \\ \vdots & \ddots & \vdots \\ a_{(n-1),2} & \cdots & a_{(n-1),(n-1)} \end{pmatrix}$$

1- ML combination:

For $n = 4$, we use this combination when Hhas no zero or zeroes are in the last column.

2- RM combination:

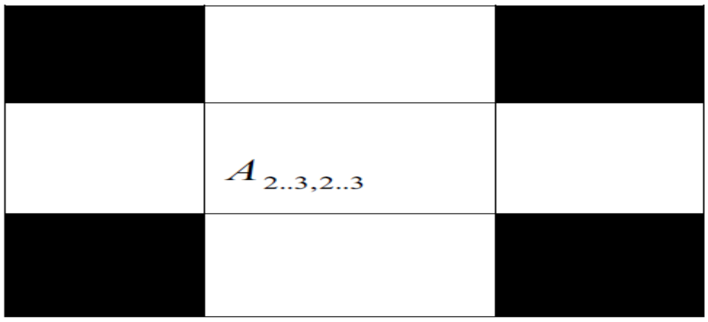

Fig. 1. Implementation of the Jacobian theorem for 4×4 matrices.

For ML, we use this combination when Hhas no zero or zeroes are in the first column.

In both of the above combination, the maximum number of zeros in the first or last column may not exceed $((n-2)-3)$.

3- RMLcombination:

For matrices of $n \geq 5$, this combination is applied. There may no zero in Hor zeroes may be at first and the last columns.

In this combination, the maximum permitted number of zeroes in the first and the last columns is $((n-2)-1)$; and no two zeroes are allowed to be adjacent and no column should be totally zero.

Consider the 4×4 matrix at the end of section 1 and 2. The last contraction would result in.

$$\det A^{(1)} = \left(\frac{|A^{(2)}|}{A_{22}^{(3)}}\right) = \left(\frac{|A_{1..3,1..3}|\,|A_{2..4,2..4}| - |A_{2..4,1..3}|\,|A_{1..3,2..4}|}{|A_{2..3,2..3}|}\right)$$

Concerning the Jacobian theorem (figure 1), we select 2×2 sub-matrices from the corners of A:

$$M = \begin{pmatrix} a_{11} & a_{14} \\ a_{41} & a_{44} \end{pmatrix}$$

Whose complement is the sub-matrix of $M^* = (A_{2..3,2..3})$.The cofactor matrix is:

$$M' = \begin{pmatrix} A'_{11} & -A'_{14} \\ -A'_{41} & A'_{44} \end{pmatrix}$$

Where, $A'_{11} = (A_{2..4,2..4})$, $A'_{44} = (A_{1..3,1..3})$, $A'_{14} = (A_{2..4,1..3})$, $A'_{41} = (A_{1..3,2..4})$; using the Jacobian theorem we would have:

$$\det\left(M'\right) = (\det A)^{2-1} \cdot \det M^* \cdot (-1)^{1+1+4+4}$$

$$\det A = \frac{\det M'}{\det M*}$$

$$\det A^{(1)} = \left(\frac{\left|A^{(2)}\right|}{A^{(3)}{}_{22}}\right) = \left(\frac{|A_{1..3,1..3}|\,|A_{2..4,2..4}| - |A_{2..4,1..3}|\,|A_{1..3,2..4}|}{|A_{2..3,2..3}|}\right)$$

Note: This is implemented just in the last step of virtual center in A. Moreover, when determinant is being calculated, the minus values of M' are not considered.

Proof (combination 1):

Assume that A is a 4×4 matrix. We show that ML combination uses Jacobian theorem

$$A^{(3)} = \begin{pmatrix} |A_{1..2,1..2}| & |A_{1..2,2..3}| & |A_{1..2,2..4}| \\ |A_{2..3,1..2}| & |A_{2..3,2..3}| & |A_{2..3,2..4}| \\ |A_{3..4,1..2}| & |A_{3..4,2..3}| & |A_{3..4,2..4}| \end{pmatrix}$$

And

$$A^{(2)} = \begin{pmatrix} \dfrac{\begin{vmatrix} |A_{1..2,1..2}| & |A_{1..2,2..3}| \\ |A_{2..3,1..2}| & |A_{2..3,2..3}| \end{vmatrix}}{a_{22}} & \dfrac{\begin{vmatrix} |A_{1..2,2..3}| & |A_{1..2,2..4}| \\ |A_{2..3,2..3}| & |A_{2..3,2..4}| \end{vmatrix}}{a_{22}} \\ \dfrac{\begin{vmatrix} |A_{2..3,1..2}| & |A_{2..3,2..3}| \\ |A_{3..4,1..2}| & |A_{3..4,2..3}| \end{vmatrix}}{a_{32}} & \dfrac{\begin{vmatrix} |A_{2..3,2..3}| & |A_{2..3,2..4}| \\ |A_{3..4,2..3}| & |A_{3..4,2..4}| \end{vmatrix}}{a_{32}} \end{pmatrix}$$

To show that A^2 is consistent with the Jacobian theorem consider a 3×3 submatrix from the upper corner of A^4:

$$A = A_{1..3,1..3} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Removing the row 2 and the column 2 from A, would result in:

$$M = \begin{pmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{pmatrix}$$

Whose complementary in A is $M^* = (a_{22})$and A' s are 2×2 matrices similar to cofactor M.

$$M' = \begin{pmatrix} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \end{pmatrix} = \begin{pmatrix} |A_{2..3,2..3}| & |A_{2..3,1..2}| \\ |A_{1..2,2..3}| & |A_{1..2,1..2}| \end{pmatrix}$$

We use Jacobian theorem and displacement of rows and columns:

$$\det\left(M'\right) = (\det A)^{2-1} \cdot \det M^* \cdot (-1)^{1+1+3+3}$$

$$\det A = \frac{\det M'}{\det M*}$$

$$\det A = \frac{\begin{vmatrix} |A_{1..2,1..2}| & |A_{1..2,2..3}| \\ |A_{2..3,1..2}| & |A_{2..3,2..3}| \end{vmatrix}}{a_{22}}$$

This determinant equals to the value of the upper right corner of A^2 . Also:

The value of the below right corner of A^2 equals $A_{2..4H2..4}$, which is calculated using the same process. The next step and the last step is:

$$\det A^{(1)} = \left(\frac{|A^{(2)}|}{A^{(3)}{}_{22}}\right) = \left(\frac{|A_{1..3,1..3}|\,|A_{2..4,2..4}| - |A_{2..4,1..3}|\,|A_{1..3,2..4}|}{|A_{2..3,2..3}|}\right)$$

The Jacobian theorem is implemented in A; we use 2×2 sub-matrices in A and achieve:

$$|A| = \frac{|A_{1..3,1..3}|\,|A_{2..4,2..4}| - |A_{2..4,1..3}|\,|A_{1..3,2..4}|}{|A_{2..3,2..3}|}$$

Proof (combination 2)

Assume that A is a 4×4 matrix. We show that RM combination uses the Jacobian theorem.

$$A^{(3)} = \begin{pmatrix} |A_{1..2,1..3}| & |A_{1..2,2..3}| & |A_{1..2,3..4}| \\ |A_{2..3,1..3}| & |A_{2..3,2..3}| & |A_{2..3,3..4}| \\ |A_{3..4,3..3}| & |A_{3..4,2..3}| & |A_{3..4,3..4}| \end{pmatrix}$$

$$A^{(2)} = \begin{pmatrix} \frac{\begin{vmatrix} |A_{1..2,1..3}| & |A_{1..2,2..3}| \\ |A_{2..3,1..3}| & |A_{2..3,2..3}| \end{vmatrix}}{a_{23}} & \frac{\begin{vmatrix} |A_{1..2,2..3}| & |A_{1..2,3..4}| \\ |A_{2..3,2..3}| & |A_{2..3,3..4}| \end{vmatrix}}{a_{23}} \\ \frac{\begin{vmatrix} |A_{2..3,1..3}| & |A_{2..3,2..3}| \\ |A_{3..4,3..3}| & |A_{3..4,2..3}| \end{vmatrix}}{a_{33}} & \frac{\begin{vmatrix} |A_{2..3,2..3}| & |A_{2..3,3..4}| \\ |A_{3..4,2..3}| & |A_{3..4,3..4}| \end{vmatrix}}{a_{33}} \end{pmatrix}$$

To show consistency of A^2 with the Jacobian theorem, consider a 3×3 matrix from the upper left corner of A^4:

$$A = A_{1..3,1..3} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Omitting row 2 and column 2, we achieve:

$$M = \begin{pmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{pmatrix}$$

Whose complement in A is $M^* = (a_{23})$and A's are 2×2 matrices similar to cofactor M.

$$M' = \begin{pmatrix} \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \\ \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \end{pmatrix} = \begin{pmatrix} |A_{1..2,1..3}| & |A_{1..2,2..3}| \\ |A_{2..3,1..3}| & |A_{2..3,2..3}| \end{pmatrix}$$

We use Jacobian theorem and displacement of rows and columns to achieve:

$$\det\left(M'\right) = (\det A)^{2-1} \cdot \det M^* \cdot (-1)^{1+1+3+3}$$

$$\det A = \frac{\det M'}{\det M*}$$

$$\det A = \frac{\begin{pmatrix} |A_{1..2,1..3}| & |A_{1..2,2..3}| \\ |A_{2..3,1..3}| & |A_{2..3,2..3}| \end{pmatrix}}{a_{23}}$$

This determinant equals to the value of the upper left corner of A^2. Moreover:

- The same method is used to calculate the value of lower left corner of A^2 which is equal to determinant $A_{2..4H1..3}$.

For $A_{1..3,2..4}$consider

$$A = A_{1..3H2..4} = \begin{pmatrix} a_{12} & a_{13} & a_{14} \\ a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \end{pmatrix}$$

Deleting row 2 and column 2, we achieve:

$$M = \begin{pmatrix} a_{22} & a_{24} \\ a_{42} & a_{44} \end{pmatrix}$$

Whose complement in A is$M^* = (a_{23})$. A's are 2×2 matrices similar to cofactor

M.

$$M' = \begin{pmatrix} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} & \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix} \\ \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix} \end{pmatrix} = \begin{pmatrix} |A_{1..2,2..3}| & |A_{1..2,3..4}| \\ |A_{2..3,2..3}| & |A_{2..3,3..4}| \end{pmatrix}$$

Using Jacobian theorem and displacement of rows and columns, we would achieve:

$$\det\left(M'\right) = (\det A)^{2-1} \cdot \det M^* \cdot (-1)^{1+1+3+3}$$

$$\det A = \frac{\det M'}{\det M*}$$

$$\det A = \frac{\begin{pmatrix} |A_{1..2,2..3}| & |A_{1..2,3..4}| \\ |A_{2..3,2..3}| & |A_{2..3,3..4}| \end{pmatrix}}{a_{23}}$$

This determinant equals to the value of the upper left corner of A^2. Also:

- The value of lower left A^2 equals to $A_{2..4H2..4}$determinant which is calculated using the same method. For$A_{1..3,2..4}$, consider:

$$A = A_{1..3H2..4} = \begin{pmatrix} a_{12} & a_{13} & a_{14} \\ a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \end{pmatrix}$$

Deleting row 2 and column 2 we would have:

$M = \begin{pmatrix} a_{22} & a_{24} \\ a_{42} & a_{44} \end{pmatrix}$ Whose complement in A is M^*= (a_{23}). A's are 2×2 matrices similar to cofactor M.

$$M' = \begin{pmatrix} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} & \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix} \\ \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix} \end{pmatrix} = \begin{pmatrix} |A_{1..2,2..3}| & |A_{1..2,3..4}| \\ |A_{2..3,2..3}| & |A_{2..3,3..4}| \end{pmatrix}$$

Using the Jacobian theorem and displacement of rows and columns, we would achieve:

$$\det\left(M'\right) = (\det A)^{2-1} \cdot \det M^{*} \cdot (-1)^{1+1+3+3}$$

$$\det A = \frac{\det M'}{\det M*}$$

$$\det A = \frac{\begin{pmatrix} |A_{1..2,2..3}| & |A_{1..2,3..4}| \\ |A_{2..3,2..3}| & |A_{2..3,3..4}| \end{pmatrix}}{a_{23}}$$

This determinant equals to the value of the upper left corner of A^2. Also:

- The lower left corner value of A^2 equals to the determinant $A_{2..4H2..4}$,Which is calculated using the same method. For $A_{1..3,2..4}$consider:

$$A = A_{1..3H2..4} = \begin{pmatrix} a_{12} & a_{13} & a_{14} \\ a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \end{pmatrix}$$

Deleting row 2 and columns, we would achieve:

$$M = \begin{pmatrix} a_{22} & a_{24} \\ a_{42} & a_{44} \end{pmatrix}$$

Whose complement is$M^{*} = (a_{23})$. A's are 2×2 matrices similar to cofactor M.

$$M' = \begin{pmatrix} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} & \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix} \\ \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix} \end{pmatrix} = \begin{pmatrix} |A_{1..2,2..3}| & |A_{1..2,3..4}| \\ |A_{2..3,2..3}| & |A_{2..3,3..4}| \end{pmatrix}$$

Using the Jacobian theorem and displacement of rows and columns. We would have:

$$\det\left(M'\right) = (\det A)^{2-1} \cdot \det M^{*} \cdot (-1)^{1+1+3+3}$$

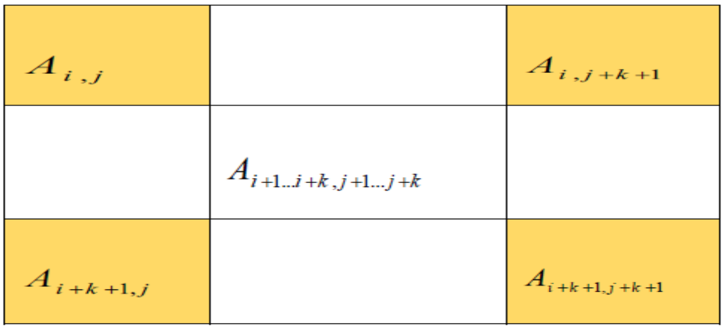

Fig. 2. Implementation of the Jacobian theorem, using the virtual center method for n×n matrices.

$$\det A = \frac{\det M'}{\det M*}$$

$$\det A = \frac{\begin{pmatrix} |A_{1..2,2..3}| & |A_{1..2,3..4}| \\ |A_{2..3,2..3}| & |A_{2..3,3..4}| \end{pmatrix}}{a_{23}}$$

This determinant equals to upper right corner of A^2. Also:

- The Value of lower right corner of A^2 equals to determinant $A_{2..4H2..4}$, which is calculated using the same method.

The next step and the last step is:

$$\det A^{(1)} = \left(\frac{|A^{(2)}|}{A^{(3)}{}_{22}}\right) = \left(\frac{|A_{1..3,1..3}|\,|A_{2..4,2..4}| - |A_{2..4,1..3}|\,|A_{1..3,2..4}|}{|A_{2..3,2..3}|}\right)$$

The Jacobian theorem is implemented on A and use M^*s of the 2×2 sub-matrices in A_1. Then:

$$|A| = \frac{|A_{1..3,1..3}|\,|A_{2..4,2..4}| - |A_{2..4,1..3}|\,|A_{1..3,2..4}|}{|A_{2..3,2..3}|}$$

Consider a $(K+1)\times(K+1)$ sub-matrix from the matrix A, so that $A =$

$A_{i...i+k+1,j...j+k+1}$. The 2×2 matrix M would be:

$$M = \begin{pmatrix} A_{i,j} & A_{i,j+k+1} \\ A_{i+k+1,j} & A_{i+k+1,j+k+1} \end{pmatrix}$$

Whose supplement is $M^* = (A_{i+1...i+k,j+1...j+k})$. The M cofactor matrix in A is

$$M' = \begin{pmatrix} A'_{i,j} & \pm A'_{i,j+k+1} \\ \pm A'_{i+k+1,j} & A'_{i+k+1,j+k+1} \end{pmatrix}$$

So that:

$$A'_{i,j} = |A_{i+1...i+k+1,j+1...j+k+1}|$$

$$A'_{i,j+k+1} = |A_{i+1...i+k+1,j...j+k}|$$
$$A'_{i+k+1,j} = |A_{i...i+k,j+1...j+k+1}|$$
$$A'_{i+k+1,j+k+1} = |A_{i...i+k,j...j+k}|$$

Using Jacobian theorem, we would achieve:

$$\det M = \frac{\left|A'_{i,j}\right| \cdot \left|A'_{i+k+1,j+k+1}\right| - \left|A'_{i,j+k+1}\right| \cdot \left|A'_{i+k+1,j}\right|}{|A_{i+1...i+k,j+1...j+k}|}$$

Proof (combination 3)

Using deduction on K, we prove that, Deduction basis: $K = 1$ is a certain theorem: provides a contraction:

$$A^{(n-1)} = \begin{pmatrix} |A_{1..2,1..3}| & |A_{1..2,2..3}| & \cdots & |A_{1..2,n-1...n}| \\ |A_{2..3,1..3}| & |A_{2..i+2,2..i+2}| & \cdots & |A_{2..3,n-2...n}| \\ \vdots & \vdots & \ddots & \vdots \\ |A_{n-1...n,1...3}| & |A_{n-i...n,2...i+2}| & \cdots & |A_{n-1...n,n-2...n}| \end{pmatrix}$$

Deduction assumption: K is constant. Assume that, for $L = 1, ..., K$, the Lth contraction results in $A^{(N-L)}$ so that for all $i \geq 1, j \leq n$,we have:

$$A_{i,j}^{(N-L)} = \left|A_{i...i+(N-L),j...j+(N-L)}\right|$$

Deduction procedures: Firstly, we assume that $K = 1$; then

$i \in \{1, ..., n-2\}$, $j = 1$

$$A_{i,j}^{(n-(k+1))} = \frac{A_{i,j}^{(n-k)} \times A_{i+1,j+1}^{(n-k)} - A_{i+1,j}^{(n-k)} \times A_{i,j+1}^{(n-k)}}{A_{i+1,j+2}^{(n-(k-1))}}$$

For $i \in \{1, ..., n-2\}$, $j \geq 2, j < n-2$

$$A_{i,j}^{(n-(k+1))} = \frac{A_{i,j}^{(n-k)} \times A_{i+1,j+1}^{(n-k)} - A_{i+1,j}^{(n-k)} \times A_{i,j+1}^{(n-k)}}{A_{i+1,j}^{(n-(k-1))}}$$

For $i \in \{1, ..., n-2\}$, $j = n-2$

$$A_{i,j}^{(n-(k+1))} = \frac{A_{i,j}^{(n-k)} \times A_{i+1,j+1}^{(n-k)} - A_{i+1,j}^{(n-k)} \times A_{i,j+1}^{(n-k)}}{A_{i+1,j}^{(n-(k-1))}}$$

And for $K \geq 2$

$$A_{i,j}^{(n-(k+1))} = \frac{A_{i,j}^{(n-k)} \times A_{i+1,j+1}^{(n-k)} - A_{i+1,j}^{(n-k)} \times A_{i,j+1}^{(n-k)}}{A_{i+1,j+1}^{(n-(k-1))}}$$

Using the deduction assumption, we can displace, when $K = 1$. For $i \in \{1, ..., n-2\}$, $j = 1$:

$$A_{i,j}^{(n-(k+1))} = \left(\frac{|A_{i...i+k,j...j+k+1}|\,|A_{i+1...i+k+1,j+1...j+k+1}| - |A_{i+1...i+k+1,j...j+k+1}|\,|A_{i...i+k,j+1...j+k+1}|}{|A_{i+1...i+k,j+2...j+k+1}|} \right)$$

For $i \in \{1, ..., n-2\}$, $j \geq 2, j < n-2$

$$A_{i,j}^{(n-(k+1))} = \left(\frac{|A_{i...i+k,j...j+k}|\,|A_{i+1...i+k+1,j+1...j+k+1}| - |A_{i+1...i+k+1,j...j+k}|\,|A_{i...i+k,j+1...j+k+1}|}{|A_{i+1...i+k,j+1...j+k}|} \right)$$

For $i \in \{1, ..., n-2\}$, $j = n-2$

$$A_{i,j}^{(n-(k+1))} = \left(\frac{|A_{i...i+k,j...j+k}|\,|A_{i+1...i+k+1,j...j+k+1}| - |A_{i+1...i+k+1,j...j+k}|\,|A_{i...i+k,j...j+1}|}{|A_{i+1...i+k,j+1...j+k}|} \right)$$

And for $K \geq 2$

$$A_{i,j}^{(n-(k+1))} = \left(\frac{|A_{i...i+k,j...j+k}|\,|A_{i+1...i+k+1,j+1...j+k+1}| - |A_{i+1...i+k+1,j...j+k}|\,|A_{i...i+k,j+1...j+k+1}|}{|A_{i+1...i+k,j+1...j+k}|} \right)$$

We use the Jacobian theorem, based on the following condition:

$$A = A_{i....i+k+1,j...j+k+1}$$

Ms are 2×2 matrices resulted from corners of A;

M's are component 2×2 matrices corresponding to A';

M* is complementary k×k matric corresponding to A;

Ultimately:

$$A_{i,j}^{(n-(k+1))} = |A_{i...i+k,j...j+k+1}|$$

Example 5

$$A = \begin{pmatrix} 1 & 0 & 3 & 2 & 1 \\ 2 & 1 & 2 & 0 & 0 \\ 1 & 3 & 2 & 0 & 2 \\ 1 & 2 & 3 & 1 & 4 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix}$$

We use RML method, because zeros are at the last column of the center; and since the center of the two sub-matrices$H_3 = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 0 & 0 \\ 2 & 0 & 2 \end{pmatrix}$, $H_6 = \begin{pmatrix} 2 & 0 & 0 \\ 2 & 0 & 2 \\ 3 & 1 & 4 \end{pmatrix}$ are zero, we cannot use the Dodgson method.

$$A^{(4)} = \begin{pmatrix} \begin{vmatrix} 1 & 3 \\ 2 & 2 \end{vmatrix} & \begin{vmatrix} 0 & 3 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 3 & 2 \\ 2 & 0 \end{vmatrix} & \begin{vmatrix} 3 & 1 \\ 2 & 0 \end{vmatrix} \\ \begin{vmatrix} 2 & 2 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 0 \\ 2 & 0 \end{vmatrix} & \begin{vmatrix} 2 & 0 \\ 2 & 2 \end{vmatrix} \\ \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} & \begin{vmatrix} 3 & 2 \\ 2 & 3 \end{vmatrix} & \begin{vmatrix} 2 & 0 \\ 3 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 2 \\ 3 & 4 \end{vmatrix} \\ \begin{vmatrix} 1 & 3 \\ 0 & 0 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ 0 & 0 \end{vmatrix} & \begin{vmatrix} 3 & 1 \\ 0 & 0 \end{vmatrix} & \begin{vmatrix} 3 & 4 \\ 0 & 5 \end{vmatrix} \end{pmatrix}$$

$$A^{(4)} = \begin{pmatrix} -4 & -3 & -4 & -2 \\ 2 & -4 & 0 & 4 \\ 1 & 5 & 2 & 2 \\ 0 & 0 & 0 & 15 \end{pmatrix}$$

$$A^{(3)} = \begin{pmatrix} \frac{1}{2}\begin{vmatrix} -4 & -3 \\ 2 & -4 \end{vmatrix} & \frac{1}{2}\begin{vmatrix} -3 & -4 \\ -4 & 0 \end{vmatrix} & \frac{1}{2}\begin{vmatrix} -4 & -2 \\ 0 & 4 \end{vmatrix} \\ \frac{1}{2}\begin{vmatrix} 2 & -4 \\ 1 & 5 \end{vmatrix} & \frac{1}{2}\begin{vmatrix} -4 & 0 \\ 5 & 2 \end{vmatrix} & \frac{1}{2}\begin{vmatrix} 0 & 4 \\ 2 & 2 \end{vmatrix} \\ \frac{1}{3}\begin{vmatrix} 1 & 5 \\ 0 & 0 \end{vmatrix} & \frac{1}{3}\begin{vmatrix} 5 & 2 \\ 0 & 0 \end{vmatrix} & \frac{1}{3}\begin{vmatrix} 2 & 2 \\ 0 & 15 \end{vmatrix} \end{pmatrix}$$

$$A^{(3)} = \begin{pmatrix} 11 & -8 & -8 \\ 7 & -4 & -4 \\ 0 & 0 & 10 \end{pmatrix}$$

$$A^{(2)} = \begin{pmatrix} \frac{1}{-4}\begin{vmatrix} 11 & -8 \\ 7 & -4 \end{vmatrix} & \frac{1}{0}\begin{vmatrix} -8 & -8 \\ -4 & -4 \end{vmatrix} \\ \frac{1}{5}\begin{vmatrix} 7 & -4 \\ 0 & 0 \end{vmatrix} & \frac{1}{2}\begin{vmatrix} -4 & -4 \\ 0 & 10 \end{vmatrix} \end{pmatrix}$$

$$A^{(2)} = \begin{pmatrix} -3 & 0 \\ 0 & -20 \end{pmatrix}$$

$$A^{(1)} = \left(\frac{1}{-4}\begin{vmatrix} -3 & 0 \\ 0 & -20 \end{vmatrix} \right) = -15$$

Example 6:

$$A = \begin{pmatrix} 1 & 3 & 3 & 4 \\ 6 & 5 & 0 & 9 \\ 1 & 3 & 8 & 10 \\ 2 & 4 & 3 & 1 \end{pmatrix}$$

We use ML method to calculate determinant:

$$center = H = \begin{pmatrix} 5 & 0 \\ 3 & 8 \end{pmatrix}$$

$$\det(H) = 40 \neq 0$$

$$A^{(3)} = \begin{pmatrix} \begin{vmatrix} 1 & 3 \\ 6 & 5 \end{vmatrix} & \begin{vmatrix} 3 & 3 \\ 5 & 0 \end{vmatrix} & \begin{vmatrix} 3 & 4 \\ 5 & 9 \end{vmatrix} \\ \begin{vmatrix} 6 & 5 \\ 1 & 3 \end{vmatrix} & \begin{vmatrix} 5 & 0 \\ 3 & 8 \end{vmatrix} & \begin{vmatrix} 5 & 9 \\ 3 & 10 \end{vmatrix} \\ \begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} & \begin{vmatrix} 3 & 8 \\ 4 & 3 \end{vmatrix} & \begin{vmatrix} 3 & 10 \\ 4 & 1 \end{vmatrix} \end{pmatrix}$$

$$A^{(3)} = \begin{pmatrix} -13 & -15 & 7 \\ 13 & 40 & 23 \\ -2 & -23 & -37 \end{pmatrix}$$

$$A^{(2)} = \begin{pmatrix} \frac{1}{5}\begin{vmatrix} -13 & -15 \\ 13 & 40 \end{vmatrix} & \frac{1}{5}\begin{vmatrix} -15 & -7 \\ 40 & 23 \end{vmatrix} \\ \frac{1}{3}\begin{vmatrix} 13 & 40 \\ -2 & -23 \end{vmatrix} & \frac{1}{3}\begin{vmatrix} 40 & 23 \\ -23 & -37 \end{vmatrix} \end{pmatrix}$$

$$A^{(2)} = \begin{pmatrix} -65 & -125 \\ -73 & -317 \end{pmatrix}$$

$$A^{(1)} = \left(\frac{1}{40} \begin{vmatrix} -65 & -125 \\ -73 & -317 \end{vmatrix} \right) = 287$$

Example 6:

$$A = \begin{pmatrix} 1 & 3 & 3 & 4 \\ 6 & 5 & 0 & 9 \\ 1 & 3 & 8 & 10 \\ 2 & 4 & 3 & 1 \end{pmatrix}$$

We use RM method to calculate determinant

$$center = H = \begin{pmatrix} 0 & 7 \\ 3 & 5 \end{pmatrix}$$

$$\det(H) = -21 \neq 0$$

$$A^{(3)} = \begin{pmatrix} \begin{vmatrix} 1 & 3 \\ 6 & 7 \end{vmatrix} & \begin{vmatrix} 3 & 3 \\ 0 & 7 \end{vmatrix} & \begin{vmatrix} 3 & 4 \\ 7 & 9 \end{vmatrix} \\ \begin{vmatrix} 6 & 7 \\ 1 & 5 \end{vmatrix} & \begin{vmatrix} 0 & 7 \\ 3 & 5 \end{vmatrix} & \begin{vmatrix} 7 & 9 \\ 5 & 6 \end{vmatrix} \\ \begin{vmatrix} 1 & 5 \\ 6 & 3 \end{vmatrix} & \begin{vmatrix} 3 & 5 \\ 4 & 3 \end{vmatrix} & \begin{vmatrix} 5 & 6 \\ 3 & 1 \end{vmatrix} \end{pmatrix}$$

$$A^{(3)} = \begin{pmatrix} -11 & 21 & -1 \\ 23 & -21 & -3 \\ -27 & -11 & -13 \end{pmatrix}$$

$$A^{(2)} = \begin{pmatrix} \frac{1}{7} \begin{vmatrix} -11 & 21 \\ 23 & -21 \end{vmatrix} & \frac{1}{7} \begin{vmatrix} 21 & -1 \\ -21 & -3 \end{vmatrix} \\ \frac{1}{5} \begin{vmatrix} 23 & -21 \\ -27 & -11 \end{vmatrix} & \frac{1}{5} \begin{vmatrix} -21 & -3 \\ -11 & -13 \end{vmatrix} \end{pmatrix}$$

$$A^{(2)} = \begin{pmatrix} -36 & -12 \\ -164 & 48 \end{pmatrix}$$

$$A^{(1)} = \left(\frac{1}{-21} \begin{vmatrix} -36 & -12 \\ -164 & 48 \end{vmatrix} \right) = 176$$

Example 7:

$$A = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 5 & 3 & 1 & 0 \\ 1 & 3 & 2 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 2 & 0 & 2 & 0 & 1 \end{pmatrix}$$

$$center = H = \begin{pmatrix} 5 & 3 & 1 \\ 3 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$\det(H) = \frac{1}{3} \begin{pmatrix} \begin{vmatrix} 5 & 3 \\ 3 & 2 \end{vmatrix} & \begin{vmatrix} 5 & 1 \\ 3 & 1 \end{vmatrix} \\ \begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} = 0$$

$\det(H) = 0$, to resolve this problem, we have to displace a row column in the center with a row and column out of the center

$$A = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 5 & 3 & 0 & 1 \\ 1 & 3 & 2 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 2 & 0 & 2 & 1 & 0 \end{pmatrix}$$

$$H = \begin{pmatrix} 0 & 3 & 1 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$\det(H) = \frac{1}{1} \begin{pmatrix} \begin{vmatrix} 0 & 3 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} \\ \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} \end{pmatrix} = \frac{1}{1} \begin{pmatrix} -3 & -1 \\ 1 & 1 \end{pmatrix} = -2$$

$\det(H) = 0$; now we can calculate RML determinant of the matrix A, using the RMLmethod

$$A^{(4)} = \begin{pmatrix} \begin{vmatrix} 1 & 1 \\ 0 & 3 \end{vmatrix} & \begin{vmatrix} 0 & 1 \\ 5 & 3 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 3 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 3 & 1 \end{vmatrix} \\ \begin{vmatrix} 0 & 3 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 5 & 3 \\ 3 & 2 \end{vmatrix} & \begin{vmatrix} 3 & 0 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 3 & 1 \\ 2 & 1 \end{vmatrix} \\ \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} \\ \begin{vmatrix} 0 & 1 \\ 2 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix} \end{pmatrix}$$

$$A^{(4)} = \begin{pmatrix} 3 & -5 & -3 & 1 \\ -3 & 1 & 3 & 1 \\ 1 & 1 & -1 & 1 \\ -2 & 2 & 1 & -2 \end{pmatrix}$$

$$A^{(3)} = \begin{pmatrix} \frac{1}{3}\begin{vmatrix} 3 & -5 \\ -3 & 1 \end{vmatrix} & \frac{1}{3}\begin{vmatrix} -5 & -3 \\ 1 & 3 \end{vmatrix} & \frac{1}{3}\begin{vmatrix} -3 & 1 \\ 3 & 1 \end{vmatrix} \\ \frac{1}{2}\begin{vmatrix} -3 & 1 \\ 1 & 1 \end{vmatrix} & \frac{1}{2}\begin{vmatrix} 1 & 3 \\ 1 & -1 \end{vmatrix} & \frac{1}{2}\begin{vmatrix} 3 & 1 \\ -1 & 1 \end{vmatrix} \\ \frac{1}{1}\begin{vmatrix} 1 & 1 \\ -2 & 2 \end{vmatrix} & \frac{1}{1}\begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} & \frac{1}{1}\begin{vmatrix} -1 & 1 \\ 1 & -2 \end{vmatrix} \end{pmatrix}$$

$$A^{(3)} = \begin{pmatrix} -4 & -4 & -2 \\ -2 & -2 & 2 \\ 4 & 3 & 1 \end{pmatrix}$$

$$A^{(2)} = \begin{pmatrix} \frac{1}{1}\begin{vmatrix} -4 & -4 \\ -2 & -2 \end{vmatrix} & \frac{1}{3}\begin{vmatrix} -4 & -2 \\ -2 & 2 \end{vmatrix} \\ \frac{1}{1}\begin{vmatrix} -2 & -2 \\ 4 & 3 \end{vmatrix} & \frac{1}{-1}\begin{vmatrix} -2 & 2 \\ 3 & 1 \end{vmatrix} \end{pmatrix}$$

$$A^{(2)} = \begin{pmatrix} 0 & -4 \\ 2 & 8 \end{pmatrix}$$

$$A^{(1)} = \left(\frac{1}{-2}\begin{vmatrix} 0 & -4 \\ 2 & 8 \end{vmatrix} \right) = -4$$

And we must multiply the determinant by (-1), in order to have a displacement in columns.

$$\det(A) = -4 \times -1 = 4$$

Example 7

$$A=\begin{pmatrix}1&2&4&0&1\\0&0&1&0&6\\4&5&5&1&2\\0&0&1&1&1\\1&2&0&3&1\end{pmatrix}$$

$$center=H=\begin{pmatrix}0&1&0\\5&5&1\\0&1&1\end{pmatrix}$$

$$\det(H)=\frac{1}{1}\begin{pmatrix}\begin{vmatrix}0&0\\5&1\end{vmatrix}&\begin{vmatrix}1&0\\5&1\end{vmatrix}\\\begin{vmatrix}5&1\\0&1\end{vmatrix}&\begin{vmatrix}5&1\\1&1\end{vmatrix}\end{pmatrix}=5\neq 0$$

We use RML

$$A^{(4)}=\begin{pmatrix}\begin{vmatrix}1&4\\0&1\end{vmatrix}&\begin{vmatrix}2&4\\0&1\end{vmatrix}&\begin{vmatrix}4&0\\1&0\end{vmatrix}&\begin{vmatrix}4&1\\1&6\end{vmatrix}\\\begin{vmatrix}0&1\\4&5\end{vmatrix}&\begin{vmatrix}0&1\\5&5\end{vmatrix}&\begin{vmatrix}1&0\\5&1\end{vmatrix}&\begin{vmatrix}1&6\\5&2\end{vmatrix}\\\begin{vmatrix}4&5\\0&1\end{vmatrix}&\begin{vmatrix}5&5\\0&1\end{vmatrix}&\begin{vmatrix}5&1\\1&1\end{vmatrix}&\begin{vmatrix}5&2\\1&1\end{vmatrix}\\\begin{vmatrix}0&1\\1&0\end{vmatrix}&\begin{vmatrix}0&1\\2&0\end{vmatrix}&\begin{vmatrix}1&1\\0&3\end{vmatrix}&\begin{vmatrix}1&1\\0&1\end{vmatrix}\end{pmatrix}$$

$$A^{(4)}=\begin{pmatrix}1&2&0&23\\-4&-5&1&-28\\4&5&4&3\\-1&-2&3&1\end{pmatrix}$$

$$A^{(3)} = \begin{pmatrix} \frac{1}{1}\begin{vmatrix} 1 & 2 \\ -4 & -5 \end{vmatrix} & \frac{1}{1}\begin{vmatrix} 2 & 0 \\ -5 & 1 \end{vmatrix} & \frac{1}{1}\begin{vmatrix} 0 & 23 \\ 1 & -28 \end{vmatrix} \\ \frac{1}{5}\begin{vmatrix} -4 & -5 \\ 4 & 5 \end{vmatrix} & \frac{1}{5}\begin{vmatrix} -5 & 1 \\ 5 & 4 \end{vmatrix} & \frac{1}{5}\begin{vmatrix} 1 & -28 \\ 4 & 3 \end{vmatrix} \\ \frac{1}{1}\begin{vmatrix} 4 & 5 \\ -1 & -2 \end{vmatrix} & \frac{1}{1}\begin{vmatrix} 5 & 4 \\ -2 & 3 \end{vmatrix} & \frac{1}{1}\begin{vmatrix} 4 & 3 \\ 3 & 1 \end{vmatrix} \end{pmatrix}$$

$$A^{(3)} = \begin{pmatrix} 3 & 2 & -23 \\ 0 & -5 & 23 \\ -3 & 23 & -5 \end{pmatrix}$$

$$A^{(2)} = \begin{pmatrix} \frac{1}{-5}\begin{vmatrix} 3 & 2 \\ 0 & -5 \end{vmatrix} & \frac{1}{1}\begin{vmatrix} 2 & -23 \\ -5 & 23 \end{vmatrix} \\ \frac{1}{5}\begin{vmatrix} 0 & -5 \\ -3 & 23 \end{vmatrix} & \frac{1}{4}\begin{vmatrix} -5 & 23 \\ 23 & -5 \end{vmatrix} \end{pmatrix}$$

$$A^{(2)} = \begin{pmatrix} 3 & -69 \\ -3 & -126 \end{pmatrix}$$

$$A^{(1)} = \left(\frac{1}{-5}\begin{vmatrix} 3 & -69 \\ -3 & -126 \end{vmatrix} \right) = 117$$

Example 8:

$$A = \begin{pmatrix} 8 & 3 & 2 & 1 & 5 & 2 \\ 6 & 0 & 5 & 4 & 0 & 1 \\ 4 & 1 & 2 & 2 & 1 & 0 \\ 2 & 8 & 1 & 9 & 8 & 2 \\ 1 & 3 & 2 & 3 & 4 & 3 \\ 1 & 2 & 2 & 3 & 8 & 5 \end{pmatrix}$$

$$
A^{(5)} = \begin{pmatrix}
\begin{vmatrix} 8 & 2 \\ 6 & 5 \end{vmatrix} & \begin{vmatrix} 3 & 2 \\ 0 & 5 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 5 & 4 \end{vmatrix} & \begin{vmatrix} 1 & 5 \\ 4 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 4 & 1 \end{vmatrix} \\
\begin{vmatrix} 6 & 5 \\ 4 & 2 \end{vmatrix} & \begin{vmatrix} 0 & 5 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 5 & 4 \\ 2 & 2 \end{vmatrix} & \begin{vmatrix} 4 & 0 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 4 & 1 \\ 2 & 0 \end{vmatrix} \\
\begin{vmatrix} 4 & 2 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 8 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 2 \\ 1 & 9 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 9 & 8 \end{vmatrix} & \begin{vmatrix} 2 & 0 \\ 9 & 2 \end{vmatrix} \\
\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 8 & 1 \\ 3 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 9 \\ 2 & 3 \end{vmatrix} & \begin{vmatrix} 9 & 8 \\ 3 & 4 \end{vmatrix} & \begin{vmatrix} 9 & 2 \\ 3 & 3 \end{vmatrix} \\
\begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 3 & 2 \\ 2 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ 2 & 3 \end{vmatrix} & \begin{vmatrix} 3 & 4 \\ 3 & 8 \end{vmatrix} & \begin{vmatrix} 3 & 3 \\ 3 & 5 \end{vmatrix}
\end{pmatrix}
$$

$$
A^{(5)} = \begin{pmatrix}
28 & 15 & 3 & -20 & -7 \\
8 & -5 & 2 & 4 & -2 \\
0 & -15 & 16 & 7 & 4 \\
3 & 13 & -15 & 12 & 21 \\
0 & 2 & 0 & 12 & 6
\end{pmatrix}
$$

$$
A^{(4)} = \begin{pmatrix}
\frac{1}{5}\begin{vmatrix} 28 & 15 \\ -8 & -5 \end{vmatrix} & \frac{1}{5}\begin{vmatrix} 15 & 3 \\ -5 & 2 \end{vmatrix} & \frac{1}{4}\begin{vmatrix} 3 & -20 \\ 2 & 4 \end{vmatrix} & \frac{1}{4}\begin{vmatrix} -20 & -7 \\ 4 & -2 \end{vmatrix} \\
\frac{1}{2}\begin{vmatrix} -8 & -5 \\ 0 & -15 \end{vmatrix} & \frac{1}{2}\begin{vmatrix} -5 & 2 \\ -15 & 16 \end{vmatrix} & \frac{1}{2}\begin{vmatrix} 2 & 4 \\ 16 & 7 \end{vmatrix} & \frac{1}{2}\begin{vmatrix} 4 & -2 \\ 7 & 4 \end{vmatrix} \\
\frac{1}{1}\begin{vmatrix} 0 & -15 \\ 3 & 13 \end{vmatrix} & \frac{1}{1}\begin{vmatrix} -15 & 16 \\ 13 & -15 \end{vmatrix} & \frac{1}{9}\begin{vmatrix} 16 & 7 \\ -15 & 12 \end{vmatrix} & \frac{1}{9}\begin{vmatrix} 7 & 4 \\ 12 & 21 \end{vmatrix} \\
\frac{1}{2}\begin{vmatrix} 3 & 13 \\ 0 & 2 \end{vmatrix} & \frac{1}{2}\begin{vmatrix} 13 & -15 \\ 2 & 0 \end{vmatrix} & \frac{1}{3}\begin{vmatrix} -15 & 12 \\ 0 & 12 \end{vmatrix} & \frac{1}{3}\begin{vmatrix} 12 & 21 \\ 12 & 6 \end{vmatrix}
\end{pmatrix}
$$

$$A^{(4)} = \begin{pmatrix} -4 & 9 & 13 & 17 \\ 60 & -25 & -25 & 15 \\ 45 & 17 & 33 & 11 \\ 3 & 15 & -60 & -60 \end{pmatrix}$$

$$A^{(3)} = \begin{pmatrix} \frac{1}{-5}\begin{vmatrix} -4 & 9 \\ 60 & -25 \end{vmatrix} & \frac{1}{2}\begin{vmatrix} 9 & 13 \\ -25 & -25 \end{vmatrix} & \frac{1}{4}\begin{vmatrix} 13 & 17 \\ -25 & 15 \end{vmatrix} \\ \frac{1}{-15}\begin{vmatrix} 60 & -25 \\ 45 & 17 \end{vmatrix} & \frac{1}{16}\begin{vmatrix} -25 & -25 \\ 17 & 33 \end{vmatrix} & \frac{1}{7}\begin{vmatrix} -25 & 15 \\ 33 & 11 \end{vmatrix} \\ \frac{1}{13}\begin{vmatrix} 45 & 17 \\ 3 & 15 \end{vmatrix} & \frac{1}{-15}\begin{vmatrix} 17 & 33 \\ 15 & -60 \end{vmatrix} & \frac{1}{12}\begin{vmatrix} 33 & 11 \\ -60 & -60 \end{vmatrix} \end{pmatrix}$$

$$A^{(3)} = \begin{pmatrix} 88 & 50 & 155 \\ -143 & -25 & -110 \\ 48 & 101 & -110 \end{pmatrix}$$

$$A^{(2)} = \begin{pmatrix} \frac{1}{-25}\begin{vmatrix} 88 & 50 \\ -143 & -25 \end{vmatrix} & \frac{1}{-25}\begin{vmatrix} 50 & 155 \\ -25 & -110 \end{vmatrix} \\ \frac{1}{17}\begin{vmatrix} -143 & -25 \\ 48 & 101 \end{vmatrix} & \frac{1}{33}\begin{vmatrix} -25 & -110 \\ 101 & -110 \end{vmatrix} \end{pmatrix}$$

$$A^{(2)} = \begin{pmatrix} -198 & 65 \\ -779 & 420 \end{pmatrix}$$

$$A^{(1)} = \left(\frac{1}{-25}\begin{vmatrix} -198 & 65 \\ -779 & 420 \end{vmatrix} \right) = 1301$$

References

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