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Higher Derivations Associated with the Cauchy-Jensen Type Mapping

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Abstract

Let H be an infinite–dimensional Hilbert space and $K(H)$ be the set of all compact operators on H. We will adopt spectral theorem for compact self-adjoint operators, to investigate of higher derivation and higher Jordan derivation on $K(H)$ associated with the following Cauchy–Jencen type functional equation

$$2f(\frac{T+S}{2}+R) = f(T) + f(S) + 2f(R)$$

for all $T, S, R \in K(H)$.

Key words: Cauchy–Jensen type higher derivation; Cauchy-Jensen type higher Jordan derivation; Approximate–strong; C^*-algebra

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1 Introduction

Let's H be a Hilbert space, an operator T in $B(H)$ is called a compact operator if the image of unit ball of H under T is a compact subset of H. Note that if the operator $T : H \longrightarrow H$ is compact, then the adjoint of T is also compact. The set of all compact operators on H is shown by $K(H)$. It is easy to see that $K(H)$ is a C^*-algebra [1]. Moreover, every operator on H with finite range is compact. We denote by $P(H)$ the set of all finite range projections on Hilbert space H.

An approximate unit for a C^*-algebra $\mathcal{A}$ is an increasing net $(u_\lambda)_{\lambda\in\Lambda}$ of positive elements in the closed unit ball of $\mathcal{A}$ such that $a = \lim_\lambda au_\lambda = \lim_\lambda u_\lambda a$ for all $a \in \mathcal{A}$. Every C^*-algebra admits an approximate unit [10].

Example 1.1 *Let H be a Hilbert space with orthonormal basis $(e_n)_{n=1}^{\infty}$. The C^*-algebra $K(H)$ is non–unital, since $dim(H) = \infty$. If P_n is a projection on $\mathbb{C}e_1 + ... + \mathbb{C}e_n$, then the increasing sequence $(P_n)_{n=1}^{\infty}$ is an approximate unit for $K(H)$.*

Theorem 1.1 *([10]). Let $T : H \longrightarrow H$ be a compact self–adjoint operator on Hilbert space H. Then there is an orthonormal basis of H consisting of eigenvectors of T. The nonzero eigenvalues of T are from finite or countably infinite set $\{\lambda_k\}_{k=1}^{\infty}$ of real numbers and $T = \sum_{k=1}^{\infty} \lambda_k P_k$, where P_k is the orthogonal projection on the finite–dimensional space of eigenvectors corresponding to eigenvalues. If the number of nonzero eigenvalues is countably infinite, then the series converges to T in the operator norm.*

The problem of stability of functional equations originated from a question of Ulam [13] concerning the stability of group homomorphisms: let $(G_1, *)$ be a group and $(G_2, \star, d)$ be a metric group with the metric $d(.,.)$. Given $\varepsilon > 0$, does there exist a $\delta(\varepsilon) > 0$ such that if a mapping $h : G_1 \longrightarrow G_2$ satisfies the inequality

$$d(h(x * y), h(x) \star h(y)) < \delta$$

for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with

$$d(h(x), H(x)) < \varepsilon$$

for all $x \in G_1$? If the answer is affirmative, we would say that the equation of homomorphism $H(x * y) = H(x) \star H(y)$ is stable. Thus, the stability question of functional equations is that how do the solutions of the inequality differ from those of the given functional equation?

Hyers [9] gave the first affirmative answer to the question of Ulam for Banach spaces. Let X and Y be Banach spaces. Assume that $f : X \longrightarrow Y$ satisfies

$$\|f(x + y) - f(x) - f(y)\| \leq \varepsilon$$

for all $x, y \in X$ and some $\varepsilon > 0$. Then, there exists a unique additive mapping $T : X \longrightarrow Y$ such that

$$\|f(x) - T(x)\| \leq \varepsilon$$

for all $x \in X$. This method is called the direct method or Hyers-Ulam stability of functional equations.

Let $\mathbb{N}$ be the set of natural numbers. For $m \in \mathbb{N} \cup \{0\} = \mathbb{N}_0$, a sequence $H = \{h_0, h_1, ..., h_m\}$ (resp. $H = \{h_0, h_1, ..., h_n, ...\}$ of linear mappings from C^*-algebra A into C^*-algebra B is called a higher derivation of rank m (resp. infinite rank) from A into B if

$$h_n(xy) = \sum_{l+j=n} h_l(x)h_j(y)$$

holds for each $n \in \{0, 1, ..., m\}$ (resp. $n \in \mathbb{N}_0$) and all $x, y \in A$. A higher derivation H from A into B is said to be continuous if each h_n is continuous on A. The higher derivation H on A is called strong if h_0 is an identity mapping on A. Of course, a higher derivation of rank 0 from A into B (resp. a strong higher derivation of rank 1 on A) is a homomorphism (resp. a derivation). So a higher derivation is a generalization of both a homomorphism and a derivation.

Definition 1.1 *Let $\mathcal{A}$ be a C^*-algebra without unit. A sequence $H = \{h_0, h_1, ..., h_m, ...\}$ of mappings from $\mathcal{A}$ into $\mathcal{A}$ is called approximate –*

strong when, $\lim_n h_0(x_n)x = x\lim_n h_0(x_n) = x$ *for all* $x \in \mathcal{A}$ *which* $\{x_n\}_n$ *is approximate unit in* $\mathcal{A}$*.*

Theorem 1.2 *[14] Let* X *be a normed space and* Y *be a Banach space. Suppose that for the mapping* $f : X \longrightarrow Y$ *there exists a function* $\psi : X^3 \longrightarrow [0,\infty)$ *such that;*

$$\sum_j \frac{1}{2^j}\psi(2^j x, 2^j y, 2^j z) < \infty$$

and

$$\|2f(\frac{x+y}{2}+z) - f(x) - f(y) - 2f(z)\| < \psi(x,y,z)$$

for all $x, y, z \in X$*. Then, there exists a unique additive mapping* $L : X \longrightarrow Y$ *such that*

$$\|f(x) - L(x)\| < \frac{1}{4}\psi(x,x,x)$$

for all $x \in X$*.*

In this paper, we will adopt spectral theorem for compact self-adjoint operators to investigate of higher derivation and higher Jordan derivation on $K(H)$ associated with the following Cauchy–Jencen type functional equation

$$2f(\frac{T+S}{2}+R) = f(T) + f(S) + 2f(R)$$

for all $T, S, R \in K(H)$.

2 Higher derivation on $K(H)$

It is easy to see that if a continuous mapping $f : X \longrightarrow Y$ satisfying $f(ix) = if(x)$ for all $x \in X$ and all conditions of Theorem 1.2, then the mapping $L : X \longrightarrow Y$ given in Theorem 1.2 is a $\mathbb{C}-$linear. We use this fact in this paper.

Lemma 1 *Assume that a mapping* $f : X \longrightarrow B$ *is additive and for each fixed* $x \in X$*,* $f(\lambda x) = \lambda f(x)$ *for all* $\lambda \in \mathbb{T}^1_{\theta_0} := \{e^{i\theta} : 0 \le \theta \le \theta_0\}$*. Then* f

is $\mathbb{C}-$*linear.*

Proof. If λ belongs to $\mathbb{T}^1$, then there exists $\theta \in [0, 2\pi]$ such that $\lambda = e^{i\theta}$. By $\frac{\theta}{n} \to 0$ as $n \to \infty$ follows that there exists $n_0 \in \mathbb{N}$ such that $\lambda_1 = e^{i\frac{\theta}{n}}$ in $\mathbb{T}^1_{\theta_0}$ satisfying $f(\lambda x) = f(\lambda_1^{n_0} x) = \lambda_1^{n_0} f(x) = \lambda f(x)$ for all $x \in X$. For $t \in (0, 1)$, take $t_1 = t + i(1 - t^2)^{\frac{1}{2}}$ and $t_2 = t - i(1 - t^2)^{\frac{1}{2}}$. Then we have $t = \frac{t_1 + t_2}{2}$ and $t_1, t_2 \in \mathbb{T}^1$. It follows that

$$f(tx) = f(\frac{t_1 + t_2}{2} x) = \frac{t_1}{2} f(x) + \frac{t_2}{2} f(x) = t f(x).$$

If $\lambda \in B_1 := \{\lambda \in \mathbb{C}; \mid \lambda \mid \leq 1\}$, then there exists $\theta \in [0, 2\pi]$ such that $\lambda = \mid \lambda \mid e^{i\theta}$ and so

$$f(\lambda x) = f(\mid \lambda \mid e^{i\theta} x) = \mid \lambda \mid f(e^{i\theta} x) = \lambda f(x).$$

for all $x \in X$. If $\lambda \in \mathbb{C}$ then, there exist $n_0 \in \mathbb{N}$ (from $\frac{\lambda}{n} \to 0$ as $\to \infty$) such that $\lambda_0 = \frac{\lambda}{n_0} \in B_1$. It follows that

$$f(\lambda x) = f(n_0 \lambda_0 x) = n_0 \lambda_0 f(x) = \lambda f(x)$$

for all $x \in X$.

□

Lemma 2.1 *Let* $\phi = \{\varphi_0, \varphi_1, ..., \varphi_m, ...\}$ *be a sequence of continuous mappings from* $K(H)$ *into* $K(H)$ *such that* $\varphi_m(TP) = \sum_{l+j=m} \varphi_l(P)\varphi_j(T)$ *for all* $T \in K(H)$ *and* $P \in P(H)$*.*

1) *If* $\phi = \{\varphi_0, \varphi_1, ..., \varphi_m, ...\}$ *is an approximate–strong, then for approximate unit* $\{P_n\} \subset P(H)$ *we have* $\lim_n \varphi_m(P_n) = 0$ *for each* $m \in \{1, 2, 3, \cdots\}$*.*

2) *If* $\varphi_0(0) = 0$*, then* $\varphi_m(0) = 0$ *for each* $m \in \{1, 2, 3, \cdots\}$*.*

Proof. It is precisely verified. □

Definition 2.1 *Let* $\mathcal{A}$ *be a* C^*-*algebra. A sequence* $H = \{h_0, h_1, ..., h_m, ...\}$ *of mappings from* $\mathcal{A}$ *into* $\mathcal{A}$ *with* $h_0(0) = 0$ *is called a Cauchy–Jensen type* *higher derivation if for each* $m \in \mathbb{N}_0$

$$h_m(xy) = \sum_{l+j=m} h_l(x)h_j(y)$$

and

$$2h_m\left(\frac{x+y}{2} + 2\lambda z\right) = h_m(x) + h_m(y) + 2\lambda h_m(z)$$

for all $x, y, z \in \mathcal{A}$ *and* $\lambda \in \mathbb{C}$.

Theorem 2.1 *Let* $\phi = \{\varphi_0, \varphi_1, ..., \varphi_m, ...\}$ *be an approximate–strong sequence of continuous mappings from* $K(H)$ *into* $K(H)$ *which for each* $m \in \mathbb{N}_0$ *there exists a function* $\psi_m : K(H)^3 \longrightarrow [0, \infty)$ *such that*

$$\sum_j \frac{1}{2^j} \psi_m(2^j T, 2^j S, 2^j R) < \infty \qquad (2.1)$$

and

$$\|2\varphi_m(\frac{T+S}{2} + \lambda R) - \varphi_m(T) - \varphi_m(S) - 2\lambda\varphi_m(R)\| < \psi_m(T, S, R) \quad (2.2)$$

for all $T, S, R \in K(H)$, $\lambda \in \{1, i\}$ *and* $m \in \mathbb{N}_0$.

If $\varphi_m(2^n TP) = \sum_{l+j=m} 2^n \varphi_l(T)\varphi_j(P)$ *for each* $T \in K(H)$ *and* $P \in P(H)$, *then* $\phi = \{\varphi_0, \varphi_1, ..., \varphi_m, ...\}$ *is a Cauchy–Jensen type higher derivation.*

Proof. From continuity of φ_m and by the same reasoning as in the proof of the theorems of [14], for each $m \in \mathbb{N}_0$, there exists $\mathbb{R}-$linear mapping $h_m : K(H) \longrightarrow K(H)$ with $h_m(T) = \lim \frac{1}{2^n}\varphi_m(2^n T)$ such that

$$\|h_m(T) - \varphi_m(T)\| < \frac{1}{4}\psi_m(T, T, T)$$

for all $T \in K(H)$. It follows from (2.2) and Lemma 1 that h_m is a $\mathbb{C}-$linear for each $m \in \mathbb{N}_0$.

We show that $h_m \equiv \varphi_m$ for each $m \in \mathbb{N}_0$. Let $\{P_k\} \subset P(H)$ be approximate unit of $K(H)$. Then by Lemma 2.1 and linearity of φ_m we get

$$\begin{aligned} h_m(T) &= \lim_n \frac{1}{2^n}\varphi_m(2^nT) = \lim_{n,k} \frac{1}{2^n}\varphi_m(2^nTP_k) \\ &= \lim_{n,k} \frac{1}{2^n} \sum_{l+j=m} 2^n \varphi_l(P_k)\varphi_j(T) \\ &= \lim_k \sum_{l+j=m} \varphi_l(P_k)\varphi_j(T) = \varphi_m(T) \end{aligned}$$

for all $T \in K(H)$ and $m \in \mathbb{N}_0$. Now, let $S, T \in K(H)$. There are compact self adjoint operators S_1, S_2 such that $S = S_1 + iS_2$. According to Theorem 1.1 we have $S = S_1 + iS_2 = \sum_{l=1}^{\infty} \alpha_l P_l + i \sum_{j=1}^{\infty} \beta_j P_j$ where $P_k \in P(H)$ and $\alpha_k, \beta_k \in \mathbb{C}$ for all $k \in \{1, 2, 3, \cdots\}$. It follows from linearity and continuity of φ and T that

$$\begin{aligned} \varphi_m(TS) &= \varphi_m\Big(T\Big\{\sum_{l=1}^{\infty}\alpha_l P_l + i\sum_{j=1}^{\infty}\beta_j P_j\Big\}\Big) \\ &= \sum_{l=1}^{\infty}\varphi_m\Big(T\alpha_l P_l\Big) + i\sum_{j=1}^{\infty}\varphi_m\Big(T\beta_j P_j\Big) \\ &= \sum_{l=1}^{\infty}\sum_{s+k=m}\varphi_k(T)\varphi_s\Big(\alpha_l P_l\Big) + i\sum_{j=1}^{\infty}\sum_{s+k=m}\varphi_k(T)\varphi_s\Big(\beta_j P_j\Big) \\ &= \sum_{s+k=m}\varphi_k(T)\sum_{l=1}^{\infty}\varphi_s\Big(\alpha_l P_l\Big) + i\sum_{s+k=m}\varphi_k(T)\sum_{j=1}^{\infty}\varphi_s\Big(\beta_j P_j\Big) \\ &= \sum_{s+k=m}\varphi_k(T)\left\{\sum_{l=1}^{\infty}\varphi_s(\alpha_l P_l) + i\sum_{j=1}^{\infty}\varphi_s(\beta_j P_j)\right\} \\ &= \sum_{k+s=m}\varphi_k(T)\varphi_s(S). \end{aligned}$$

This means that ϕ is a Cauchy–Jensen type higher derivation.

□

Corollary 2.1 *Let $p \in (0,1)$ and $\theta \in [0,\infty)$ be real numbers and suppose that $\phi = \{\varphi_0, \varphi_1, ..., \varphi_m, ...\}$ is an approximate–strong sequence of continuous mappings from $K(H)$ into $K(H)$ with $\varphi_0(0) = 0$ such that, $\varphi_m(2^nTP) = \sum_{l+j=m} 2^n\varphi_l(T)\varphi_j(P)$ for each $m \in \mathbb{N}_0$, $T \in K(H)$ and $P \in P(H)$.*

If

$$\|2\varphi_m(\frac{T+S}{2}+\lambda R)-\varphi_m(T)-\varphi_m(S)-2\lambda\varphi_m(R)\| < \theta(\|T\|^p+\|S\|^p+\|S\|^p)$$

for all $\lambda \in \{1, i\}$ *and all* $T, S, R \in K(H)$ *and for each* $m \in \mathbb{N}_0$*, then* $\phi = \{\varphi_0, \varphi_1, ..., \varphi_m, ...\}$ *is a Cauchy–Jensen type higher derivation.*

Proof. Setting $\phi(T, S, R) := \theta(\|T\|^p+\|S\|^p+\|R\|^p)$ for all $T, S, R \in K(H)$. Then by Theorem 2.1 we get the desired result. □

Corollary 2.2 *Let* p_1, p_2, p_3 *and* θ *be positive real numbers with* $p_1 + p_2 + p_3 < 1$ *and let* $\phi = \{\varphi_0, \varphi_1, ..., \varphi_m, ...\}$ *be an approximate–strong sequence of continuous mappings from* $K(H)$ *into* $K(H)$ *with* $\varphi_0(0) = 0$ *such that* $\varphi_m(2^n TP) = \sum_{l+j=m} 2^n \varphi_l(T)\varphi_j(P)$ *for each* $m \in \mathbb{N}_0$*,* $T \in K(H)$ *and* $P \in P(H)$*.*

If

$$\|2\varphi_m(\frac{T+S}{2}+\lambda R)-\varphi_m(T)-\varphi_m(S)-2\lambda\varphi_m(R)\| < \theta(\|T\|^p_1.\|S\|^p_2.\|S\|^p_3)$$

for all $\lambda \in \{1, i\}$ *and all* $T, S, R \in K(H)$ *and for each* $m \in \mathbb{N}_0$*, then* $\phi = \{\varphi_0, \varphi_1, ..., \varphi_m, ...\}$ *is a Cauchy–Jensen type higher derivation.*

Proof. Setting $\phi(T, S, R) := \theta(\|T\|^p_1.\|S\|^p_2.\|R\|^p_3)$ all $T, S, R \in K(H)$. Then by Theorem 2.1 we get the desired result. □

3 Higher Jordan derivations on $K(H)$

Definition 3.1 *Let* $\mathcal{A}$ *be a* C^*-*algebra. A sequence* $H = \{h_0, h_1, ..., h_m, ...\}$ *of mappings from* $\mathcal{A}$ *into* $\mathcal{A}$ *with* $h_0(0) = 0$ *is called a Cauchy–Jensen type higher Jordan derivation if for each* $m \in \mathbb{N}_0$*,*

$$h_m(xy) = \sum_{l+j=m} \Big[h_l(x)h_j(y) + h_j(x)h_l(y)\Big]$$

and

$$2h_m(\frac{x+y}{2}+\lambda z) = h_m(x)+h_m(y)+2\lambda h_m(z)$$

for all $x, y, z \in \mathcal{A}$ *and* $\lambda \in \mathbb{C}$.

Theorem 3.1 *Let* $\phi = \{\varphi_0, \varphi_1, ..., \varphi_m, ...\}$ *be an approximate–strong sequence of continuous mappings from* $K(H)$ *into* $K(H)$ *which for each* $m \in \mathbb{N}_0$ *there exists a function* $\psi_m : K(H)^3 \longrightarrow [0, \infty)$ *such that*

$$\sum_j \frac{1}{2^t}\psi_m(2^tT, 2^tS, 2^tR) < \infty$$

and

$$\|2\varphi_m(\frac{T+S}{2}+\lambda R) - \varphi_m(T) - \varphi_m(S) - 2\lambda\varphi_m(R)\| < \psi_m(T, S, R)$$

for all $T, S, R \in K(H)$ *and* $m \in \mathbb{N}_0$. *If* $\varphi_m(2^nTP+2^nPT) = \sum_{l+j=m}\Big[2^n\varphi_l(T)\varphi_j(P)+ 2^n\varphi_j(P)\varphi_l(T)\Big]$ *for all* $T \in K(H)$ *and* $P \in P(H)$, *then* $\phi = \{\varphi_0, \varphi_1, ..., \varphi_m, ...\}$ *is a higher Jordan derivation.*

Proof. By the same reasoning in the proof of Theorem 2.1, for each $m \in \mathbb{N}_0$ there exists a unique $\mathbb{C}-$linear mapping $h_m : K(H) \longrightarrow K(H)$ with $h_m(T) = \lim_{n\to\infty} \frac{1}{2^n}\varphi(2^nT)$ such that

$$\|h_m(T) - \varphi_m(T)\| < \frac{1}{4}\psi(T, T, T)$$

for all $T \in K(H)$ and $m \in \mathbb{N}_0$.

We show that $h_m \equiv \varphi_m$ for each $m \in \mathbb{N}_0$. Let $\{P_k\} \subset P(H)$ be approximate unit of $K(H)$. Then by Lemma 2.1 and linearity of φ_m we get

$$\begin{aligned} h_m(T) &= \lim_n \frac{1}{2^n}\varphi_m(2^nT) = \lim_{n,k}\frac{1}{2^n}\varphi_m(2^{n-1}TP_k + 2^{n-1}P_kT) \\ &= \lim_{n,k}\frac{1}{2^n}\sum_{l+j=m}[2^{n-1}\varphi_l(T)\varphi_j(P_k) + 2^{n-1}\varphi_j(T)\varphi_l(P_k)] \\ &= \lim_{n,k}\frac{1}{2}\sum_{l+j=m}[\varphi_l(T)\varphi_j(P_k) + \varphi_j(T)\varphi_l(P_k)] = \varphi_m(T) \end{aligned}$$

for all $T \in K(H)$ and $m \in \mathbb{N}_0$. For $S, T \in K(H)$ there are compact self–adjoint operators S_1, S_2 such that $S = S_1 + iS_2 = \sum_{l=1}^{\infty} \alpha_l P_l + i \sum_{j=1}^{\infty} \beta_j P_j$ where $P_k \in P(H)$ and $\alpha_k, \beta_k \in \mathbb{C}$ for all $k \in \{1, 2, 3, \cdots\}$. It follows from linearity and continuity of φ and T that

$$\begin{aligned}
\varphi_m(TS + ST) &= \varphi_m \left(T\Big\{ \sum_{l=1}^{\infty} \alpha_l P_l + i \sum_{j=1}^{\infty} \beta_j P_j \Big\} + \Big\{ \sum_{l=1}^{\infty} \alpha_l P_l + i \sum_{j=1}^{\infty} \beta_j P_j \Big\} T \right) \\
&= \sum_{l=1}^{\infty} \varphi_m \Big(\alpha_l T P_l + \alpha_l P_l T \Big) + i \sum_{j=1}^{\infty} \varphi_m \Big(\beta_j T P_j + \beta_j P_j T \Big) \\
&= \sum_{l=1}^{\infty} \sum_{s+k=m} \Big[\varphi_s(T) \varphi_k(\alpha_l P_l) + \varphi_s(\alpha_l P_l)) \varphi_k(T) \Big] \\
&+ i \sum_{j=1}^{\infty} \sum_{s+k=m} \Big[\varphi_s(T) \varphi_k(\beta_j P_j) + \varphi_s(\beta_j P_j)) \varphi_k(T) \Big] \\
&= \sum_{s+k=m} \varphi_s(T) \left\{ \sum_{l=1}^{\infty} \varphi_k(\alpha_l P_l) + i \sum_{j=1}^{\infty} \varphi_k(\beta_j P_j) \right\} \\
&+ \left\{ \sum_{l=1}^{\infty} \varphi_s(\alpha_l P_l) + i \sum_{j=1}^{\infty} \varphi_s(\beta_j P_j) \right\} \sum_{s+k=m} \varphi_k(T) \\
&= \sum_{s+k=m} \varphi_s(T) \varphi_k(S) + \sum_{s+k=m} \varphi_k(T) \varphi_s(S)
\end{aligned}$$

This means that ϕ is a Cauchy–Jensen type higher Jordan derivation. □

Corollary 3.1 *Let $p \in (0, 1), \theta \in [0, \infty)$ be real numbers and suppose that $\phi = \{\varphi_0, \varphi_1, ..., \varphi_m, ...\}$ is an approximate–strong sequence of continuous mappings from $K(H)$ into $K(H)$ with $\varphi_0(0) = 0$ such that $\varphi_m(2^n TP + 2^n PT) = \sum_{l+j=m} \Big[2^n \varphi_l(T) \varphi_j(P) + 2^n \varphi_j(P) \varphi_l(T) \Big]$ for each $m \in \mathbb{N}_0$, $T \in K(H)$ and $P \in P(H)$.*

If

$$\|2\varphi_m(\frac{T+S}{2} + \lambda R) - \varphi_m(T) - \varphi_m(S) - 2\lambda \varphi_m(R)\| < \theta(\|T\|^p + \|S\|^p + \|S\|^p)$$

for all $\lambda \in \{1, i\}$ and all $T, S, R \in K(H)$ and for each $m \in \mathbb{N}_0$, then $\phi = \{\varphi_0, \varphi_1, ..., \varphi_m, ...\}$ is a Cauchy–Jensen type higher derivation.

Proof. Setting $\phi(T,S,R) := \theta(\|T\|^p+\|S\|^p+\|R\|^p)$ for all $T,S,R \in K(H)$. Then by Theorem 3.1 we get the desired result. □

Corollary 3.2 *Let p_1, p_2, p_3 and θ be positive real numbers with $p_1+p_2+p_3 < 1$ and suppose that $\phi = \{\varphi_0, \varphi_1, ..., \varphi_m, ...\}$ is an approximate–strong sequence of continuous mappings from $K(H)$ into $K(H)$ with $\varphi_0(0) = 0$ such that $\varphi_m(2^nTP + 2^nPT) = \sum_{l+j=m}\left[2^n\varphi_l(T)\varphi_j(P) + 2^n\varphi_j(P)\varphi_l(T)\right]$ for each $m \in \mathbb{N}_0$, $T \in K(H)$ and $P \in P(H)$.*

If

$$\|2\varphi_m(\frac{T+S}{2} + \lambda R) - \varphi_m(T) - \varphi_m(S) - 2\lambda\varphi_m(R)\| < \theta(\|T\|_1^p.\|S\|_2^p.\|S\|_3^p)$$

for all $\lambda \in \{1, i\}$ and all $T, S, R \in K(H)$ and for each $m \in \mathbb{N}_0$, then $\phi = \{\varphi_0, \varphi_1, ..., \varphi_m, ...\}$ is a Cauchy–Jensen type higher derivation.

Proof. Setting $\phi(T,S,R) := \theta(\|T\|_1^p.\|S\|_2^p.\|R\|_3^p)$ for all $T,S,R \in K(H)$. Then by Theorem 3.1 we get the desired result. □

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