

Exact Closed-Form Solution for Vibration Analysis of Truncated Conical and Tapered Beams Carrying Multiple Concentrated Masses

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Received 13 July 2017; accepted 18 September 2017

ABSTRACT

In this paper, an exact closed-form solution is presented for free vibration analysis of Euler-Bernoulli conical and tapered beams carrying any desired number of attached masses. The concentrated masses are modeled by Dirac's delta functions which creates no need for implementation of compatibility conditions. The proposed technique explicitly provides frequency equation and corresponding mode as functions with only two integration constants which leads to solution of a two by two eigenvalue problem for any number of attached masses. Using Basic functions which are made of the appropriate linear composition of Bessel functions leads to make implementation of boundary conditions much easier. The proposed technique is employed to study effect of quantity, position and translational inertia of the concentrated masses on the natural frequencies and corresponding modes of conical and tapered beams for all standard boundary conditions. Unlike many of previous exact approaches, presented solution has no limitation in number of concentrated masses. In other words, by increase in number of attached masses, there is no considerable increase in computational effort.

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Keywords : Exact solution; Transverse vibration; Concentrated mass; Conical beam; Tapered beam.

1 INTRODUCTION

DYNAMIC characteristics of the rotors and flexible shafts can be strongly affected by mounted elements such as gears, sprockets, flywheels and cams; thus studying vibration of beams carrying concentrated masses is an essential research which can provide successful design for mechanisms and structures. So far, many researches have been focused on the vibration characteristics of beams carrying various concentrated elements such as translational and rotational springs, point masses, rotary inertias, spring-mass systems and multi-span beams. Many researchers studied vibration analysis of uniform beams carrying point masses. Chen [1] analytically studied dynamic behavior of a simply supported beam carrying a concentrated mass at its center. This mass was modeled by the Dirac's delta function. Using the modified Dunkerley formula, a frequency analysis for a Euler-Bernoulli beam carrying a concentrated mass at an arbitrary position was presented by Low [2]. Laura et al. [3] obtained an analytical solution

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for determining natural frequencies and mode shapes of a clamped-free beam carrying a mass at the free end. In a comprehensive paper, Dowell [4] focused on the effects of mass and stiffness added to a dynamical system. Laura et al. [5] presented a note on the transverse vibration of continuous beams subjecting an axial force and carrying concentrated masses by applying the Rayleigh–Ritz method. Gurgoze [6] determined the fundamental frequency and first mode shape of a beam with local springs and point masses. Also, in another paper, he investigated the vibration of restrained beams with heavy masses [7]. Liu et al. [8] employed Laplace transformation technique to formulate frequency equation for beams with elastically restrained ends and carrying intermediate concentrated masses. Some authors extended researches to vibration analysis of multi-step beams carrying concentrated elements. Torabi et al. [9] used transfer matrix method (TMM) and studied free vibration analysis of multi-step Euler-Bernoulli and Timoshenko beams carrying concentrated masses having rotary inertia. Depending on the type of boundary conditions, natural frequencies were obtained through solution of a determinant of order two or four for any number of lumped elements. Farghaly and El-Sayed [10] presented an exact solution for the analysis of the natural frequencies and mode shapes of an axially loaded multi-step Timoshenko beam carrying several point elements.

In order to achieve better mass distribution and flexural stiffness, non-uniform beams are used widely in structures; e.g. robots, rotating shafts, blades, etc. Thus, studying dynamical behavior of non-uniform beams is one of the most important problems concerned by many researchers; Using Bessel functions and power series, Cranch and Adler [11] presented a closed-form solution for the vibration analysis of non-uniform beams with four kinds of rectangular cross-sections: linear depth-any power width, quadratic depth-any power width, cubic depth-any power width and constant depth-any exponential width. This solution extended by Conway and Dubil [12] for truncated conical beams and truncated wedge ones. Mabie and Rogers [13] studied transverse vibration of tapered cantilever beams with the end of support. They treated two configurations of interest: constant width and linearly variable thickness and constant thickness and linearly variable width. Heidebrecht [14] obtained the approximate natural frequencies and mode shapes of a non-uniform simply supported beam from the frequency equation and a Fourier sine series, where the frequency equation was derived from the Lagrangian equation by expanding the beam's sectional area and moment of inertia in terms of the Fourier cosine series. Similarly, Mabie and Rogers [15] used the second and fourth order polynomials so as to express the sectional area and moment of inertia, respectively; But they transformed the partial differential equation for free vibration of a double-tapered beam into the ordinary one and then solved the last equation to derive the natural frequencies. Transverse vibrations of linearly tapered beams, elastically restrained against rotation at either end, have been investigated by Goel [16]. Using the dynamic discretion technique, Downs [17] obtained the natural frequencies and corresponding mode shapes of the cantilever beams with 36 combinations of linear depth and breadth taper. His work was based on both Euler-Bernoulli and Timoshenko beam theories. Free vibration analysis of non-uniform cantilever beams solved numerically by Bailey [18]; He derived the frequency equation from the Hamilton's law. Gupta [19] derived the stiffness and consistent mass matrices for the linearly tapered beam element and then derived the natural frequencies and corresponding mode shapes with the finite element method. With the direct solution of the mode shape equation based on the Frobenius method, Naguleswaran determined the approximate natural frequencies of the single-tapered beams [20] and double-tapered ones [21]. Abrate [22] found that if the sectional area and the moment of inertia take the special forms, the equation of motion of a non-uniform beam may be transformed to the uniform beam and then determined the natural frequencies and corresponding mode shapes. Laura et al. [23] used three approximate numerical methods: Rayleigh–Ritz method, differential quadrature method and finite element method; He derived the natural frequencies of Bernoulli beams with constant width and bilinear various thicknesses. Datta and Sil [24] used the reverse procedures of Ref. [11] and obtained the natural frequencies of cantilever beams with constant width and linearly various depths. Hoffmann and Wertheimer [25] presented a simple formula for determining the fundamental frequency of tapered cantilever beams with linear tapers as a function of the first-mode-deflection beam stiffness, beam mass and a mass distribution parameter. Based on the Timoshenko beam theory, Genta and Gugliotta [26] derived a conical beam element for finite element analysis of nonuniform rotors. Attarnejad et al. [27] presented an exact solution for the free vibration of a tapered beam with elastic end rotational restraints problem. Their solution was in terms of the Bessel functions. Torabi et al. [28] used differential transform method (DTM) and presented a semi-analytical solution for vibration analysis of both conical and truncated conical cantilever beams. Vibration of truncated conical beam modeled by Euler-Bernoulli beam theory was presented analytically by Yan et al. [29] and results were used to study the vibration behaviour of a rat whisker. They used translational and rotational springs to better represent the constraint conditions at the base of the whiskers in a living rat. Boiangiu et al. [30] solved vibration analysis of a conical beam analytically in terms of Bessel functions and used transfer matrix method to present an exact solution for vibration analysis of multi-step conical beams.

In the natural frequencies and mode shapes of the non-uniform beams with concentrated attachments at one end or both ends, the solution procedures are exactly the same as those for the non-uniform bare beams. The only

difference is to change the boundary conditions; In other words, inertial force due to attached mass should be considered in boundary conditions. Using Bessel functions, Lau [31] presented exact solution of a cantilever tapered beam with a tip mass. He considered both translational and rotational inertias of attached mass. Grossi and Aranda [32] focused on the vibration of tapered beams with one end spring hinged and the other end with tip mass. Auciello [33] generalized problem to the transverse vibration of a linearly tapered cantilever beam with tip mass and flexible constraint. He considered rotary inertia of the concentrated mass and its eccentricity. Wu and Chen [34] presented an exact solution for the natural frequencies and mode shapes of an immersed elastically restrained wedge beam carrying an eccentric tip mass with mass moment of inertia.

In the cases that attachments are located at arbitrary positions along the length of the beam, the literatures are fewer, particularly for the cases with more than two attachments. Auciello and Maurizi [35] focused on the vibration of tapered beams with attached inertia elements. Wu and Hsieh [36] studied vibration analysis of a non-uniform beam with multiple point masses. Wu and Lin [37] presented an analytical and numerical combined method for Free vibration analysis of a uniform cantilever beam with point masses. Their method employed by Wu and Chen [38] to investigate vibrations of wedge beams by any number of point masses.

Because of increasing number of masses leads to increase in computational effort and complexity, most of the above mentioned works was limited to a finite number of masses attached on the beam, while by using delta functions, any attachments at the beam can be modeled without imposing any compatibility conditions. The most advantage of this method is that frequency equation leads to a two by two matrix for any number of attachments [39,40]. In the present paper, formulation of governing equations in the presented technique is derived as an infinite series for terms including the effect of concentrated masses. Therefore, using this technique, a beam carrying an unlimited number of masses can be solved with the less calculation. A parametric analysis is presented for different boundary conditions in order to investigate the effect of the quantity, value and position of the attached masses on the natural frequencies and mode shapes of the truncated conical and tapered Euler-Bernoulli beams.

2 VIBRATION ANALYSIS OF TRUNCATED CONICAL AND TAPERED BEAMS

Consider a non-uniform beam as depicted in Fig.1; cross-sectional area and cross-sectional moment of inertia about the neutral axis can be expressed respectively as:

$$A(x) = A_1 \left(\frac{x}{L_1} \right)^2 \quad I(x) = I_1 \left(\frac{x}{L_1} \right)^4. \quad (1)$$

For a conical beam which its diameter at $x = L_1$ is denoted by d_1 :

$$A_1 = \frac{\pi}{4} d_1^2 \quad I_1 = \frac{\pi}{64} d_1^4, \quad (2)$$

and for a tapered one

$$A_1 = b_1 h_1 \quad I_1 = \frac{1}{12} b_1 h_1^3; \quad (3)$$

where b_1 and h_1 are width and height of the beam at $x = L_1$, respectively.

Eq. (1) can be transformed to the following dimensionless form:

$$A(\xi) = A_1 \xi^2 \quad I(\xi) = I_1 \xi^4, \quad (4)$$

where

$$\xi = \frac{x}{L_1}. \quad (5)$$

It is obvious that

$$\frac{L_0}{L_1} = \xi_0 \leq \xi \leq 1. \quad (6)$$

In which ξ_0 is called taper ratio. According to Euler-Bernoulli beam theory, the governing differential equation for free vibration analysis can be written as follows [41]:

$$\frac{\partial^2}{\partial x^2} \left[EI(x) \frac{\partial^2 y(x,t)}{\partial x^2} \right] + \rho A(x) \frac{\partial^2 y(x,t)}{\partial t^2} = 0, \quad (7)$$

where $y(x,t)$, ρ and E denote transverse displacement, mass density and Young's modulus of material, respectively. The displacement $y(x,t)$ can be assumed as the product of the function $w(x)$ which only depends on the spatial coordinate x and a time dependent harmonic function as:

$$y(x,t) = w(x)e^{i\omega t}, \quad (8)$$

where ω is natural frequency of vibration; by definition of dimensionless parameters as:

$$W = \frac{w}{L_1} \quad \beta^4 = \frac{\rho A_1 L_1^4 \omega^2}{EI_1}, \quad (9)$$

Eq. (7) can be rewritten in the following dimensionless form:

$$\frac{1}{\xi^2} \frac{d^2}{d\xi^2} \left[\xi^4 \frac{d^2 W(\xi)}{d\xi^2} \right] - \beta^4 W(\xi) = 0. \quad (10)$$

It is worth mentioning that L_1 appeared in definition of dimensionless frequency (β) is not length of the beam; thus, following definition of dimensionless frequency is more suitable:

$$\lambda^4 = \frac{\rho A_1 L^4 \omega^2}{EI_1} = (1 - \xi_0)^4 \beta^4. \quad (11)$$

Solution of the Eq. (10) can be considered as [42]

$$W(\xi) = \left[c_1 J_2(z) + c_2 Y_2(z) + c_3 I_2(z) + c_4 K_2(z) \right] / \xi, \quad (12)$$

where

$$z = 2\beta\sqrt{\xi}. \quad (13)$$

In Eq. (12) J_n and Y_n are the n^{th} order Bessel functions of type one and two, respectively, while I_n and K_n are the modified n^{th} order Bessel functions of type one and two, respectively. Also $c_1 - c_4$ are integration constants.

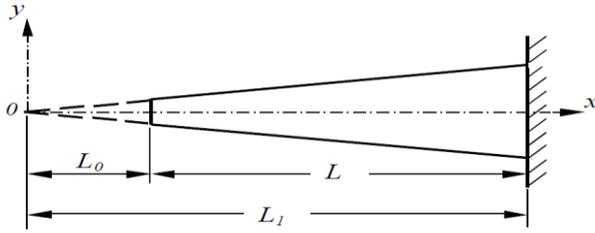


Fig.1
Geometry parameters of conical and tapered beams.

3 BASIC FUNCTIONS

Instead of combination of standard Bessel functions (Eq. (12)), displacement may be expressed in a more convenient form in terms of four Basic functions which are a better choice of eligible functions than standard Bessel functions. These functions have several useful properties such that help implementation of boundary conditions. In order to obtain Basic functions, consider Eq. (12) in a new form as:

$$W(\xi) = e_1 g_1(\xi) + e_2 g_2(\xi) + e_3 g_3(\xi) + e_4 g_4(\xi), \tag{14}$$

where $g_i(\xi)$ are defined in order that they have following properties:

$$\begin{aligned} g_1(\xi_0) = 1 \quad g_1'(\xi_0) = 0 \quad \xi_0^4 g_1''(\xi_0) = 0 \quad \left[\xi_0^4 g_1''(\xi) \right]' \Big|_{\xi=\xi_0} &= 0 \\ g_2(\xi_0) = 0 \quad g_2'(\xi_0) = 1 \quad \xi_0^4 g_2''(\xi_0) = 0 \quad \left[\xi_0^4 g_2''(\xi) \right]' \Big|_{\xi=\xi_0} &= 0 \\ g_3(\xi_0) = 0 \quad g_3'(\xi_0) = 0 \quad \xi_0^4 g_3''(\xi_0) = 1 \quad \left[\xi_0^4 g_3''(\xi) \right]' \Big|_{\xi=\xi_0} &= 0 \\ g_4(\xi_0) = 0 \quad g_4'(\xi_0) = 0 \quad \xi_0^4 g_4''(\xi_0) = 0 \quad \left[\xi_0^4 g_4''(\xi) \right]' \Big|_{\xi=\xi_0} &= 1 \end{aligned} \tag{15}$$

where prime indicates the derivative with respect to the dimensionless spatial variable ξ . It is noticeable that $\xi^4 W''(\xi)$ and $\left[\xi^4 W''(\xi) \right]'$ are proportional with bending moment and shear force, respectively. Consider i^{th} Basic functions as a combination of the standard Bessel functions as:

$$g_i(\xi) = \left[c_{i1} J_2(z) + c_{i2} Y_2(z) + c_{i3} I_2(z) + c_{i4} K_2(z) \right] / \xi; \tag{16}$$

where c_{ij} ($i, j = 1, 2, 3, 4$) are presented in Appendix A.

For convenience in notification, let us rewrite Eq. (14) as:

$$W(\xi) = e_1 {}^2g_1(\xi) + e_2 {}^2g_2(\xi) + e_3 {}^2g_3(\xi) + e_4 {}^2g_4(\xi), \tag{17}$$

where

$${}^2g_i(\xi) = \left[c_{i1} J_2(z) + c_{i2} Y_2(z) + c_{i3} I_2(z) + c_{i4} K_2(z) \right] / \xi. \tag{18}$$

Now, derivatives of $W(\xi)$ can be written as:

$$\begin{aligned}
W'(\xi) &= e_1 {}^3g_1(\xi) + e_2 {}^3g_2(\xi) + e_3 {}^3g_3(\xi) + e_4 {}^3g_4(\xi) \\
\xi^4 W''(\xi) &= e_1 {}^4g_1(\xi) + e_2 {}^4g_2(\xi) + e_3 {}^4g_3(\xi) + e_4 {}^4g_4(\xi), \\
[\xi^4 W''(\xi)]' &= e_1 {}^5g_1(\xi) + e_2 {}^5g_2(\xi) + e_3 {}^5g_3(\xi) + e_4 {}^5g_4(\xi)
\end{aligned} \tag{19}$$

where

$$\begin{aligned}
{}^3g_i(\xi) &= -\beta \xi^{-1.5} [c_{i1} J_3(z) + c_{i2} Y_3(z) - c_{i3} I_3(z) + c_{i4} K_3(z)] \\
{}^4g_i(\xi) &= \beta^2 \xi^2 [c_{i1} J_4(z) + c_{i2} Y_4(z) + c_{i3} I_4(z) + c_{i4} K_4(z)]. \\
{}^5g_i(\xi) &= \beta^3 \xi^{1.5} [c_{i1} J_3(z) + c_{i2} Y_3(z) + c_{i3} I_3(z) - c_{i4} K_3(z)]
\end{aligned} \tag{20}$$

4 ATTACHED MASSES

According to Fig.2, a non-uniform beam with p concentrated masses of translational inertia $m_1 - m_p$ located at $x_1 - x_p$ is considered. The translational inertia of the any attached mass can be assumed as a function of the spatial coordinates x as:

$$M_i(x) = \bar{m}_i [u(x - x_i) - u(x - x_i - dx)], \tag{21}$$

where $u(x - x_i)$ is the well-known Heaviside function and

$$\bar{m}_i = \frac{m_i}{dx}. \tag{22}$$

By considering the attached masses as point elements, differential length dx should be led to zero, thus

$$\lim_{dx \rightarrow 0} M_i(x) = \lim_{dx \rightarrow 0} \frac{m_i}{dx} [u(x - x_i) - u(x - x_i - dx)] = m_i \delta(x - x_i). \tag{23}$$

By adding this term to the mass of the beam in Eq. (7), one can write

$$\frac{\partial^2}{\partial x^2} \left[EI(x) \frac{\partial^2 y(x,t)}{\partial x^2} \right] + \left[\rho A(x) + \sum_{i=1}^p m_i \delta(x - x_i) \right] \frac{\partial^2 y(x,t)}{\partial t^2} = 0. \tag{24}$$

Using Eqs. (8) and (9), Eq. (24) can be rewritten in dimensionless form as:

$$\frac{1}{\xi^2} \frac{d^2}{d\xi^2} \left(\xi^4 \frac{d^2 W(\xi)}{d\xi^2} \right) - \beta^4 W(\xi) = B(\xi), \tag{25}$$

where

$$B(\xi) = \frac{\beta^4 (1 - \xi_0) W(\xi)}{\xi^2} \sum_{i=1}^p \alpha_i \delta(\xi - \xi_i) \quad \alpha_i = \frac{m_i}{\rho A_i L}. \tag{26}$$

In Eq. (26) α_i is dimensionless intensity of each attached mass. It should be noted that in deriving last equation, the following property of Dirac's delta function is used [43,44]:

$$\delta[L_1(\xi - \xi_i)] = \frac{1}{L_1} \delta(\xi - \xi_i). \tag{27}$$

In order to solve Eq. (25) it can be observed that the solution of $W(\xi)$ must be in the same form with the Eigenmode of the bare beam which is presented in Eq. (12). Therefore, a solution for the overall beam is assumed as a combination of the standard Bessel functions in which the coefficients of the combination are functions of spatial coordinate as:

$$W(\xi) = [c_1(\xi)J_2(z) + c_2(\xi)Y_2(z) + c_3(\xi)I_2(z) + c_4(\xi)K_2(z)]/\xi. \tag{28}$$

The coefficients $c_1(\xi) - c_4(\xi)$ appeared in Eq. (28), correspondent to the integration constant in the case of bare beams are unknown generalized functions determined according to the procedure outlined in Appendix B. The expressions of $c_1(\xi) - c_4(\xi)$ depended on four integration constants d_1, d_2, d_3 and d_4 are defined as:

$$\begin{aligned} c_1(\xi) &= \frac{1-\xi_0}{4} \sum_{i=1}^p \left\{ \frac{\alpha_i b_1(\xi_i)}{\xi_i^2} W(\xi_i) u(\xi - \xi_i) \right\} + d_1 & c_2(\xi) &= \frac{1-\xi_0}{4} \sum_{i=1}^p \left\{ \frac{\alpha_i b_2(\xi_i)}{\xi_i^2} W(\xi_i) u(\xi - \xi_i) \right\} + d_2 \\ c_3(\xi) &= \frac{1-\xi_0}{4} \sum_{i=1}^p \left\{ \frac{\alpha_i b_3(\xi_i)}{\xi_i^2} W(\xi_i) u(\xi - \xi_i) \right\} + d_3 & c_4(\xi) &= \frac{1-\xi_0}{4} \sum_{i=1}^p \left\{ \frac{\alpha_i b_4(\xi_i)}{\xi_i^2} W(\xi_i) u(\xi - \xi_i) \right\} + d_4 \end{aligned}, \tag{29}$$

and substituting Eq. (29) into the Eq. (28) leads to

$$W(\xi) = \frac{1-\xi_0}{4} \sum_{i=1}^p \left\{ \frac{\alpha_i T_2(\xi, \xi_i)}{\xi_i^2} W(\xi_i) u(\xi - \xi_i) \right\} + C(\xi), \tag{30}$$

where

$$T_2(\xi, \xi_i) = [b_1(\xi_i)J_2(z) + b_2(\xi_i)Y_2(z) + b_3(\xi_i)I_3(z) + b_4(\xi_i)K_2(z)]/\xi, \tag{31a}$$

$$C(\xi) = [d_1 J_2(z) + d_2 Y_2(z) + d_3 I_2(z) + d_4 K_2(z)]/\xi. \tag{31b}$$

For convenience in applying boundary conditions, Eq. (31b) can be rewritten in terms of the Basic functions similar to Eq. (17) as:

$$C(\xi) = e_1 {}^2g_1(\xi) + e_2 {}^2g_2(\xi) + e_3 {}^2g_3(\xi) + e_4 {}^2g_4(\xi). \tag{32}$$

Using Eq. (30) and known property of Dirac's delta function, $W(\xi_j)$ can be calculated as:

$$W(\xi_j) = \int_{-\infty}^{+\infty} W(\xi) \delta(\xi - \xi_j) d\xi = \frac{1-\xi_0}{4} \sum_{i=1}^{j-1} \left\{ \frac{\alpha_i T_2(\xi_j, \xi_i)}{\xi_i^2} W(\xi_i) \right\} + C(\xi_j). \tag{33}$$

Now $W(\xi_j)$ can be given by the following explicit form:

$$W(\xi_j) = e_1 \mu_j + e_2 \eta_j + e_3 \gamma_j + e_4 \kappa_j. \quad (34)$$

and substituting this presented form into the Eq. (33) leads to

$$\begin{aligned} \mu_j &= \frac{1-\xi_0}{4} \sum_{i=1}^{j-1} \left\{ \frac{\alpha_i T_2(\xi_j, \xi_i) \mu_i W(\xi_i)}{\xi_i^2} \right\} + {}^2g_1(\xi_j) & \eta_j &= \frac{1-\xi_0}{4} \sum_{i=1}^{j-1} \left\{ \frac{\alpha_i T_2(\xi_j, \xi_i) \eta_i W(\xi_i)}{\xi_i^2} \right\} + {}^2g_2(\xi_j) \\ \gamma_j &= \frac{1-\xi_0}{4} \sum_{i=1}^{j-1} \left\{ \frac{\alpha_i T_2(\xi_j, \xi_i) \gamma_i W(\xi_i)}{\xi_i^2} \right\} + {}^2g_3(\xi_j) & \kappa_j &= \frac{1-\xi_0}{4} \sum_{i=1}^{j-1} \left\{ \frac{\alpha_i T_2(\xi_j, \xi_i) \kappa_i W(\xi_i)}{\xi_i^2} \right\} + {}^2g_4(\xi_j) \end{aligned} \quad (35)$$

Finally, the exact solution of the Eigen-mode in explicit form with using of Basic functions can be derived by using Eqs. (30), (34) and (35) as:

$$\begin{aligned} W(\xi) &= e_1 \left[\frac{1-\xi_0}{4} \sum_{i=1}^p \left\{ \frac{\alpha_i \mu_i T_2(\xi, \xi_i)}{\xi_i^2} u(\xi - \xi_i) \right\} + {}^2g_1(\xi) \right] + e_2 \left[\frac{1-\xi_0}{4} \sum_{i=1}^p \left\{ \frac{\alpha_i \eta_i T_2(\xi, \xi_i)}{\xi_i^2} u(\xi - \xi_i) \right\} + {}^2g_2(\xi) \right] + \\ & e_3 \left[\frac{1-\xi_0}{4} \sum_{i=1}^p \left\{ \frac{\alpha_i \gamma_i T_2(\xi, \xi_i)}{\xi_i^2} u(\xi - \xi_i) \right\} + {}^2g_3(\xi) \right] + e_4 \left[\frac{1-\xi_0}{4} \sum_{i=1}^p \left\{ \frac{\alpha_i \kappa_i T_2(\xi, \xi_i)}{\xi_i^2} u(\xi - \xi_i) \right\} + {}^2g_4(\xi) \right]. \end{aligned} \quad (36)$$

Also derivatives of $W(\xi)$ appeared in boundary conditions can be written as:

$$\begin{aligned} W'(\xi) &= e_1 \left[\frac{1-\xi_0}{4} \sum_{i=1}^p \left\{ \frac{\alpha_i \mu_i T_3(\xi, \xi_i)}{\xi_i^2} u(\xi - \xi_i) \right\} + {}^3g_1(\xi) \right] + e_2 \left[\frac{1-\xi_0}{4} \sum_{i=1}^p \left\{ \frac{\alpha_i \eta_i T_3(\xi, \xi_i)}{\xi_i^2} u(\xi - \xi_i) \right\} + {}^3g_2(\xi) \right] + \\ & e_3 \left[\frac{1-\xi_0}{4} \sum_{i=1}^p \left\{ \frac{\alpha_i \gamma_i T_3(\xi, \xi_i)}{\xi_i^2} u(\xi - \xi_i) \right\} + {}^3g_3(\xi) \right] + e_4 \left[\frac{1-\xi_0}{4} \sum_{i=1}^p \left\{ \frac{\alpha_i \kappa_i T_3(\xi, \xi_i)}{\xi_i^2} u(\xi - \xi_i) \right\} + {}^3g_4(\xi) \right], \end{aligned} \quad (37a)$$

$$\begin{aligned} \xi^4 W''(\xi) &= e_1 \left[\frac{1-\xi_0}{4} \sum_{i=1}^p \left\{ \frac{\alpha_i \mu_i T_4(\xi, \xi_i)}{\xi_i^2} u(\xi - \xi_i) \right\} + {}^4g_1(\xi) \right] + e_2 \left[\frac{1-\xi_0}{4} \sum_{i=1}^p \left\{ \frac{\alpha_i \eta_i T_4(\xi, \xi_i)}{\xi_i^2} u(\xi - \xi_i) \right\} + {}^4g_2(\xi) \right] + \\ & e_3 \left[\frac{1-\xi_0}{4} \sum_{i=1}^p \left\{ \frac{\alpha_i \gamma_i T_4(\xi, \xi_i)}{\xi_i^2} u(\xi - \xi_i) \right\} + {}^4g_3(\xi) \right] + e_4 \left[\frac{1-\xi_0}{4} \sum_{i=1}^p \left\{ \frac{\alpha_i \kappa_i T_4(\xi, \xi_i)}{\xi_i^2} u(\xi - \xi_i) \right\} + {}^4g_4(\xi) \right], \end{aligned} \quad (37b)$$

$$\begin{aligned} [\xi^4 W''(\xi)]' &= e_1 \left[\frac{1-\xi_0}{4} \sum_{i=1}^p \left\{ \frac{\alpha_i \mu_i T_5(\xi, \xi_i)}{\xi_i^2} u(\xi - \xi_i) \right\} + {}^5g_1(\xi) \right] + e_2 \left[\frac{1-\xi_0}{4} \sum_{i=1}^p \left\{ \frac{\alpha_i \eta_i T_5(\xi, \xi_i)}{\xi_i^2} u(\xi - \xi_i) \right\} + {}^5g_2(\xi) \right] + \\ & e_3 \left[\frac{1-\xi_0}{4} \sum_{i=1}^p \left\{ \frac{\alpha_i \gamma_i T_5(\xi, \xi_i)}{\xi_i^2} u(\xi - \xi_i) \right\} + {}^5g_3(\xi) \right] + e_4 \left[\frac{1-\xi_0}{4} \sum_{i=1}^p \left\{ \frac{\alpha_i \kappa_i T_5(\xi, \xi_i)}{\xi_i^2} u(\xi - \xi_i) \right\} + {}^5g_4(\xi) \right]. \end{aligned} \quad (37c)$$

where

$$\begin{aligned} T_3(\xi, \xi_i) &= -\beta \xi^{-1.5} \left[b_1(\xi_i) J_3(z) + b_2(\xi_i) Y_3(z) - b_3(\xi_i) I_3(z) + b_4(\xi_i) K_3(z) \right] \\ T_4(\xi, \xi_i) &= \beta^2 \xi^2 \left[b_1(\xi_i) J_4(z) + b_2(\xi_i) Y_4(z) + b_3(\xi_i) I_4(z) + b_4(\xi_i) K_4(z) \right]. \\ T_5(\xi, \xi_i) &= \beta^3 \xi^{1.5} \left[b_1(\xi_i) J_3(z) + b_2(\xi_i) Y_3(z) + b_3(\xi_i) I_3(z) + b_4(\xi_i) K_3(z) \right] \end{aligned} \quad (38)$$

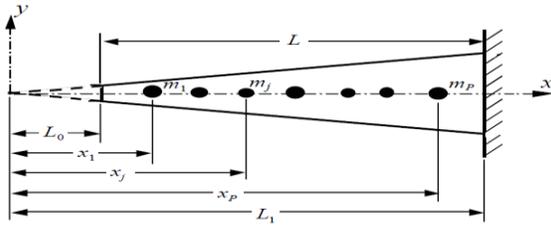


Fig.2
Geometry and parameters of a non-uniform beam with attached masses.

5 FREQUENCY EQUATION

In this section, frequency equation of the beam carrying multiple concentrated masses will be derived by enforcing nine standard boundary conditions such as pinned–pinned (PP), pinned–clamped (PC), pinned–free (PF), clamped–pinned (CP), clamped–clamped (CC), clamped–free (CF), free–pinned (FP), free–clamped (FC) and free–free (FF). Then, the frequency equations will be derived from determinant of a matrix 2×2 for each boundary conditions and any number of point masses. This equation will be solved in order to obtain the dimensionless frequency (λ) and corresponding vibration modes. It should be noted that in these paper each mode is normalized as the maximum displacement be equal to unit.

5.1 Pinned-Pinned (PP)

The boundary conditions of the pinned-pinned beam can be expressed as follows:

$$W(\xi_0) = 0 \quad \xi_0^4 W''(\xi_0) = 0 \quad W(1) = 0 \quad W''(1) = 0. \tag{39}$$

According to Eqs. (15), (36) and (37), the following conditions for the integration constants e_1, e_2, e_3 and e_4 can be indicated:

$$e_1 = e_3 = 0, \tag{40a}$$

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{Bmatrix} e_2 \\ e_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}, \tag{40b}$$

where

$$\begin{aligned} A_{11} &= \frac{1-\xi_0}{4} \sum_{i=1}^p \left\{ \frac{\alpha_i \eta_i T_2(1, \xi_i)}{\xi_i^2} \right\} + {}^2g_2(1) & A_{12} &= \frac{1-\xi_0}{4} \sum_{i=1}^p \left\{ \frac{\alpha_i \kappa_i T_2(1, \xi_i)}{\xi_i^2} \right\} + {}^2g_4(1) \\ A_{21} &= \frac{1-\xi_0}{4} \sum_{i=1}^p \left\{ \frac{\alpha_i \eta_i T_4(1, \xi_i)}{\xi_i^2} \right\} + {}^4g_2(1) & A_{22} &= \frac{1-\xi_0}{4} \sum_{i=1}^p \left\{ \frac{\alpha_i \kappa_i T_4(1, \xi_i)}{\xi_i^2} \right\} + {}^4g_4(1) \end{aligned} \tag{41}$$

5.2 Pinned-Clamped (PC)

The boundary conditions of the pinned-clamped beam can be expressed as follows:

$$W(\xi_0) = 0 \quad \xi_0^4 W''(\xi_0) = 0 \quad W(1) = 0 \quad W'(1) = 0. \tag{42}$$

In a similar manner Eqs. (40a) and (40b) can be obtained again; where

$$\begin{aligned}
 A_{11} &= \frac{1-\xi_0}{4} \sum_{i=1}^p \left\{ \frac{\alpha_i \eta_i T_2(1, \xi_i)}{\xi_i^2} \right\} + {}^2g_2(1) & A_{12} &= \frac{1-\xi_0}{4} \sum_{i=1}^p \left\{ \frac{\alpha_i \kappa_i T_2(1, \xi_i)}{\xi_i^2} \right\} + {}^2g_4(1) \\
 A_{21} &= \frac{1-\xi_0}{4} \sum_{i=1}^p \left\{ \frac{\alpha_i \eta_i T_3(1, \xi_i)}{\xi_i^2} \right\} + {}^3g_2(1) & A_{22} &= \frac{1-\xi_0}{4} \sum_{i=1}^p \left\{ \frac{\alpha_i \kappa_i T_3(1, \xi_i)}{\xi_i^2} \right\} + {}^3g_4(1)
 \end{aligned} \tag{43}$$

5.3 Pinned-Free (PF)

The boundary conditions of the pinned-free beam can be expressed as follows:

$$W(\xi_0) = 0 \quad \xi_0^4 W''(\xi_0) = 0 \quad W''(1) = 0 \quad \left[\xi^4 W''(\xi) \right] \Big|_{\xi=1} = 0. \tag{44}$$

Again in a similar manner Eqs. (40a) and (40b) can be obtained again; where

$$\begin{aligned}
 A_{11} &= \frac{1-\xi_0}{4} \sum_{i=1}^p \left\{ \frac{\alpha_i \eta_i T_4(1, \xi_i)}{\xi_i^2} \right\} + {}^4g_2(1) & A_{12} &= \frac{1-\xi_0}{4} \sum_{i=1}^p \left\{ \frac{\alpha_i \kappa_i T_4(1, \xi_i)}{\xi_i^2} \right\} + {}^4g_4(1) \\
 A_{21} &= \frac{1-\xi_0}{4} \sum_{i=1}^p \left\{ \frac{\alpha_i \eta_i T_5(1, \xi_i)}{\xi_i^2} \right\} + {}^5g_2(1) & A_{22} &= \frac{1-\xi_0}{4} \sum_{i=1}^p \left\{ \frac{\alpha_i \kappa_i T_5(1, \xi_i)}{\xi_i^2} \right\} + {}^5g_4(1)
 \end{aligned} \tag{45}$$

5.4 Clamped-Pinned (CP)

The boundary conditions of the clamped-pinned beam can be expressed as follows:

$$W(\xi_0) = 0 \quad W'(\xi_0) = 0 \quad W(1) = 0 \quad W''(1) = 0. \tag{46}$$

According to Eqs. (15), (36) and (37), the following conditions for the integration constants e_1, e_2, e_3 and e_4 can be indicated:

$$e_1 = e_2 = 0, \tag{47a}$$

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} e_3 \\ e_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{47b}$$

where

$$\begin{aligned}
 A_{11} &= \frac{1-\xi_0}{4} \sum_{i=1}^p \left\{ \frac{\alpha_i \gamma_i T_2(1, \xi_i)}{\xi_i^2} \right\} + {}^2g_3(1) & A_{12} &= \frac{1-\xi_0}{4} \sum_{i=1}^p \left\{ \frac{\alpha_i \kappa_i T_2(1, \xi_i)}{\xi_i^2} \right\} + {}^2g_4(1) \\
 A_{21} &= \frac{1-\xi_0}{4} \sum_{i=1}^p \left\{ \frac{\alpha_i \gamma_i T_3(1, \xi_i)}{\xi_i^2} \right\} + {}^4g_3(1) & A_{22} &= \frac{1-\xi_0}{4} \sum_{i=1}^p \left\{ \frac{\alpha_i \kappa_i T_3(1, \xi_i)}{\xi_i^2} \right\} + {}^4g_4(1)
 \end{aligned} \tag{48}$$

5.5 Clamped-Clamped (CC)

The boundary conditions of the clamped-clamped beam can be expressed as follows:

$$W(\xi_0) = 0 \quad W'(\xi_0) = 0 \quad W(1) = 0 \quad W'(1) = 0. \tag{49}$$

In a similar manner Eqs. (47a) and (47b) can be obtained again; where

$$\begin{aligned} A_{11} &= \frac{1-\xi_0}{4} \sum_{i=1}^p \left\{ \frac{\alpha_i \gamma_i T_2(1, \xi_i)}{\xi_i^2} \right\} + {}^2g_3(1) & A_{12} &= \frac{1-\xi_0}{4} \sum_{i=1}^p \left\{ \frac{\alpha_i \kappa_i T_2(1, \xi_i)}{\xi_i^2} \right\} + {}^2g_4(1) \\ A_{21} &= \frac{1-\xi_0}{4} \sum_{i=1}^p \left\{ \frac{\alpha_i \gamma_i T_2(1, \xi_i)}{\xi_i^2} \right\} + {}^3g_3(1) & A_{22} &= \frac{1-\xi_0}{4} \sum_{i=1}^p \left\{ \frac{\alpha_i \kappa_i T_3(1, \xi_i)}{\xi_i^2} \right\} + {}^3g_4(1) \end{aligned} \tag{50}$$

5.6 Clamped-Free (CF)

The boundary conditions of the clamped-free beam can be expressed as follows:

$$W(\xi_0) = 0 \quad W'(\xi_0) = 0 \quad W''(1) = 0 \quad \left[\xi^4 W''(\xi) \right]' \Big|_{\xi=1} = 0. \tag{51}$$

Again in a similar manner, Eqs. (47a) and (47b) can be obtained again; where

$$\begin{aligned} A_{11} &= \frac{1-\xi_0}{4} \sum_{i=1}^p \left\{ \frac{\alpha_i \gamma_i T_4(1, \xi_i)}{\xi_i^2} \right\} + {}^4g_3(1) & A_{12} &= \frac{1-\xi_0}{4} \sum_{i=1}^p \left\{ \frac{\alpha_i \kappa_i T_4(1, \xi_i)}{\xi_i^2} \right\} + {}^4g_4(1) \\ A_{21} &= \frac{1-\xi_0}{4} \sum_{i=1}^p \left\{ \frac{\alpha_i \gamma_i T_3(1, \xi_i)}{\xi_i^2} \right\} + {}^5g_3(1) & A_{22} &= \frac{1-\xi_0}{4} \sum_{i=1}^p \left\{ \frac{\alpha_i \kappa_i T_5(1, \xi_i)}{\xi_i^2} \right\} + {}^5g_4(1) \end{aligned} \tag{52}$$

5.7 Free-Pinned (FP)

The boundary conditions of the free-pinned beam can be expressed as follows:

$$\xi_0^4 W''(\xi_0) = 0 \quad \left[\xi^4 W''(\xi) \right]' \Big|_{\xi=\xi_0} = 0 \quad W(1) = 0 \quad W''(1) = 0. \tag{53}$$

According to Eqs. (15), (36) and (37), the following conditions for the integration constants e_1, e_2, e_3 and e_4 can be indicated:

$$e_3 = e_4 = 0, \tag{54a}$$

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{Bmatrix} e_1 \\ e_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}, \tag{54b}$$

where

$$\begin{aligned} A_{11} &= \frac{1-\xi_0}{4} \sum_{i=1}^p \left\{ \frac{\alpha_i \mu_i T_2(1, \xi_i)}{\xi_i^2} \right\} + {}^2g_1(1) & A_{12} &= \frac{1-\xi_0}{4} \sum_{i=1}^p \left\{ \frac{\alpha_i \eta_i T_2(1, \xi_i)}{\xi_i^2} \right\} + {}^2g_2(1) \\ A_{21} &= \frac{1-\xi_0}{4} \sum_{i=1}^p \left\{ \frac{\alpha_i \mu_i T_2(1, \xi_i)}{\xi_i^2} \right\} + {}^4g_1(1) & A_{22} &= \frac{1-\xi_0}{4} \sum_{i=1}^p \left\{ \frac{\alpha_i \eta_i T_2(1, \xi_i)}{\xi_i^2} \right\} + {}^4g_2(1) \end{aligned} \tag{55}$$

5.8 Free-clamped (FC)

The boundary conditions of the free-clamped beam can be expressed as follows:

$$\xi_0^4 W''(\xi_0) = 0 \quad \left[\xi^4 W''(\xi) \right] \Big|_{\xi=\xi_0} = 0 \quad W(1) = 0 \quad W'(1) = 0. \quad (56)$$

In a similar manner, Eqs. (54a) and (54b) can be obtained again; where

$$\begin{aligned} A_{11} &= \frac{1-\xi_0}{4} \sum_{i=1}^p \left\{ \frac{\alpha_i \mu_i T_2(1, \xi_i)}{\xi_i^2} \right\} + {}^2g_1(1) & A_{12} &= \frac{1-\xi_0}{4} \sum_{i=1}^p \left\{ \frac{\alpha_i \eta_i T_2(1, \xi_i)}{\xi_i^2} \right\} + {}^2g_2(1) \\ A_{21} &= \frac{1-\xi_0}{4} \sum_{i=1}^p \left\{ \frac{\alpha_i \mu_i T_3(1, \xi_i)}{\xi_i^2} \right\} + {}^3g_1(1) & A_{22} &= \frac{1-\xi_0}{4} \sum_{i=1}^p \left\{ \frac{\alpha_i \eta_i T_3(1, \xi_i)}{\xi_i^2} \right\} + {}^3g_2(1) \end{aligned} \quad (57)$$

5.9 Free-Free (FF)

The boundary conditions of the free-free beam can be expressed as follows:

$$\xi_0^4 W''(\xi_0) = 0 \quad \left[\xi^4 W''(\xi) \right] \Big|_{\xi=\xi_0} = 0 \quad W''(1) = 0 \quad \left[\xi^4 W''(\xi) \right] \Big|_{\xi=1} = 0. \quad (58)$$

Again in a similar manner, Eqs. (54a) and (54b) can be obtained again; where

$$\begin{aligned} A_{11} &= \frac{1-\xi_0}{4} \sum_{i=1}^p \left\{ \frac{\alpha_i \mu_i T_4(1, \xi_i)}{\xi_i^2} \right\} + {}^4g_1(1) & A_{12} &= \frac{1-\xi_0}{4} \sum_{i=1}^p \left\{ \frac{\alpha_i \eta_i T_4(1, \xi_i)}{\xi_i^2} \right\} + {}^4g_2(1) \\ A_{21} &= \frac{1-\xi_0}{4} \sum_{i=1}^p \left\{ \frac{\alpha_i \mu_i T_5(1, \xi_i)}{\xi_i^2} \right\} + {}^5g_1(1) & A_{22} &= \frac{1-\xi_0}{4} \sum_{i=1}^p \left\{ \frac{\alpha_i \eta_i T_5(1, \xi_i)}{\xi_i^2} \right\} + {}^5g_2(1) \end{aligned} \quad (59)$$

Depended on the type of boundary conditions, the frequency equation can be derived using Eqs. (40), (47) or (54) as:

$$A_{11}A_{22} - A_{12}A_{21} = 0, \quad (60)$$

Also using obtained dimensionless frequencies, Eqs. (40), (47) or (54) and Eq. (36), corresponding modes can be derived.

6 RESULTS AND DISCUSSIONS

In this section effect of quantity, position and intensity of the point masses on the dimensionless frequencies and corresponding mode shapes are investigated using various numerical examples.

First, in order to validate proposed method, consider a bare FC beam; Table 1. shows value of the first five dimensionless frequencies for various values of taper ratio. As this table shows, results of proposed solution are in excellent agreement with those presented by Lau [31].

A bare beam is considered to study effect of taper ratio (ξ_0); Figs. 3(a)-3(c) show value of the first three dimensionless frequencies versus taper ratio for various boundary conditions. As shown, increase in value of the taper ratio, increases value of the frequencies except for first dimensionless frequency of the FC beam. These figures also show that as value of taper ration grows and the beam becomes more similar to a uniform one, value of

dimensionless frequencies get close to the corresponding values of a uniform beam which are presented in Table 2. [41].

Table 1

First five dimensionless frequencies of a bare FC beam for various values of taper ratio.

ξ_0		λ_1	λ_2	λ_3	λ_4	λ_5
0.1	present	2.68418	4.321934	6.092701	7.968475	9.907901
	Lau [31]	2.684189	4.322055	6.092932	7.968996	9.907861
0.3	Present	2.347119	4.317476	6.54229	8.860386	11.22162
	Lau [31]	2.347181	4.317541	6.542966	8.861199	11.222747
0.5	Present	2.150391	4.421094	6.969648	9.581652	12.22146
	Lau [31]	2.150616	4.421268	6.969857	9.581898	12.222516
0.7	Present	2.016504	4.533779	7.348485	10.19622	13.05803
	Lau [31]	2.016664	4.533818	7.349502	10.196823	13.060525
0.9	Present	1.916602	4.641855	7.69276	10.74172	13.79558
	Lau [31]	1.916690	4.642225	7.693415	10.742334	13.796986

Table 2

First three dimensionless non-zero frequencies of uniform beam for various boundary conditions [41].

Boundary conditions	λ_1	λ_2	λ_3
PP	π	2π	3π
CC & FF	4.730041	7.853205	10.995608
FC & CF	1.875104	4.694091	7.854757
PC & CP & PF & FP	3.926602	7.068583	10.210176

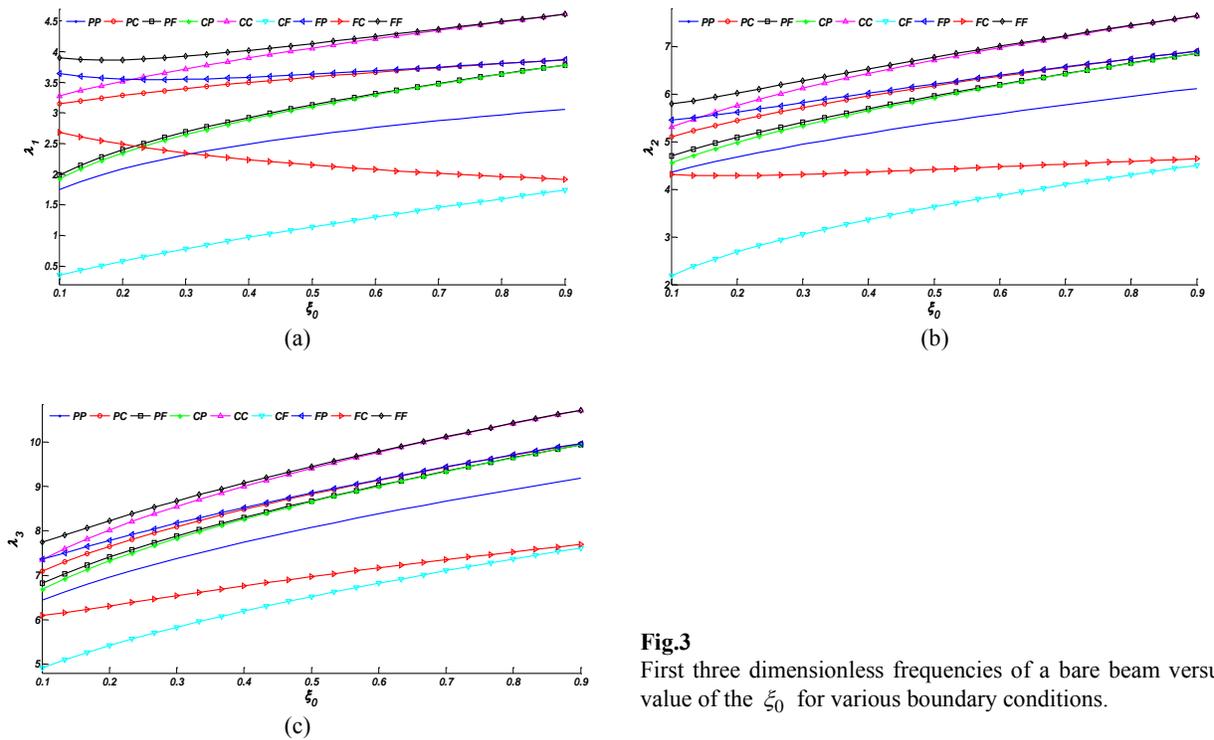


Fig.3 First three dimensionless frequencies of a bare beam versus value of the ξ_0 for various boundary conditions.

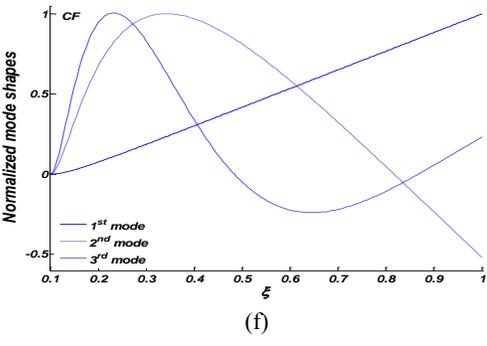
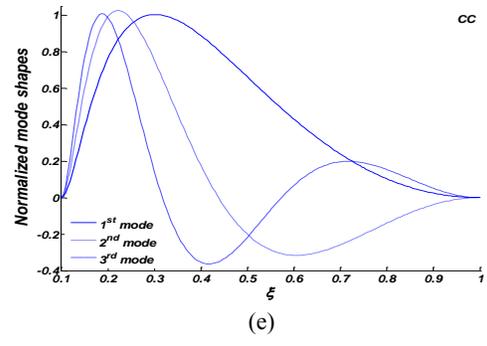
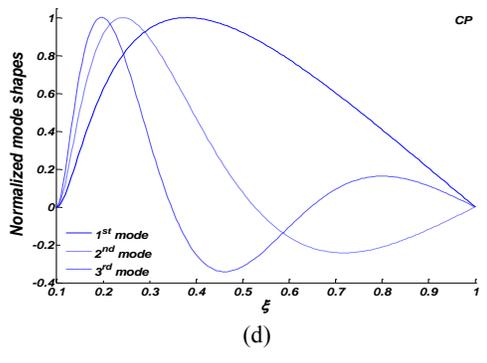
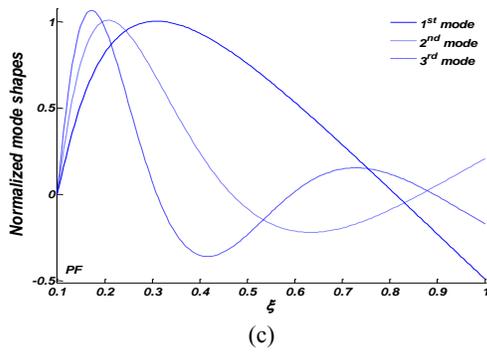
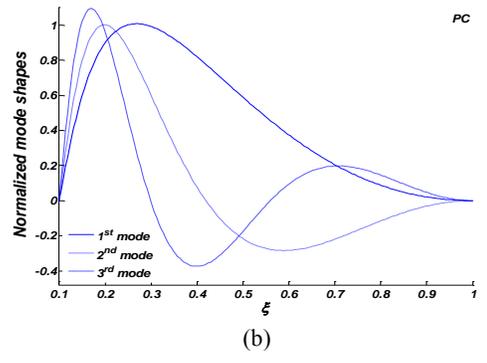
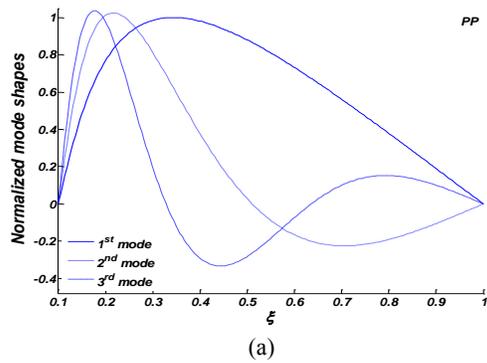
In order to investigate effect of point mass on the natural frequencies, consider a non-uniform beam ($\xi_0 = 0.1$) with a single point mass ($\alpha = 1$) at its midpoint, Table 3. shows value of the first three dimensionless frequencies and corresponding values of a bare beam for various boundary conditions. It is clear that existence of a point mass leads to decreases in all frequencies; it can be explained by increase in total mass of the system. Also corresponding

normal mode shapes are presented for bare beam in Figs. 4(a)-4(i) for all boundary conditions. These figures confirm accuracy of the proposed solution.

Table 3

First three dimensionless frequencies of a non-uniform beam ($\xi_0 = 0.1$) with a single point mass ($\alpha = 1$) at its midpoint and corresponding values of a bare one.

Boundary conditions	Bare beam			$\xi_1=0.5, \alpha=1$		
	λ_1	λ_2	λ_3	λ_1	λ_2	λ_3
PP	1.746606	4.3659	6.440923	1.021069	4.363214	5.478232
PC	3.147363	5.108915	7.097992	1.790112	4.778717	6.703847
PF	1.986768	4.709272	6.827068	1.332202	4.652848	6.075744
CP	1.923047	4.562297	6.68917	1.120386	4.519773	5.678545
CC	3.280518	5.312834	7.354928	1.833398	5.072924	6.793575
CF	0.347849	2.187378	4.919799	0.298081	1.439209	4.90155
FP	3.639111	5.465834	7.364438	3.565723	5.004114	6.602788
FC	2.68418	4.321934	6.092701	1.661792	3.731095	6.092235
FF	3.899268	5.798026	7.743018	3.685693	5.657023	6.79289



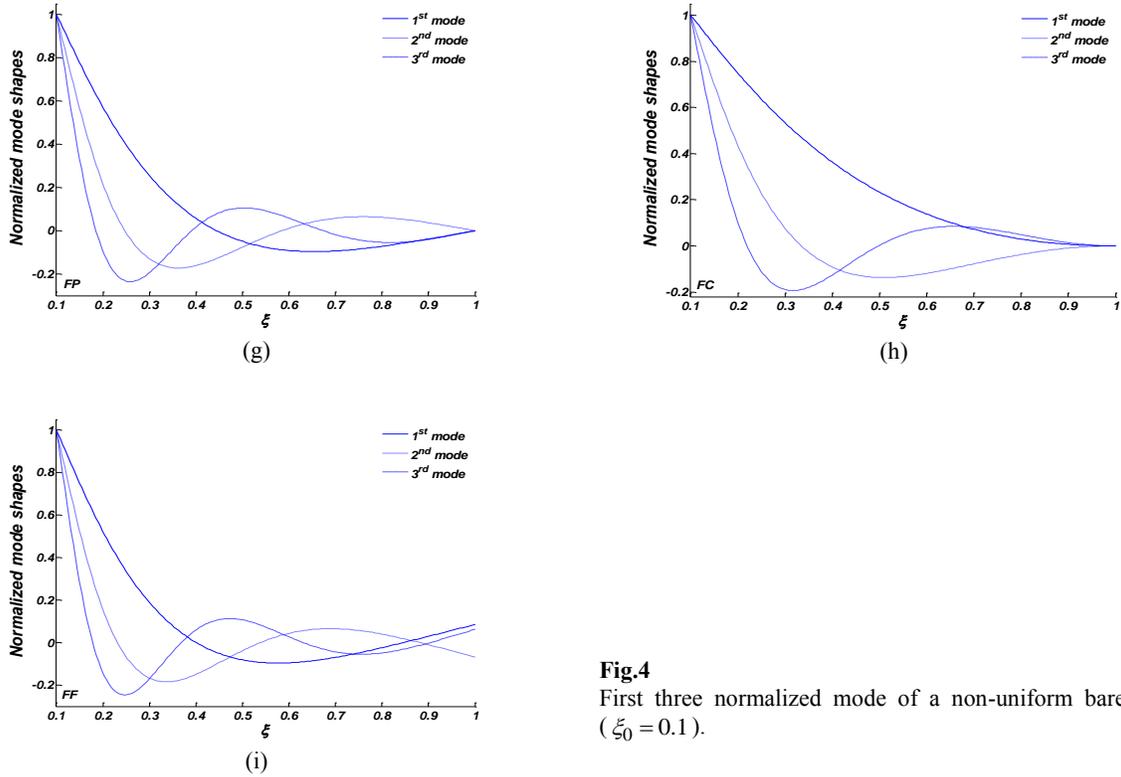


Fig.4 First three normalized mode of a non-uniform bare beam ($\xi_0 = 0.1$).

Consider PP, CC, PC and CF non-uniform beams ($\xi_0 = 0.5$) in three cases, a bare beam, a beam with a point mass ($\alpha = 0.5$) located at $\xi_1 = 0.6$ and a beam with two similar point mass ($\alpha_1 = \alpha_2 = 0.5$) located at $\xi_1 = 0.6$ and $\xi_2 = 0.7$. For various boundary conditions Tables 4-7. show value of the first four dimensionless frequencies. These tables indicate that as number of point masses increases, more decrease in all natural frequencies can be detected.

Table 4 First four dimensionless frequencies of a non-uniform PP beam ($\xi_0 = 0.5$) for three cases.

PP	λ_1	λ_2	λ_3	λ_4
Bare beam	2.637207	5.395068	8.076177	10.75294
$\xi_1=0.6, \alpha=0.5$	2.204834	4.230005	7.397146	10.63491
$\xi_{1,2}=[0.6 \ 0.7], \alpha=[0.5 \ 0.5]$	1.88501	3.859155	5.315383	9.667234

Table 5 First four dimensionless frequencies of a non-uniform CC beam ($\xi_0 = 0.5$) for three cases.

CC	λ_1	λ_2	λ_3	λ_4
Bare beam	4.059082	6.72085	9.398794	12.07695
$\xi_1=0.6, \alpha=0.5$	3.331055	5.331152	8.432627	11.70753
$\xi_{1,2}=[0.6 \ 0.7], \alpha=[0.5 \ 0.5]$	2.835938	4.226465	6.490908	10.90758

Table 6 First four dimensionless frequencies of a non-uniform PC beam ($\xi_0 = 0.5$) for three cases.

PC	λ_1	λ_2	λ_3	λ_4
Bare beam	3.584473	6.180225	8.822661	11.48032
$\xi_1=0.6, \alpha=0.5$	2.713379	5.157568	8.31688	11.45911
$\xi_{1,2}=[0.6 \ 0.7], \alpha=[0.5 \ 0.5]$	2.391113	3.995947	6.532261	10.92572

Table 7

First three dimensionless frequencies of a non-uniform CF beam ($\xi_0 = 0.5$) for three cases.

CF	λ_1	λ_2	λ_3	λ_4
Bare beam	1.139282	3.630652	6.518131	9.256858
$\xi_1=0.6, \alpha=0.5$	1.13374	3.11186	5.029492	8.239472
$\xi_{1,2}=[0.6 \ 0.7], \alpha=[0.5 \ 0.5]$	1.086621	2.708955	4.265007	6.095804

In order to investigate effect of value of the translational inertia of the point masses on the natural frequencies, consider a non-uniform beam ($\xi_0 = 0.1$) with a point mass located at midpoint of the beam. Figs. 5(a)-5(c) show ratio of the first three frequencies to the corresponding value of a bare beam (r_i) versus value of the dimensionless intensity of the point mass; As shown in these figures, increase in value of the translational inertia of the point mass leads to decrease in value of the natural frequencies.

Position of the point masses has a significant effect on the value of the decrease in frequencies; for this purpose consider a non-uniform beam ($\xi_0 = 0.1$) with a single point mass ($\alpha = 0.5$); Figs. 6(a)-6(i) show value of the ratio of the first three frequencies to the corresponding value of a bare beam (r_i) versus the position of the point mass. As these figures show, in each mode of any boundary conditions, there are some points which mass is located on them, there is no reduction in the natural frequency; actually these points are nodes in corresponding mode. On the other hand, in each mode there are some points that when mass is located on them, the highest reduction in the natural frequency happens; actually these points are antinodes in the corresponding mode. It is worth mentioning that quantity of nodes and antinodes increase at higher modes.

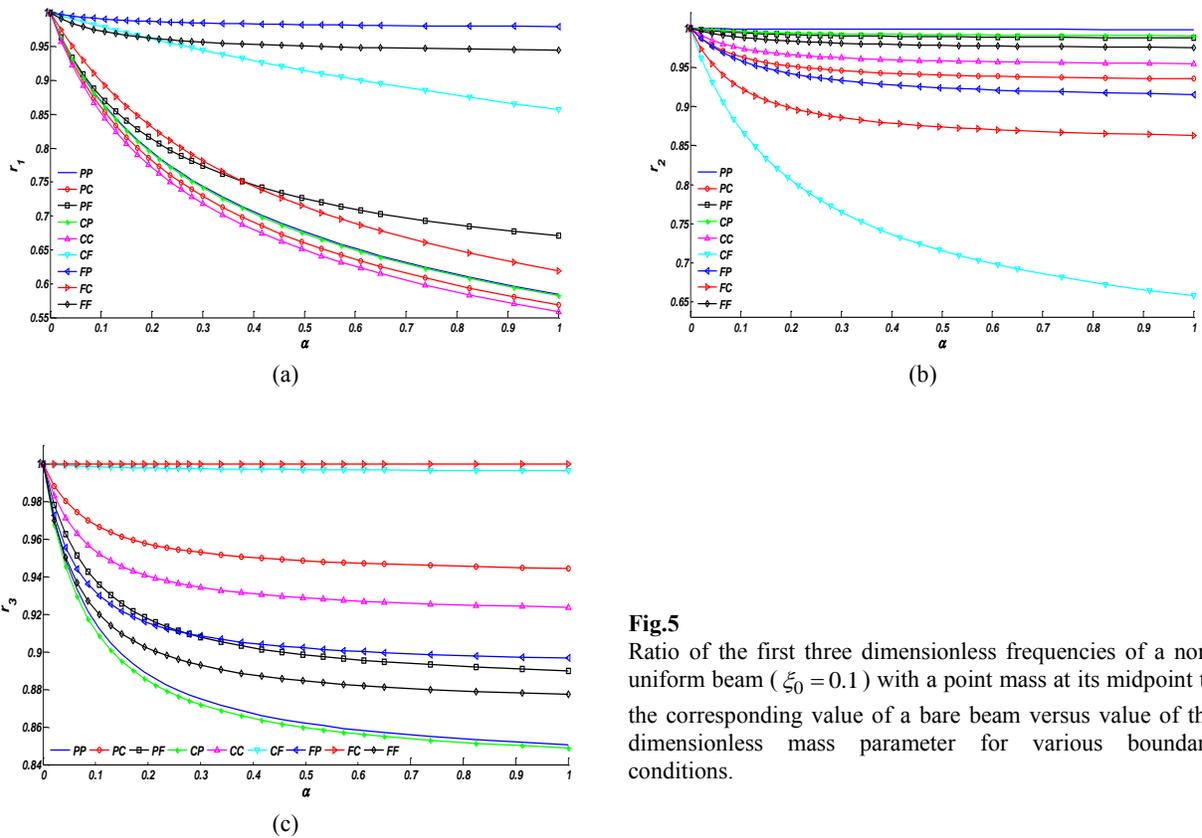
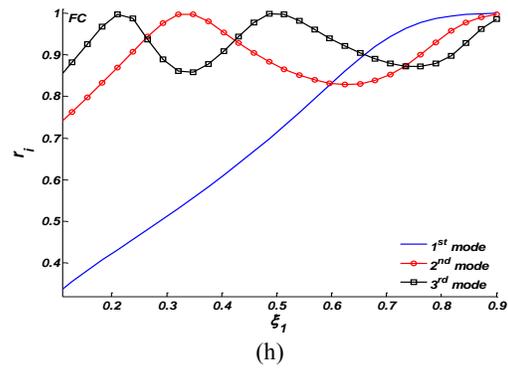
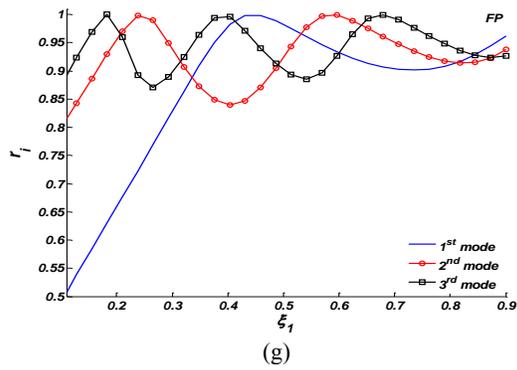
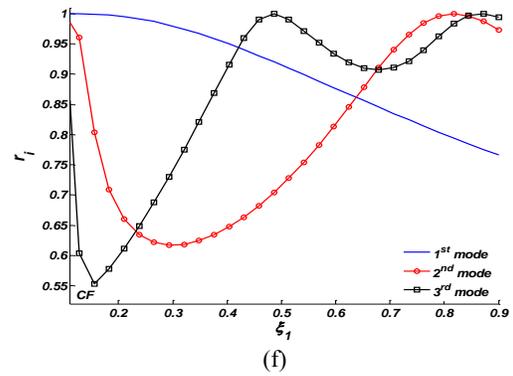
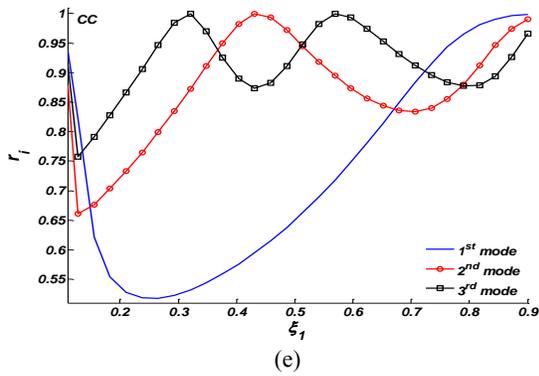
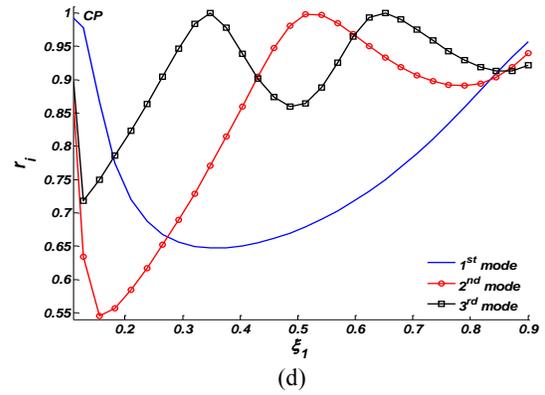
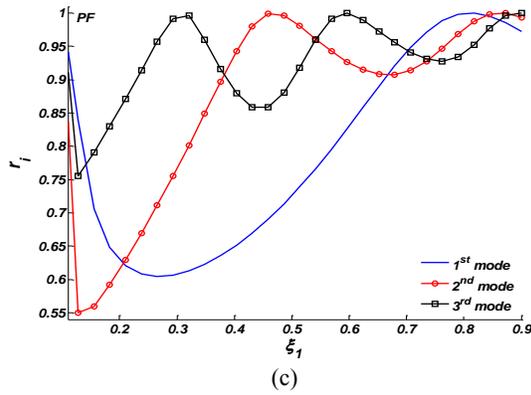
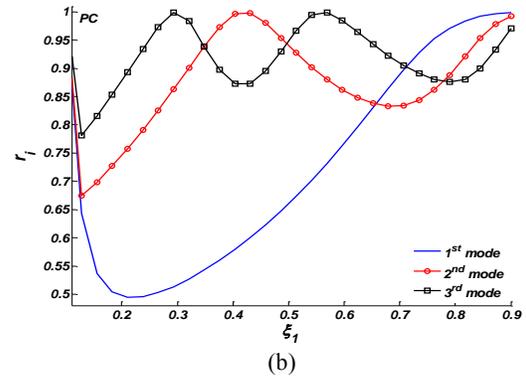
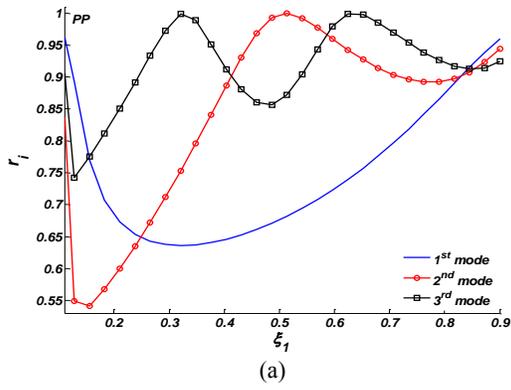


Fig.5 Ratio of the first three dimensionless frequencies of a non-uniform beam ($\xi_0 = 0.1$) with a point mass at its midpoint to the corresponding value of a bare beam versus value of the dimensionless mass parameter for various boundary conditions.



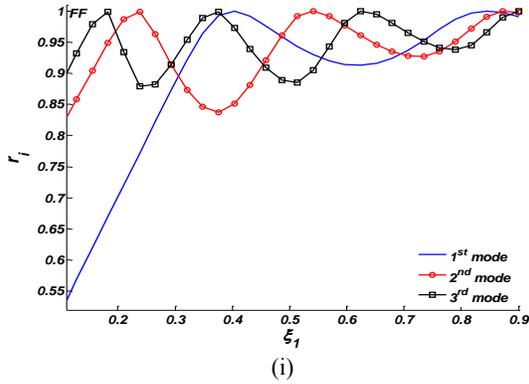


Fig.6
Ratio of the first three dimensionless frequencies of a non-uniform beam ($\xi_0 = 0.1$) with a point mass ($\alpha = 0.5$) to the corresponding value of a bare beam versus position of the mass for various boundary conditions.

Finally consider a non-uniform ($\xi_0 = 0.2$) FC beam with two symmetric point masses ($\alpha = 0.8$); Table 8. shows value of the first three dimensionless frequencies for the various position of the masses. Also corresponding mode shapes are depicted in Figs. 7(a)-7(c). These figures indicate that position of point masses has a significant effect on the mode shapes.

Table 8
First three dimensionless frequencies of a non-uniform FC beam ($\xi_0 = 0.2$) with two symmetric masses ($\alpha = 0.8$).

ξ_i	λ_1	λ_2	λ_3
[0.3 0.9]	1.130391	2.839645	6.411314
[0.35 0.85]	1.239961	2.73985	4.980748
[0.4 0.8]	1.355586	2.729225	4.258655
[0.45 0.75]	1.471016	2.836363	3.878181
[0.5 0.7]	1.570625	3.134703	3.734798
[0.55 0.65]	1.635664	3.388541	4.409531

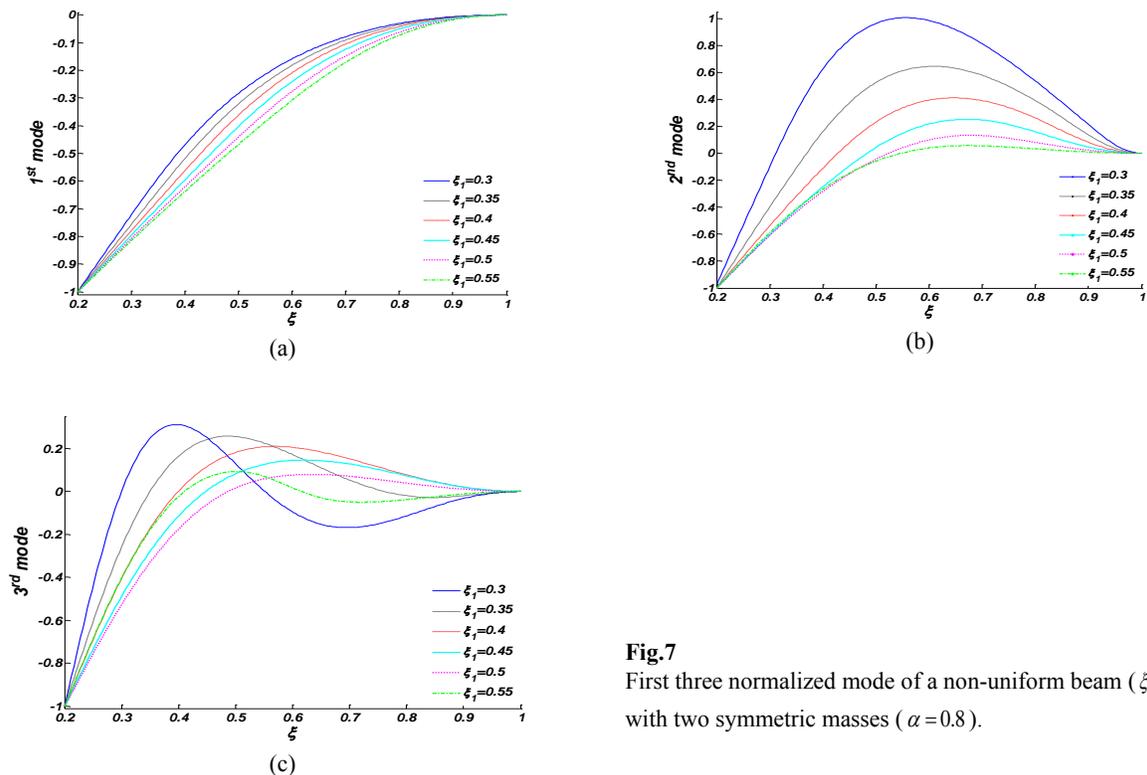


Fig.7
First three normalized mode of a non-uniform beam ($\xi_0 = 0.2$) with two symmetric masses ($\alpha = 0.8$).

8 CONCLUSIONS

Vibration analysis of truncated conical and tapered Euler-Bernoulli beams carrying multiple concentrated masses was investigated for nine standard boundary conditions: pinned-pinned, pinned-clamped, pinned-free, clamped-pinned, clamped-clamped, clamped-free, free-pinned, free-clamped and free-free. Using Bessel functions, the fourth-order partial differential equation was transformed to a quadratic eigenvalue problem. Some typical results calculated by the presented model, showed excellent coincidence with the presented results of the other authors. The influence of the quantity, intensity and position of point masses on the dimensionless frequencies were studied. Based on the results discussed earlier, several conclusions can be addressed as follows:

In general, for a beam with concentrated masses the value of frequencies are less than corresponding ones of a bare beam. Therefore, it can be obviously concluded that increase in the number of concentrated masses always causes more decrease in frequencies.

It is observed that all frequencies decrease with respect to the intensity of the point masses except for the cases which the masses are located at nodal points of the corresponding modes.

The concentrated mass has its highest influence over a natural frequency, when is located at an antinode of the corresponding mode. It is worth mentioning that proposed solution has some advantages against previous exact approaches; e.g. for any number of concentrated masses, natural frequencies and corresponding modes can be derived by solution of a determinant of order two; therefore, this method can be applied easily for beams with desired number of concentrated masses without considerable additional computational effort.

APPENDIX A

Substituting Eq. (16) in Eq. (15), leads to calculate c_{ij} coefficients as:

$$\begin{aligned}
 c_{11} &= \pi z_0 \frac{(z_0^2 - 8)Y_1(z_0) + 4z_0Y_0(z_0)}{16\beta^2} & c_{12} &= -\pi z_0 \frac{(z_0^2 - 8)J_1(z_0) + 4z_0J_0(z_0)}{16\beta^2} \\
 c_{13} &= z_0 \frac{(z_0^2 + 8)K_1(z_0) + 4z_0K_0(z_0)}{8\beta^2} & c_{14} &= z_0 \frac{(z_0^2 + 8)I_1(z_0) - 4z_0I_0(z_0)}{8\beta^2} \\
 c_{21} &= -\pi z_0 \frac{8(z_0^2 - 6)Y_1(z_0) - z_0(z_0^2 - 24)Y_0(z_0)}{32\beta^4} & c_{22} &= \pi z_0 \frac{8(z_0^2 - 6)J_1(z_0) - z_0(z_0^2 - 24)J_0(z_0)}{32\beta^4} \\
 c_{23} &= z_0 \frac{8(z_0^2 + 6)K_1(z_0) + z_0(z_0^2 + 24)K_0(z_0)}{16\beta^4} & c_{24} &= z_0 \frac{8(z_0^2 + 6)I_1(z_0) - z_0(z_0^2 + 24)I_0(z_0)}{16\beta^4} \\
 c_{31} &= -\pi z_0 \frac{(z_0^2 - 8)Y_1(z_0) + 4z_0Y_0(z_0)}{16\beta^4 \xi_0^3} & c_{32} &= \pi z_0 \frac{(z_0^2 - 8)J_1(z_0) + 4z_0J_0(z_0)}{16\beta^4 \xi_0^3} \\
 c_{33} &= z_0 \frac{(z_0^2 + 8)K_1(z_0) + 4z_0K_0(z_0)}{8\beta^4 \xi_0^3} & c_{34} &= z_0 \frac{(z_0^2 + 8)I_1(z_0) - 4z_0I_0(z_0)}{8\beta^4 \xi_0^3} \\
 c_{41} &= \pi z_0 \frac{2Y_1(z_0) - z_0Y_0(z_0)}{8\beta^4 \xi_0^2} & c_{42} &= -\pi z_0 \frac{2J_1(z_0) - z_0J_0(z_0)}{8\beta^4 \xi_0^2} \\
 c_{43} &= z_0 \frac{2K_1(z_0) + z_0K_0(z_0)}{4\beta^4 \xi_0^2} & c_{44} &= z_0 \frac{2I_1(z_0) - z_0I_0(z_0)}{4\beta^4 \xi_0^2}
 \end{aligned} \tag{A.1}$$

where

$$z_0 = 2\beta\sqrt{\xi_0} \tag{A.2}$$

It is noticeable that in deriving Eq. (A.1), following relations are used:

$$Y_v(z)J_{v+1}(z) - Y_{v+1}(z)J_v(z) = \frac{2}{\pi z} \quad K_v(z)I_{v+1}(z) + K_{v+1}(z)I_v(z) = \frac{1}{z}. \quad (\text{A.3})$$

APPENDIX B

Eq. (28) is given here for convenience:

$$W(\xi) = [c_1(\xi)J_2(z) + c_2(\xi)Y_2(z) + c_3(\xi)I_2(z) + c_4(\xi)K_2(z)]/\xi. \quad (\text{B.1})$$

Differentiation of Eq. (B.1) respect to the ξ leads to

$$W'(\xi) = -\beta [c_1(\xi)J_3(z) + c_2(\xi)Y_3(z) - c_3(\xi)I_3(z) + c_4(\xi)K_3(z)]/\xi^{1.5} \\ + [c_1'(\xi)J_2(z) + c_2'(\xi)Y_2(z) + c_3'(\xi)I_2(z) + c_4'(\xi)K_2(z)]/\xi. \quad (\text{B.2})$$

By imposing the following condition:

$$c_1'(\xi)J_2(z) + c_2'(\xi)Y_2(z) + c_3'(\xi)I_2(z) + c_4'(\xi)K_2(z) = 0, \quad (\text{B.3})$$

Proportional parameter with bending moment $[\xi^4 W''(\xi)]$ can be obtained as:

$$\xi^4 W''(\xi) = \beta^2 \xi^2 [c_1(\xi)J_4(z) + c_2(\xi)Y_4(z) + c_3(\xi)I_4(z) + c_4(\xi)K_4(z)] \\ - \beta \xi^{2.5} [c_1'(\xi)J_3(z) + c_2'(\xi)Y_3(z) - c_3'(\xi)I_3(z) + c_4'(\xi)K_3(z)]. \quad (\text{B.4})$$

Furthermore, by imposing the next condition as:

$$c_1'(\xi)J_3(z) + c_2'(\xi)Y_3(z) - c_3'(\xi)I_3(z) + c_4'(\xi)K_3(z) = 0, \quad (\text{B.5})$$

Proportional parameter with transverse force $[\xi^4 W'''(\xi)]$ can be written as follows:

$$[\xi^4 W'''(\xi)]' = \beta^3 \xi^{1.5} [c_1(\xi)J_3(z) + c_2(\xi)Y_3(z) + c_3(\xi)I_3(z) - c_4(\xi)K_3(z)] \\ + \beta^2 \xi^2 [c_1'(\xi)J_4(z) + c_2'(\xi)Y_4(z) + c_3'(\xi)I_4(z) + c_4'(\xi)K_4(z)]. \quad (\text{B.6})$$

and finally by imposing the next condition as:

$$c_1'(\xi)J_4(z) + c_2'(\xi)Y_4(z) + c_3'(\xi)I_4(z) + c_4'(\xi)K_4(z) = 0, \quad (\text{B.7})$$

one can write

$$\frac{1}{\xi^2} [\xi^4 W'''(\xi)]'' = \beta^4 [c_1(\xi)J_2(z) + c_2(\xi)Y_2(z) + c_3(\xi)I_2(z) + c_4(\xi)K_2(z)]/\xi \\ + \beta^3 [c_1'(\xi)J_3(z) + c_2'(\xi)Y_3(z) + c_3'(\xi)I_3(z) - c_4'(\xi)K_3(z)]/\sqrt{\xi}. \quad (\text{B.8})$$

Substituting Eq. (B.8) in governing Eq. (25), leads to

$$c_1'(\xi)J_3(z) + c_2'(\xi)Y_3(z) + c_3'(\xi)I_3(z) - c_4'(\xi)K_3(z) = \frac{\sqrt{\xi}B(\xi)}{\beta^3} \tag{B.9}$$

Incorporating the assumed conditions of the Eqs. (B.3), (B.5), (B.7) and (B.9), the generalized functions $c_1'(\xi), c_2'(\xi), c_3'(\xi)$, and $c_4'(\xi)$ can be achieved by integrating the following system of four differential equations:

$$\begin{bmatrix} J_2(z) & Y_2(z) & I_2(z) & K_2(z) \\ J_3(z) & Y_3(z) & -I_3(z) & K_3(z) \\ J_4(z) & Y_4(z) & I_4(z) & K_4(z) \\ J_3(z) & Y_3(z) & I_3(z) & -K_3(z) \end{bmatrix} \begin{Bmatrix} c_1'(\xi) \\ c_2'(\xi) \\ c_3'(\xi) \\ c_4'(\xi) \end{Bmatrix} = \frac{\sqrt{\xi}B(\xi)}{\beta^3} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{Bmatrix}. \tag{B.10}$$

The system of differential equation matrix, Eq. (B.10), can be written under the following uncoupled form:

$$c_1'(\xi) = b_1(\xi)A(\xi) \quad c_2'(\xi) = -b_2(\xi)A(\xi) \quad c_3'(\xi) = b_3(\xi)A(\xi) \quad c_4'(\xi) = b_4(\xi)A(\xi) \tag{B.11}$$

where

$$\begin{aligned} b_1(\xi) &= \frac{\pi z}{2} [2Y_1(z) - zY_0(z)] & b_2(\xi) &= -\frac{\pi z}{2} [2J_1(z) - zJ_0(z)] \\ b_3(\xi) &= z [2K_1(z) + zK_0(z)] & b_4(\xi) &= z [2I_1(z) - zI_0(z)] \end{aligned} \tag{B.12}$$

$$A(\xi) = \frac{(1-\xi_0)W(\xi)}{4\xi^2} \sum_{i=1}^p \alpha_i \delta(\xi - \xi_i)$$

It is noticeable that in deriving Eq. (B.12), Eq. (A.3) is used. Substituting $A(\xi)$ from Eqs. (B.12) in Eq. (B.11) and integration of the obtained equations leads to

$$\begin{aligned} c_1(\xi) &= \frac{1-\xi_0}{4} \sum_{i=1}^p \left\{ \frac{\alpha_i b_1(\xi_i)}{\xi_i^2} W(\xi_i) u(\xi - \xi_i) \right\} + d_1 & c_2(\xi) &= \frac{1-\xi_0}{4} \sum_{i=1}^p \left\{ \frac{\alpha_i b_2(\xi_i)}{\xi_i^2} W(\xi_i) u(\xi - \xi_i) \right\} + d_2 \\ c_3(\xi) &= \frac{1-\xi_0}{4} \sum_{i=1}^p \left\{ \frac{\alpha_i b_3(\xi_i)}{\xi_i^2} W(\xi_i) u(\xi - \xi_i) \right\} + d_3 & c_4(\xi) &= \frac{1-\xi_0}{4} \sum_{i=1}^p \left\{ \frac{\alpha_i b_4(\xi_i)}{\xi_i^2} W(\xi_i) u(\xi - \xi_i) \right\} + d_4 \end{aligned} \tag{B.13}$$

Substituting Eqs. (B.13) into Eq. (B.1) provides a suitable form of the Eigen-mode to be used to obtain the explicit closed- form solution of the problem of interest.

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