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One-Dimensional Transient Thermal and Mechanical Stresses in FGM Hollow Cylinder with Piezoelectric Layers

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ABSTRACT

In this paper, an analytical method is developed to obtain the solution for the one dimensional transient thermal and mechanical stresses in a hollow cylinder made of functionally graded material (FGM) and piezoelectric layers. The FGM properties are assumed to depend on the variable r and they are expressed as power functions of r but the Poisson's ratio is assumed to be constant. Transient temperature distribution, as a function of radial direction and time with general thermal boundary conditions on the inside and outside surfaces, is analytically obtained for different layers, using the method of separation of variables and generalized Bessel function. A direct method is used to solve the Navier equations, using the Euler equation and complex Fourier series. This method of solution does not have the limitations of the potential function or numerical methods as to handle more general types of the mechanical and thermal boundary conditions.

Keywords: Transient; Symmetric thermal stress; Hollow cylinder; Functionally graded material; Piezoelectric.

1 INTRODUCTION

FUNCTIONALLY graded materials (FGMs) are composites with material properties varying smoothly in one or more directions which exhibit preferred structural responses. These materials are useful to withstand high thermal stresses where high heat fluxes and large temperature gradients exist. Therefore, these materials are chosen to use in structure components of aircraft, aerospace vehicles, nuclear plants as well as various temperature shielding structures widely used in industries [1]. On the other hand, piezoelectric materials are widely used in modern engineering due to its direct and inverse effects. The usage of piezoelectric layer as distributed sensors and actuators in active structure control as noise attenuation, deformation control, and vibration suppression have attracted serious attention. Piezoelectric materials are, in fact, capable of altering the response of the structures through sensing and actuation [2], by integrating the surface bonded and embedded actuators in structural systems, the desired localized strains may be induced in the structures thanks to the application of an appropriate voltage to the actuators. Such an electromechanical coupling allows closed-loop control systems to be built up, in which piezoelectric materials play the role of both the actuators and the sensors. The theoretical analysis of a three-dimensional transient thermal stress problem for a nonhomogeneous hollow circular cylinder due to a moving heat source in the axial direction from the inner and /or outer surfaces is developed by Ootao and Tanigawa [3]. Using perturbation techniques, Obata and

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Noda presented a solution for the transient thermal stresses in a plate made of FGM [4]. Jabbari et al. [5] studied a general solution for mechanical and thermal stresses in a functionally graded hollow cylinder due to non-axisymmetric steady-state load. They applied separation of variables and the complex Fourier series to solve the heat conduction and Navier equations. These authors [6] also studied the mechanical and thermal stresses in functionally graded hollow cylinder due to radially symmetric loads. Poultangari et al. [7] presented a solution for the functionally graded hollow spheres under non-axisymmetric thermo mechanical loads. He et al. [8] derived the active control of FGM plates with integrated piezoelectric sensors and actuators. Fesharaki et al. [9] presented 2D solution for electro-mechanical behavior of functionally graded piezoelectric hollow cylinder by using the separation of variables method and complex Fourier series; the Navier equations in term of displacements are derived and solved. Hosseini and Akhlaghi [10] presented transient heat conduction in a cylindrical shell of functionally graded material by using analytical method. Chu and Tzou [11] presented the transient response of a composite finite hollow cylinder heated by a moving line source on its inner boundary and cooled convectively on the exterior boundary using Eigen function expansion method and the Fourier series. Vaghari et al. [12] presented an analytical method to obtain the transient thermal and mechanical stresses in a functionally graded hollow cylinder subjected to the two-dimensional asymmetric loads. Mohazzab [13] presents the analytical solution of one-dimensional mechanical and thermal stresses for a hollow cylinder made of functionally graded material.

In this paper, an analytical method is presented to obtain the transient thermal and mechanical stresses in a functionally graded hollow cylinder with piezoelectric internal and external layers subjected to the one-dimensional transient symmetric loads. Temperature distribution is considered in transient symmetric case and mechanical and thermal boundary conditions are considered in general forms.

2 GOVERNING EQUATION

2.1 Stress distribution

Consider a functionally graded hollow cylinder of inner radius *b* and outer radius *c*, with external and internal piezoelectric layers of radius *d*, a respectively as shown in Fig.1. Symmetric cylindrical coordinate (*r*) is considered along the thickness. The FGM layer is graded through the *r*-direction thus the material properties are function of *r*.

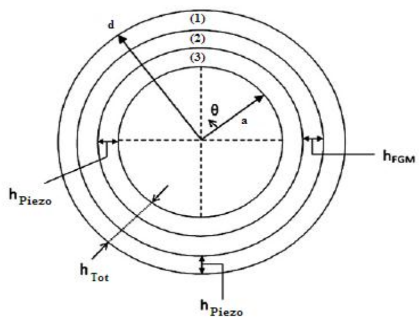

Fig.1
Geometry of the problem studied.

The governing one-dimensional strain-displacement relations in cylindrical coordinate and electric field-electric potential relations are

$$\varepsilon_{rr}=\frac{\partial u}{\partial r} \qquad \varepsilon_{\theta\theta}=\frac{u}{r} \qquad E_r=-\frac{\partial \psi}{\partial r} \tag{1}$$

In which u,ψ as displacement component along the radial direction, and the electric potential, respectively. The constitutive relations describing the mechanical and electrical interaction for a piezoelectric material are

$$\begin{aligned}
\sigma_{rr} &= C_{11}\varepsilon_{rr}+C_{12}\varepsilon_{\theta\theta}-e_{11}E_r-\alpha_r T(r,t)\\
\sigma_{\theta\theta} &= C_{12}\varepsilon_{rr}+C_{22}\varepsilon_{\theta\theta}-e_{21}E_r-\alpha_\theta T(r,t)\\
D_r &= e_{11}\varepsilon_{rr}+e_{21}\varepsilon_{\theta\theta}+\eta_{11}E_r+P_r T(r,t)
\end{aligned} \tag{2}$$

and the symmetric stress-strain relations for the FGM layer are

$$\sigma_{2rr}=\frac{E_2(r)}{(1+\upsilon)(1-2\upsilon)}\left[(1-\upsilon)\varepsilon_{rr}+\upsilon\varepsilon_{\theta\theta}\right]-\frac{E_2(r)\,\alpha_2(r)}{(1-2\upsilon)}T_2(r,t)$$
$$\sigma_{2\theta\theta}=\frac{E_2(r)}{(1+\upsilon)(1-2\upsilon)}\left[(1-\upsilon)\varepsilon_{\theta\theta}+\upsilon\varepsilon_{rr}\right]-\frac{E_2(r)\,\alpha_2(r)}{(1-2\upsilon)}T_2(r,t) \tag{3}$$
$$\sigma_{2zz}=\frac{E_2(r)}{(1+\upsilon)(1-2\upsilon)}\left[\upsilon\varepsilon_{rr}+\upsilon\varepsilon_{\theta\theta}\right]-\frac{E_2(r)\,\alpha_2(r)}{(1-2\upsilon)}T_2(r,t)$$

where $\sigma_{ij},\varepsilon_{ij}$ and $T(r,t)$ are the stress and strain tensors and the temperature distribution, $C_{ij},e_{ij},\eta_{ij},D_i,P_i$ and α_i are elastic and piezoelectric coefficients, dielectric constants, electric displacements, pyroelectric constant and thermal modulus respectively for the piezoelectric layers and $\alpha(r),E(r)$ are the coefficient of thermal expansion and the Young's modules respectively, v is Poisson's ratio that assumed to be a constant for the FGM layer and

$$E_2(r)=E_{\circ}r^{m_1} \qquad \alpha_2(r)=\alpha_{\circ}r^{m_2} \tag{4}$$

Here $E_0=E(b)/b^{m_1},\alpha_0=\alpha(b)/b^{m_2}$, where $E(b)$ and $\alpha(b)$ are the modulus of elasticity and the coefficient of thermal expansion of the inner FG material at $r=b$, m_1 and m_2 are the material constants. The equilibrium equations in the radial direction, disregarding the body forces and inertia terms and equation of electrostatic are

$$\frac{\partial\sigma_{rr}}{\partial r}+\frac{1}{r}(\sigma_{rr}-\sigma_{\theta\theta})=0 \qquad \frac{\partial D_{rr}}{\partial r}+\frac{1}{r}D_{rr}=0 \tag{5}$$

2.2 Heat conduction problem

The heat conduction equation for the functionally graded cylinder is

$$\dot{T}_2-\frac{k_2(r)}{\rho_2(r)c_2(r)}\left[T_{2,rr}+\left(\frac{k_{2,r}}{k_2}+\frac{1}{r}\right)T_{2,r}\right]=\frac{R(r,t)}{\rho_2(r)\,c_2(r)} \tag{6}$$

where $k(r)$ is the thermal conductivity, $c(r)$ is specific heat capacity, $\rho(r)$ is mass density, and $R(r,t)$ is the energy source. A comma denotes partial differentiation with respect to the space variable. The symbol dot (·) denotes derivative with respect to time. The initial and boundary conditions are assumed as:

$$k_2(r)\frac{\partial T_2}{\partial r}|_{r=c}=k_1\frac{\partial T_1}{\partial r}|_{r=c} \tag{7a}$$

$$T_2(b,t)=T_3(b,t) \tag{7b}$$

$$T_2(r,0)=g_3(r) \tag{7c}$$

The thermal FGM properties are assumed to be described with the power law functions as:

$$k_2(r)=k_0r^{m_3} \qquad \rho_2(r)=\rho_0r^{m_4} \qquad c_2(r)=c_0r^{m_5} \tag{8}$$

Here, $k_0=k(b)/b^{m_1},\rho_0=\rho(b)/b^{m_2},c_0=c(b)/b^{m_3}$ where $k(b),\rho(b)$ and $c(b)$ are the coefficient of thermal conduction, specific heat capacity and mass density of the inner FG material at $r=b$ and m_3,m_4 and m_5 are the material constants. As material properties in piezoelectric material is constant the heat conduction equation in one-dimensional problem for the outside piezoelectric layer $(j=1)$ and the inside piezoelectric layer $(j=3)$ leads to

$$\dot{T}_j - \frac{k_j}{\rho_j c_j}\left[T_{j,rr} + \frac{1}{r}T_{j,r}\right] = \frac{R(r,t)}{\rho_j c_j} \qquad (j=1,3) \tag{9}$$

and initial and boundary conditions for the 1st and 3rd piezoelectric layers are assumed as:

$$X_{11}T_1(d,t) + X_{12}T_{1,r}(d,t) = g_1(t) \tag{10a}$$

$$T_1(c,t) = T_2(c,t) \tag{10b}$$

$$T_1(r,0) = g_2(r) \tag{10c}$$

$$X_{21}T_3(a,t) + X_{22}T_{3,r}(a,t) = g_4(t) \tag{11a}$$

$$k_2(r)\frac{\partial T_2}{\partial r}\Big|_{r=b} = k_3\frac{\partial T_3}{\partial r}\Big|_{r=b} \tag{11b}$$

$$T_3(r,0) = g_5(r) \tag{11c}$$

where $X_{ij}\,(i,j=1,2)$ are Robin-type constants related to the thermal boundary condition parameters, and $g_2(r), g_3(r), g_5(r)$ are the known initial condition. The solution of the heat conduction equations for temperature distribution in FGM and piezoelectric layers may be assumed to be of the form

$$T(r,t) = W(r,t) + Y(r,t) \tag{12}$$

where $W(r,t)$ is considered in a way that the boundary conditions of $Y(r,t)$ become zero. Thus $W(r,t)$ is assumed a second order polynomial as:

$$W(r,t) = A(t)r^2 + B(t)r \tag{13}$$

Substituting Eq. (13) into Eqs. (7a), (10a) and (11a) for FGM, 1st and 3rd layers respectively yields

$$k_2(c)\left[2A_2c + B_2\right] = k_1\left[Y_{1,r}(c,t) + 2A_1c + B_1\right] \tag{14}$$

$$A_1(X_{11}d^2 + 2X_{12}d) + B_1(X_{11}d + X_{12}) = g_1(t) \tag{15}$$

$$A_3(X_{21}a^2 + 2X_{22}a) + B_3(X_{21}a + X_{22}) = g_4(t) \tag{16}$$

and substituting Eqs. (12), (13) into the heat conduction equation Eq. (9) for piezoelectric layers and Eq. (6) for the FGM layer yield

$$\dot{Y}_j - \frac{k_j}{\rho_j c_j}\left[Y_{j,rr} + \frac{1}{r}Y_{j,r}\right] = R_j \qquad (j=1,3) \tag{17a}$$

$$\dot{Y}_2 - \frac{k_2}{\rho_2 c_2}\left[Y_{2,rr} + (m_3+1)\frac{1}{r}Y_{2,r}\right] = R_2 \tag{17b}$$

where

$$R_j = \frac{R(r,t)}{\rho_j c_j} - A_{j,t} r^2 - B_{j,t} r + \frac{k_j}{\rho_j c_j}\left[4A_j + \frac{B_j}{r}\right] \qquad (j=1,3) \tag{18a}$$

$$R_2 = \frac{R(r,t)}{\rho_2 c_2} - A_{2,t} r^2 - B_{2,t} r + \frac{k_2}{\rho_2 c_2}\left[2A_2(2+m_3) + \frac{B_2}{r}(m_3+1)\right] \tag{18b}$$

The solution of Eqs. (17) may be obtained by the method of separation of variables, generalized Bessel function as:

$$Y(r,t) = F(r)G(t) \tag{19}$$

$$F_j(r) = C_j(\lambda_j r) \qquad (j=1,3) \tag{20a}$$

$$F_2(r) = r^{-\frac{m_3}{2}} C_P(\xi_n \frac{r^f}{f}) \tag{20b}$$

where $F_n(r)$ is derived from the general solution of energy equation without heat source and substituting Eqs. (20) into the Eq. (19) yield

$$Y_j(r,t) = C_j(\lambda_j r)\, G_j(t) \tag{21a}$$

$$Y_2(r,t) = \sum_{n=0}^{\infty} r^{-\frac{m_3}{2}} C_P\left(\xi_n \frac{r^f}{f}\right) G_n(t) \tag{21b}$$

and substituting Eq. (21) into the Eqs. (17) yield

$$G_j(t) = e^{-\int \tau_j dt}\left[b_j + \int \frac{R_j^*(t)}{\| C_j(\lambda_j r) \|^2} e^{\int \tau_j dt} dt\right] \qquad (j=1,3) \tag{22a}$$

$$G_2(t) = e^{-\int \tau_2 dt}\left[b_{2n} + \int \frac{R_2^*(t)}{\left\| C_P(\xi_n \frac{r^f}{f}) \right\|^2} e^{\int \tau_2 dt} dt\right] \tag{22b}$$

where $\tau_j = \frac{k_j}{\rho_j c_j}\lambda_j^2$ and $\| C_j(\lambda_j r) \|$ is the norm of the cylindrical function for piezoelectric layers as:

$$\| C_1(\lambda_1 r) \|^2 = \int_c^d \left[C_1(\lambda_1 r)\right]^2 r\, dr \tag{23a}$$

$$\| C_3(\lambda_3 r)\|^2 = \int_a^b \left[C_3(\lambda_3 r)\right]^2 r\,dr \tag{23b}$$

$$R_1^*(t) = \int_c^d r\,R_1(r,t)C_1(\lambda_1 r)dr \tag{24a}$$

$$R_3^*(t) = \int_a^b r\,R_3(r,t)C_3(\lambda_3 r)dr \tag{24b}$$

and $\tau_2 = \dfrac{k_0}{\rho_0 c_0}\xi_n^2$. $\left\| C_P\left(\xi_n \dfrac{r^f}{f}\right)\right\|$ is the norm of the cylindrical function of the FGM layer as:

$$\left\| C_P\left(\xi_n \frac{r^f}{f}\right)\right\|^2 = \int_b^c \left[C_P\left(\xi_n \frac{r^f}{f}\right)\right]^2 r\,dr \tag{25a}$$

$$R_2^*(t) = \int_b^c r^{\frac{m_3}{2}+1} R_2(r,t)\,C_P\left(\xi_n \frac{r^f}{f}\right)dr \tag{25b}$$

and b_n is derived from the initial thermal boundary condition defined by Eq. (7c), (10c), (11c) for the first, FGM, and third layer respectively as:

$$b_1 = \frac{1}{\|C_1(\lambda_1 r)\|^2}\int_c^d r\left[g_2(r) - W_1(r,0)\right]C_1(\lambda_1 r)dr - G_1^*(0) \tag{26a}$$

$$b_{2n} = \frac{1}{\left\| C_P\left(\xi_n \frac{r^f}{f}\right)\right\|^2}\int_b^c r^{\frac{m_3}{2}+1}\left[g_3(r) - W_2(r,0)\right]C_P\left(\xi_n \frac{r^f}{f}\right)dr - G_2^*(0) \tag{26b}$$

$$b_3 = \frac{1}{\|C_3(\lambda_3 r)\|^2}\int_a^b r\left[g_5(r) - W_3(r,0)\right]C_3(\lambda_3 r)dr - G_3^*(0) \tag{26c}$$

$$G_j^*(t) = \int \frac{R_j^*(t)}{\|C_j(\lambda_j r)\|^2} e^{\int \tau_j dt} dt \qquad (j = 1,3) \tag{27a}$$

$$G_2^*(t) = \int \frac{R_2^*(t)}{\left\| C_P(\xi_n \frac{r^f}{f})\right\|^2} e^{\int \tau_2 dt} dt \tag{27b}$$

where $C_j(\lambda_j r), j = 1,3, C_P\left(\xi_n \dfrac{r^f}{f}\right)$ are the mathematical Cylindrical Function given by

$$C_j(\lambda_j r) = J_0(\lambda_j r) + c_j Y_0(\lambda_j r) \qquad (j = 1,3) \tag{28a}$$

$$C_P\left(\xi_n \frac{r^f}{f}\right)=J_P\left(\xi_n \frac{r^f}{f}\right)+c_{2n}J_{-P}\left(\xi_n \frac{r^f}{f}\right) \tag{28b}$$

$$c_1=-\frac{X_{11}J_0(\lambda_1 d)+X_{12}J_0'(\lambda_1 d)}{X_{11}Y_0(\lambda_1 d)+X_{12}Y_0'(\lambda_1 d)} \tag{29a}$$

$$c_3=-\frac{X_{21}J_0(\lambda_3 a)+X_{22}J_0'(\lambda_3 a)}{X_{21}Y_0(\lambda_3 a)+X_{22}Y_0'(\lambda_3 a)} \tag{29b}$$

$$c_{2n}=-\frac{J_P(\xi_n \frac{b^f}{f})}{J_{-P}(\xi_n \frac{b^f}{f})} \tag{29c}$$

$$f=\frac{1}{2}(m_5+m_4-m_3+2) \qquad P=\frac{m_3}{2f} \tag{30}$$

Here, J_0 is the Bessel function of the first kind of order zero, Y_0 is the Bessel function of the second kind of order zero, the symbol (') denotes derivative with respect to *r*, and the eigenvalues λ_1,λ_3 and ξ_n are respectively the positive roots of

$$\left[X_{11}J_0(\lambda_1 d)+X_{12}J_0'(\lambda_1 d)\right]Y_0(\lambda_1 c)-\left[X_{11}Y_0(\lambda_1 d)+X_{12}Y_0'(\lambda_1 d)\right]J_0(\lambda_1 c)=0 \tag{31a}$$

$$\left[X_{21}J_0(\lambda_3 a)+X_{22}J_0'(\lambda_3 a)\right]Y_0'(\lambda_3 b)-\left[X_{21}Y_0(\lambda_3 a)+X_{22}Y_0'(\lambda_3 a)\right]J_0'(\lambda_3 b)=0 \tag{31b}$$

$$\begin{aligned}&\left[J_{-P}\left(\xi_n \frac{b^f}{f}\right)\right]\left[\left(-\frac{m_3}{2}\right)c^{\frac{m_3}{2}-1}J_P\left(\xi_n \frac{c^f}{f}\right)+c^{\frac{m_3}{2}}J_P'\left(\xi_n \frac{c^f}{f}\right)\right]-\left[J_P\left(\xi_n \frac{b^f}{f}\right)\right]\\ &\times\left[\left(-\frac{m_3}{2}\right)c^{\frac{m_3}{2}-1}J_{-P}\left(\xi_n \frac{c^f}{f}\right)+c^{\frac{m_3}{2}}J_P'\left(\xi_n \frac{c^f}{f}\right)\right]=0\end{aligned} \tag{31c}$$

where A_1,A_2,A_3,B_1,B_2,B_3 are six unknowns to be obtained from Eqs. (14) -(16) and (32) -(34) by system of linear equations

$$A_1c^2+B_1c=Y_2(c,t)+A_2c^2+B_2c \tag{32}$$

$$A_2b^2+B_2b=Y_3(b,t)+A_3b^2+B_3b \tag{33}$$

$$k_2(b)\left[Y_{2,r}(b,t)+2A_2b+B_2\right]=k_3\left[2A_3b+B_3\right] \tag{34}$$

The solution of this system of equation is given in the Appendix.

3 SOLUTION OF THE PROBLEM

3.1 Piezoelectric layer

Using the relations (1), (2) and (5) the Navier equations in term of the displacements are

$$
\begin{aligned}
&u_{,rr}+\frac{1}{r}u_{,r}-\frac{C_{22}}{C_{11}}\frac{1}{r^2}u+\frac{e_{11}}{C_{11}}\psi_{,rr}+\frac{e_{11}-e_{21}}{C_{11}}\frac{1}{r}\psi_{,r}=\frac{\alpha_r}{C_{11}}T_{,r}+\frac{\alpha_r-\alpha_\theta}{C_{11}}\frac{1}{r}T\\
&u_{,rr}+\left(1+\frac{e_{21}}{e_{11}}\right)\frac{1}{r}u_{,r}-\frac{\eta_{11}}{e_{11}}\psi_{,rr}-\frac{\eta_{11}}{e_{11}}\frac{1}{r}\psi_{,r}=-\frac{P_r}{e_{11}}T_{,r}-\frac{P_r}{e_{11}}\frac{1}{r}T
\end{aligned}
\tag{35}
$$

Eqs. (35) are a system of differential equations having general and particular solutions. The general solutions are assumed as:

$$
\begin{Bmatrix}u_1^g(r)\\u_3^g(r)\end{Bmatrix}=\begin{Bmatrix}R\\R'\end{Bmatrix}r^\eta \qquad \begin{Bmatrix}\psi_1^g(r)\\\psi_3^g(r)\end{Bmatrix}=\begin{Bmatrix}W\\W'\end{Bmatrix}r^\eta
\tag{36}
$$

where R, W and R', W' are the unknown constants for the 1st and 3rd layers respectively and by using the specified boundary conditions are determined. Substituting Eqs. (36) into Eqs. (35) yield

$$
\begin{aligned}
&\left[\eta^2-\frac{C_{22}}{C_{11}}\right]\begin{Bmatrix}R\\R'\end{Bmatrix}+\left[\frac{e_{11}}{C_{11}}\eta^2-\frac{e_{21}}{C_{11}}\eta\right]\begin{Bmatrix}W\\W'\end{Bmatrix}=0\\
&\left[\eta^2+\frac{e_{21}}{e_{11}}\eta\right]\begin{Bmatrix}R\\R'\end{Bmatrix}+\left[-\frac{\eta_{11}}{e_{11}}\eta^2\right]\begin{Bmatrix}W\\W'\end{Bmatrix}=0
\end{aligned}
\tag{37}
$$

Eqs. (37) are a system of algebraic equations. For obtaining the nontrivial solution of the equations, the determinant of system should be equal to zero. So the four roots η_1 to η_4 for the equations are achieved which include repeated roots of zero, hence a solution of the form of ln$r$ must be considered in the general solutions as:

$$
\begin{Bmatrix}u_1^g(r)\\u_3^g(r)\end{Bmatrix}=\sum_{k=1}^{2}N_k\begin{Bmatrix}W_k\\W_k'\end{Bmatrix}r^{\eta_k}+N^*\begin{Bmatrix}W_4\\W_4'\end{Bmatrix} \qquad \begin{Bmatrix}\psi_1^g(r)\\\psi_3^g(r)\end{Bmatrix}=\sum_{k=1}^{2}\begin{Bmatrix}W_k\\W_k'\end{Bmatrix}r^{\eta_k}+\begin{Bmatrix}W_3+W_4\ln r\\W_3'+W_4'\ln r\end{Bmatrix}
\tag{38}
$$

where N_k is the relation between constants W_k, W_k' and R_k, R_k' respectively and are obtained from Eq. (37) as:

$$
N_k=-\frac{b_k}{a_k}\quad (k=1,2) \qquad N^*=-\frac{e_{21}}{C_{22}}
\tag{39}
$$

where a_k, b_k, a_k', b_k' are given in the Appendix. The particular solutions u^p and ψ^p of Eqs. (35) for piezoelectric layers $(j=1,3)$ are assumed as:

$$
\begin{aligned}
&\begin{Bmatrix}u_1^p(r,t)\\u_3^p(r,t)\end{Bmatrix}=r\sum_{k=0}^{\infty}\left[\begin{Bmatrix}D_1\\D_9\end{Bmatrix}J_0(\lambda_j r)+\begin{Bmatrix}D_2\\D_{10}\end{Bmatrix}Y_0(\lambda_j r)\right]G_j(t)+\begin{Bmatrix}D_3\\D_{11}\end{Bmatrix}r^2+\begin{Bmatrix}D_4\\D_{12}\end{Bmatrix}r^3\\
&\begin{Bmatrix}\psi_1^p(r,t)\\\psi_3^p(r,t)\end{Bmatrix}=r\sum_{k=0}^{\infty}\left[\begin{Bmatrix}D_5\\D_{13}\end{Bmatrix}J_0(\lambda_j r)+\begin{Bmatrix}D_6\\D_{14}\end{Bmatrix}Y_0(\lambda_j r)\right]G_j(t)+\begin{Bmatrix}D_7\\D_{15}\end{Bmatrix}r^2+\begin{Bmatrix}D_8\\D_{16}\end{Bmatrix}r^3
\end{aligned}
\tag{40}
$$

We consider the definition of Bessel function of the first and second type as:

$$
\begin{aligned}
&J_n(\lambda r)=\sum_{k=0}^{\infty}\frac{(-1)^k\left(\frac{1}{2}\lambda r\right)^{2k+n}}{k!\,\Gamma(k+n+1)}\\
&Y_n(\lambda r)=\frac{J_n(\lambda r)\cos n\pi-J_{-n}(\lambda r)}{\sin n\pi}=\frac{J_n(\lambda r)\cos n\pi-(-1)^n J_n(\lambda r)}{\sin n\pi}
\end{aligned}
\tag{41}
$$

Substituting Eqs. (40) and (41) and using heat distribution in piezoelectric layers into Eqs. (35) yield

$$
\begin{cases}
\begin{Bmatrix}D_1\\D_9\end{Bmatrix}x_1+\begin{Bmatrix}D_5\\D_{13}\end{Bmatrix}x_2=x_3 \\
\begin{Bmatrix}D_1\\D_9\end{Bmatrix}x_8+\begin{Bmatrix}D_5\\D_{13}\end{Bmatrix}x_9=x_{10}
\end{cases}
\qquad
\begin{cases}
\begin{Bmatrix}D_2\\D_{10}\end{Bmatrix}x_1+\begin{Bmatrix}D_6\\D_{14}\end{Bmatrix}x_2=\begin{Bmatrix}c_1\\c_3\end{Bmatrix}x_3 \\
\begin{Bmatrix}D_2\\D_{10}\end{Bmatrix}x_8+\begin{Bmatrix}D_6\\D_{14}\end{Bmatrix}x_9=\begin{Bmatrix}c_1\\c_3\end{Bmatrix}x_{10}
\end{cases}
\tag{42}
$$

$$
\begin{cases}
\begin{Bmatrix}D_3\\D_{11}\end{Bmatrix}x_4+\begin{Bmatrix}D_7\\D_{15}\end{Bmatrix}x_5=0 \\
\begin{Bmatrix}D_3\\D_{11}\end{Bmatrix}x_{11}+\begin{Bmatrix}D_7\\D_{15}\end{Bmatrix}x_{12}=0
\end{cases}
\qquad
\begin{cases}
\begin{Bmatrix}D_4\\D_{12}\end{Bmatrix}x_6+\begin{Bmatrix}D_8\\D_{16}\end{Bmatrix}x_7=0 \\
\begin{Bmatrix}D_4\\D_{12}\end{Bmatrix}x_{13}+\begin{Bmatrix}D_8\\D_{16}\end{Bmatrix}x_{14}=0
\end{cases}
\tag{43}
$$

Eqs. (42) and (43) are four systems of algebraic equations. The determinant of coefficients Eqs. (43) are zero, so the obvious answer is the only possible. x_1 to x_{14} and D_1 to D_{16} are given in the Appendix. The complete solutions $u(r,t)$ and $\psi(r,t)$ for piezoelectric layers $(j=1,3)$ are the sum of the general and particular solutions and are

$$
\begin{aligned}
&\begin{Bmatrix}u_1(r,t)\\u_3(r,t)\end{Bmatrix}=\sum_{k=1}^{2}N_k\begin{Bmatrix}W_k\\W_k'\end{Bmatrix}r^{\eta_k}+N^*\begin{Bmatrix}W_4\\W_4'\end{Bmatrix}+r\sum_{k=0}^{\infty}\left[\begin{Bmatrix}D_1\\D_9\end{Bmatrix}J_0(\lambda_j r)+\begin{Bmatrix}D_2\\D_{10}\end{Bmatrix}Y_0(\lambda_j r)\right]G_j(t)\\
&\begin{Bmatrix}\psi_1(r,t)\\\psi_3(r,t)\end{Bmatrix}=\sum_{k=1}^{2}\begin{Bmatrix}W_k\\W_k'\end{Bmatrix}r^{\eta_k}+\begin{Bmatrix}W_3+W_4\ln r\\W_3'+W_4'\ln r\end{Bmatrix}+r\sum_{k=0}^{\infty}\left[\begin{Bmatrix}D_5\\D_{13}\end{Bmatrix}J_0(\lambda_j r)+\begin{Bmatrix}D_6\\D_{14}\end{Bmatrix}Y_0(\lambda_j r)\right]G_j(t)
\end{aligned}
\tag{44}
$$

Substituting Eqs. (44) into Eq. (1), the strains for piezoelectric layers are obtained as:

$$
\begin{aligned}
&\begin{Bmatrix}\varepsilon_{1rr}\\\varepsilon_{3rr}\end{Bmatrix}=\sum_{k=1}^{2}N_k\begin{Bmatrix}W_k\\W_k'\end{Bmatrix}\eta_k r^{\eta_k-1}+\sum_{k=0}^{\infty}\left[(2k+1)\begin{Bmatrix}D_1\\D_9\end{Bmatrix}J_0(\lambda_j r)+(2k+1)\begin{Bmatrix}D_2\\D_{10}\end{Bmatrix}Y_0(\lambda_j r)\right]G_j(t)\\
&\begin{Bmatrix}\varepsilon_{1\theta\theta}\\\varepsilon_{3\theta\theta}\end{Bmatrix}=\sum_{k=1}^{2}N_k\begin{Bmatrix}W_k\\W_k'\end{Bmatrix}r^{\eta_k-1}+\frac{N^*}{r}\begin{Bmatrix}W_4\\W_4'\end{Bmatrix}+\sum_{k=0}^{\infty}\left[\begin{Bmatrix}D_1\\D_9\end{Bmatrix}J_0(\lambda_j r)+\begin{Bmatrix}D_2\\D_{10}\end{Bmatrix}Y_0(\lambda_j r)\right]G_j(t)\\
&\begin{Bmatrix}E_{1r}\\E_{3r}\end{Bmatrix}=-\sum_{k=1}^{2}\eta_k\begin{Bmatrix}W_k\\W_k'\end{Bmatrix}r^{\eta_k-1}-\frac{1}{r}\begin{Bmatrix}W_4\\W_4'\end{Bmatrix}-\sum_{k=0}^{\infty}\left[\begin{Bmatrix}D_5\\D_{13}\end{Bmatrix}(2k+1)J_0(\lambda_j r)+\begin{Bmatrix}D_6\\D_{14}\end{Bmatrix}(2k+1)\,Y_0(\lambda_j r)\right]G_j(t)\\
&\begin{Bmatrix}\sigma_{1rr}\\\sigma_{3rr}\end{Bmatrix}=\sum_{k=1}^{2}\left[(C_{11}\eta_k+C_{12})N_k+e_{11}\eta_k\right]\begin{Bmatrix}W_k\\W_k'\end{Bmatrix}r^{\eta_k-1}+\frac{1}{r}\left(C_{12}N^*+e_{11}\right)\begin{Bmatrix}W_4\\W_4'\end{Bmatrix}+\sum_{k=0}^{\infty}\Bigg[\left(C_{11}(2k+1)+C_{12}\right)\\
&\qquad\times\begin{Bmatrix}D_1\\D_9\end{Bmatrix}J_0(\lambda_j r)+\left(C_{11}(2k+1)+C_{12}\right)\begin{Bmatrix}D_2\\D_{10}\end{Bmatrix}Y_0(\lambda_j r)\Bigg]G_j(t)+e_{11}\sum_{k=0}^{\infty}\Bigg[\begin{Bmatrix}D_5\\D_{13}\end{Bmatrix}(2k+1)J_0(\lambda_j r)\\
&\qquad+\begin{Bmatrix}D_6\\D_{14}\end{Bmatrix}(2k+1)Y_0(\lambda_j r)\Bigg]G_j(t)-\alpha_r C_j(\lambda_j r)\left\{e^{-\int\tau_j dt}\left[b_j+\int\frac{R_j^*(t)}{\|C_j(\lambda_j r)\|^2}e^{\int\tau_j dt}dt\right]\right\}-\alpha_r A_j r^2-\alpha_r B_j r
\end{aligned}
\tag{45}
$$

$$\begin{Bmatrix}\sigma_{1\theta\theta}\\ \sigma_{3\theta\theta}\end{Bmatrix}=\sum_{k=1}^{2}\Big[\left(C_{12}\eta_k+C_{22}\right)N_k+e_{21}\eta_k\Big]\begin{Bmatrix}W_k\\ W'_k\end{Bmatrix}r^{\eta_k-1}+\frac{1}{r}\left(C_{22}N^*+e_{21}\right)\begin{Bmatrix}W_4\\ W'_4\end{Bmatrix}+\sum_{k=0}^{\infty}\Big[\left(C_{12}\left(2k+1\right)+C_{22}\right)$$
$$\times\begin{Bmatrix}D_1\\ D_9\end{Bmatrix}J_0\left(\lambda_j r\right)+\left(C_{12}\left(2k+1\right)+C_{22}\right)\begin{Bmatrix}D_2\\ D_{10}\end{Bmatrix}Y_0\left(\lambda_j r\right)\Big]G_j\left(t\right)+e_{21}\sum_{k=0}^{\infty}\left[\begin{Bmatrix}D_5\\ D_{13}\end{Bmatrix}\left(2k+1\right)J_0\left(\lambda_j r\right)\right.$$
$$\left.+\begin{Bmatrix}D_6\\ D_{14}\end{Bmatrix}\left(2k+1\right)Y_0(\lambda_j r)\right]G_j\left(t\right)-\alpha_\theta C_j\left(\lambda_j r\right)\left\{e^{-\int\tau_j dt}\left[b_j+\int\frac{R_j^*(t)}{\|C_j(\lambda_j r)\|^2}e^{\int\tau_j dt}dt\right]\right\}-\alpha_\theta A_j r^2-\alpha_\theta B_j r \tag{45}$$
$$\begin{Bmatrix}D_{1rr}\\ D_{3rr}\end{Bmatrix}=\sum_{k=1}^{2}\Big[\left(e_{11}\eta_k+e_{21}\right)N_k-\eta_{11}\eta_k\Big]\begin{Bmatrix}W_k\\ W'_k\end{Bmatrix}r^{\eta_k-1}+\frac{1}{r}\left(e_{21}N^*-\eta_{11}\right)\begin{Bmatrix}W_4\\ W'_4\end{Bmatrix}+\sum_{k=0}^{\infty}\Big[\left(e_{11}\left(2k+1\right)+e_{21}\right)$$
$$\times\begin{Bmatrix}D_1\\ D_9\end{Bmatrix}J_0(\lambda_j r)+\left(e_{11}\left(2k+1\right)+e_{21}\right)\begin{Bmatrix}D_2\\ D_{10}\end{Bmatrix}Y_0(\lambda_j r)\Big]G_j(t)-\eta_{11}\sum_{k=0}^{\infty}\left[\begin{Bmatrix}D_5\\ D_{13}\end{Bmatrix}\left(2k+1\right)J_0(\lambda_j r)\right.$$
$$\left.+\begin{Bmatrix}D_6\\ D_{14}\end{Bmatrix}\left(2k+1\right)Y_0(\lambda_j r)\right]G_j(t)+P_r C_j(\lambda_j r)\left\{e^{-\int\tau_j dt}\left[b_j+\int\frac{R_j^*(t)}{\|C_j(\lambda_j r)\|^2}e^{\int\tau_j dt}dt\right]\right\}+P_r A_j r^2+P_r B_j r$$

3.2 FGM layer

Using the relations (1), (3), (4) and (5), the Navier equations in term of the displacements are

$$u_{,rr}+(m_1+1)\frac{1}{r}u_{,r}+\left(\frac{\upsilon m_1}{1-\upsilon}-1\right)\frac{1}{r^2}u=\left(\frac{1+\upsilon}{1-\upsilon}\right)\alpha_0\left((m_1+m_2)r^{m_2-1}T+r^{m_2}T_{,r}\right) \tag{46}$$

Eqs. (46) is a differential equation having general and particular solutions. The general solutions are assumed as:

$$u^g(r)=E\,r^\delta \tag{47}$$

where E is the unknown constant and by using the specified boundary conditions is determined. Substituting Eq. (47) into Eq. (46) yields

$$\delta^2+m_1\delta+\frac{\upsilon m_1}{1-\upsilon}-1=0 \tag{48}$$

Eq. (48) has two roots δ_1,δ_2 as:

$$\delta_{1,2}=-\frac{m_1}{2}\pm\sqrt{\frac{m_1^2}{4}-\frac{\upsilon m_1}{1-\upsilon}+1} \tag{49}$$

The particular solution u^p of Eq. (46) for FGM layer is assumed as:

$$u^p(r,t)=r^{m_2-\beta+1}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\left[D_{17}J_P\left(\xi_n\frac{r^f}{f}\right)+D_{18}J_{-P}\left(\xi_n\frac{r^f}{f}\right)\right]G_{2n}(t)+D_{19}r^{m_2+2}+D_{20}r^{m_2+3} \tag{50}$$

where $\beta=\dfrac{m_3}{2}$. Substituting Eq. (50) and (41) and using heat distribution in FGM layers into Eq. (46) yield

$$D_{17}y_1=y_2 \qquad\qquad D_{18}y_3=y_4 \tag{51}$$

$$D_{19}y_5 = y_6 \qquad\qquad D_{20}y_7 = y_8 \tag{52}$$

where constants y_1 to y_3 and D_{17} to D_{20} are given in the Appendix. The complete solutions $u(r,\theta,t)$ for FGM layer is

$$u(r,t) = \sum_{j=1}^{2} E_j r^{\delta_j} + r^{m_2-\beta+1} \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\left[D_{17}J_P(\xi_n \frac{r^f}{f}) + D_{18}J_{-P}(\xi_n \frac{r^f}{f})\right] G_{2n}(t) + D_{19}r^{m_2+2} + D_{20}r^{m_2+3} \tag{53}$$

Substituting Eq. (53) into Eq. (1), the strains for FGM layer are obtained as:

$$\begin{aligned}
\varepsilon_{rr} &= \sum_{j=1}^{2} E_j \delta_j r^{\delta_j-1} + r^{m_2-\beta}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\Bigg[\big(m_2-\beta+1+(2k+P)\big)D_{17}J_P\left(\xi_n\frac{r^f}{f}\right) + \big(m_2-\beta+1+(2k-P)\big)D_{18} \\
&\quad \times J_{-P}\left(\xi_n\frac{r^f}{f}\right)\Bigg]G_{2n}(t) + (m_2+2)D_{19}r^{m_2+1} + (m_2+3)D_{20}r^{m_2+2} \\
\varepsilon_{\theta\theta} &= \sum_{j=1}^{2} E_j r^{\delta_j-1} + r^{m_2-\beta}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\left[D_{17}J_P\left(\xi_n\frac{r^f}{f}\right) + D_{18}J_{-P}\left(\xi_n\frac{r^f}{f}\right)\right]G_{2n}(t) + D_{19}r^{m_2+1} + D_{20}r^{m_2+2} \\
\sigma_{rr} &= \frac{E_0}{(1+\upsilon)(1-2\upsilon)}\Bigg\langle \sum_{j=1}^{2}\big((1-\upsilon)\delta_j+\upsilon\big)E_j r^{m_1+\delta_j-1} + r^{m_1+m_2-\beta}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\Bigg\langle \big[(1-\upsilon)\big(m_2-\beta+1+(2k+P)\big)+\upsilon\big] \\
&\quad \times D_{17}J_P\left(\xi_n\frac{r^f}{f}\right) + \big[(1-\upsilon)\big(m_2-\beta+1+(2k-P)\big)+\upsilon\big]D_{18}J_{-P}\left(\xi_n\frac{r^f}{f}\right)\Bigg]\Bigg\rangle G_{2n}(t) + \big[(1-\upsilon)(m_2+2)+\upsilon\big] \\
&\quad \times D_{19}r^{m_1+m_2+1} + \big[(1-\upsilon)(m_2+3)+\upsilon\big]D_{20}r^{m_1+m_2+2} - \alpha_0(1+\upsilon)\sum_{n=0}^{n=+\infty} r^{m_1+m_2-\frac{m_3}{2}} C_P\left(\xi_n\frac{r^f}{f}\right) \\
&\quad \times \left\{ e^{-\int \tau_2 dt}\left[b_{2n} + \int \frac{R_2^*(t)}{\| C_P(\xi_n \frac{r^f}{f})\|^2} e^{\int \tau_2 dt}dt\right]\right\} - \alpha_0(1+\upsilon)A_2 r^{m_1+m_2+2} - \alpha_0(1+\upsilon)B_2 r^{m_1+m_2+1}\Bigg\rangle \\
\sigma_{\theta\theta} &= \frac{E_0}{(1+\upsilon)(1-2\upsilon)}\Bigg\langle \sum_{j=1}^{2}\big((1-\upsilon)+\upsilon\delta_j\big)E_j r^{m_1+\delta_j-1} + r^{m_1+m_2-\beta}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\Bigg\langle \big[\upsilon\big(m_2-\beta+1+(2k+P)\big)+(1-\upsilon)\big] \\
&\quad \times D_{17}J_P\left(\xi_n\frac{r^f}{f}\right) + \big[\upsilon\big(m_2-\beta+1+(2k-P)\big)+(1-\upsilon)\big]D_{18}J_{-P}\left(\xi_n\frac{r^f}{f}\right)\Bigg]\Bigg\rangle G_{2n}(t) + \big[\upsilon(m_2+2)+(1-\upsilon)\big] \\
&\quad \times D_{19}r^{m_1+m_2+1} + \big[\upsilon(m_2+3)+(1-\upsilon)\big]D_{20}r^{m_1+m_2+2} - \alpha_0(1+\upsilon)\sum_{n=0}^{n=+\infty} r^{m_1+m_2-\frac{m_3}{2}} C_P\left(\xi_n\frac{r^f}{f}\right) \\
&\quad \times \left\{ e^{-\int \tau_2 dt}\left[b_{2n} + \int \frac{R_2^*(t)}{\| C_P(\xi_n \frac{r^f}{f})\|^2} e^{\int \tau_2 dt}dt\right]\right\} - \alpha_0(1+\upsilon)A_2 r^{m_1+m_2+2} - \alpha_0(1+\upsilon)B_2 r^{m_1+m_2+1}\Bigg\rangle
\end{aligned} \tag{54}$$

It is recalled that $E_j\,(j=1,2), W_k\,(k=1,...,4)$ and $W_k'\,(k=1,...,4)$ are ten unknown constants for the FGM shell and the outer and the inner piezoelectric layers respectively and can be evaluated by satisfying the boundary conditions and continuity requirements on the interface. Therefore, displacements, electric potential, stress and other responses can be evaluated. The corresponding boundary conditions can be written as:

$$\begin{array}{lll} \sigma_{1rr}\left(r=d\right)=-p_1 & \psi_1\left(r=c\right)=0 & \psi_3\left(r=b\right)=0 \\ \sigma_{3rr}\left(r=a\right)=-p_2 & D_r\left(r=d\right)=0 & \psi_3\left(r=a\right)=v \end{array} \tag{55}$$

where p_1 and p_2 are the outer and inner pressure, respectively. In addition, the continuity requirements for the stresses and displacements on the interfaces must be satisfied, therefore we have

$$\begin{array}{ll} u_1\left(r=c\right)=u_2\left(r=c\right) & \sigma_{1rr}\left(r=c\right)=\sigma_{2rr}\left(r=c\right) \\ u_2\left(r=b\right)=u_3\left(r=b\right) & \sigma_{2rr}\left(r=b\right)=\sigma_{3rr}\left(r=b\right) \end{array} \tag{56}$$

4 NUMERICAL RESULTS AND DISCUSSION

Assume material properties for piezoelectric on first layer as an actuator and for third layer piezoelectric as a sensor PZT-4 from following table

Table1
Material properties of piezoelectric.

Material	Elastic constants, *Gpa*						
	C_{11}	C_{12}	C_{13}	C_{22}	C_{23}	C_{33}	C_{44}
PZT-4	139	78	74	139	74	115	25.6

Piezoelectric constants, C/m^2 Permittivity, $10^{-9}C/Nm^2$ Pyroelectric constants, $10^{-5}C/Km^2$ Coefficient of thermal expansion, $10^{-6} 1/K$							
	e_{11}	e_{12}	e_{22}	η_{11}	η_{22}	$P_r=P_\theta=P_0$	$\alpha_r=\alpha_\theta=\alpha_0$
PZT-4	-5.2	15.1	12.7	6.5	6.5	5.4	2.62

Let us consider a thick hollow cylinder of radii $\alpha_r=0.96m, b=1m, c=1.24m$ and $d=1.28m$ .The Poisson's ratio is assumed 0.3 and modulus of elasticity, coefficient of thermal expansion, thermal conductivity, density and specific heat capacity for FGM layer are $E_0=200GPa$, $\alpha_0=1.2\times10^{-6}/K$, $k_0=2W/mK$, $\rho_0=7800kg/m^3$, $c_0=420J/kgK$ and thermal conductivity, density and specific heat capacity for piezoelectric layers are $k=1.5W/mK$, $\rho=7500kg/m^3$ and $c=350J/kgK$ respectively. For simplicity of analysis, we consider the power law of material properties be the same as $m_1=m_2=m_3=m_4=m_5=m$. As the example Consider a hollow cylinder subjected to thermo mechanical loads, and its mechanical, thermal and electrical boundary conditions are respectively, taken as:

$$\begin{array}{llll} \sigma_{1rr}\left(r=d\right)=-30\ MPa & \psi_1\left(r=c\right)=0 & \psi_3\left(r=b\right)=0 & g_1\left(t\right)=20\sin\left(t\right)\ K \\ \sigma_{3rr}\left(r=a\right)=0 & D_{1r}\left(r=d\right)=0 & \psi_3\left(r=a\right)=20\ v & g_4\left(t\right)=0 \end{array} \tag{57}$$

The initial temperature for the FGM shell and the inner and the outer piezoelectric layers are zero. The cylinder is heated by the rate of energy generation per unit time and unit volume of $R(r,t)=6\times10^6\times\frac{1}{r}\sin(t)\frac{W}{m^3}$. Figs. 2-8 illustrates the temperature profile, radial displacement, radial and circumferential stress for FGM layer and electric potential and radial electrical displacement for piezoelectric layers at the middle radius of the layers over the course of 10 seconds for different power law indices respectively. The value of $m=0$ corresponds to pure metal. The curves associated with the non-zero heat source follow the sine-form pattern of the assumed heat source. Temperature distribution are zero at $t=0$ due to the initial temperature. It can easily be seen as the graded index m increase the values of temperature, radial displacement, radial and circumferential stresses for FGM layer and electric potential and radial electrical displacement for piezoelectric layers increase at the same time.

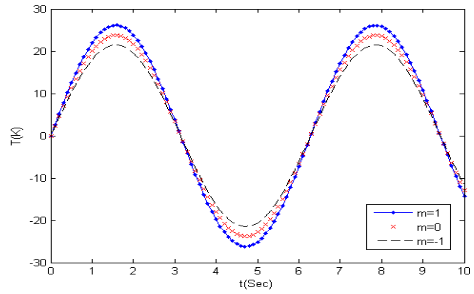

Fig.2
Transient temperature distribution in the FGM layer.

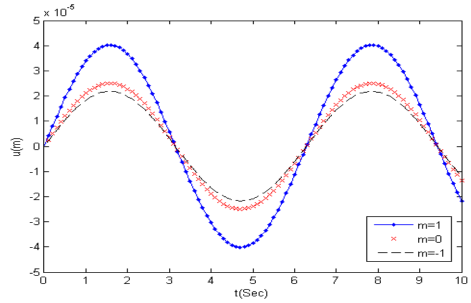

Fig.3
Radial displacement in the FGM layer.

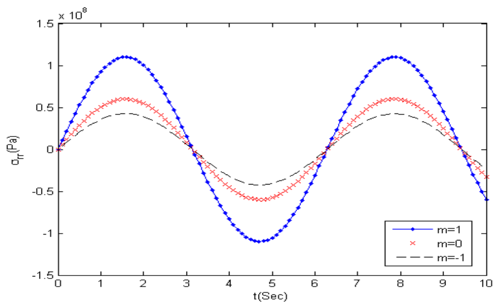

Fig.4
Radial stress in the FGM layer.

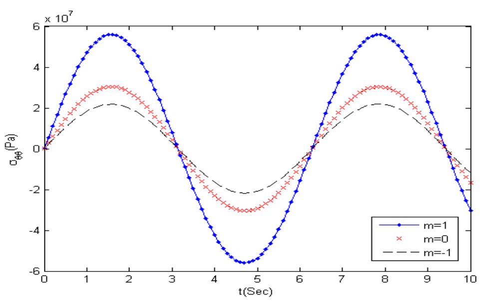

Fig.5
Circumferential stress in the FGM layer.

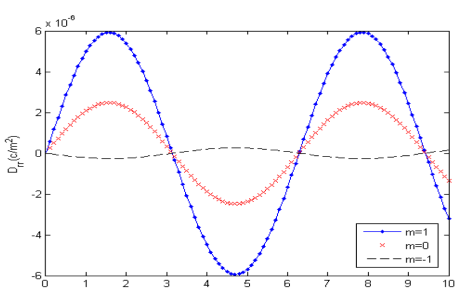

Fig.6
Radial electrical displacement in the actuator layer.

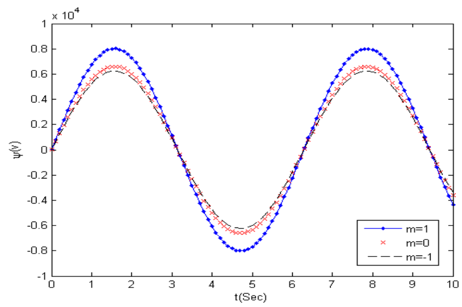

Fig.7
Electric potential distribution in the actuator layer.

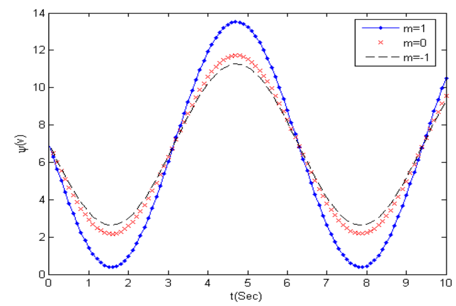

Fig.8
Electric potential distribution in the sensor layer.

For different values of *m*, the temperature profile for piezo FGM hollow cylinder, radial displacement, radial and circumferential stresses for FGM layer, electric potential and radial electrical displacement for piezoelectric layers along the radial direction at $t = 2$ seconds are plotted in Figs. 9-15. From Fig. 9, one can see the temperatures satisfy the prescribed thermal boundary conditions at the internal and external boundaries, the temperatures decrease along the thickness and increase as the graded index *m* increases at the same radial point. Fig. 10, shows that the radial displacements for FGM layer increase gradually from the inner surface to the outer surface and the radial displacements increase as the graded index *m* increases at the same radial point.

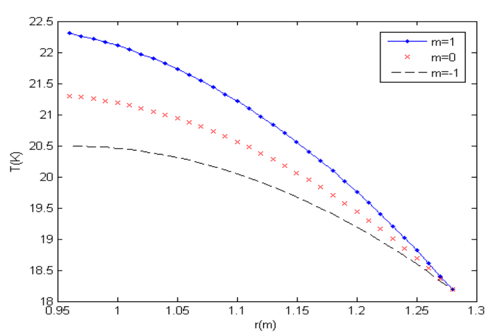

Fig.9
Temperature distribution in the piezo FGM hollow cylinder along the thickness.

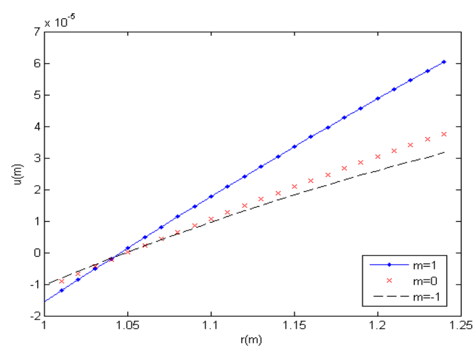

Fig.10
Radial displacement in the FGM hollow cylinder along the thickness.

It can easily be seen from Fig. 11 that the radial stresses increase with the increase of power law parameters m at the same radial point. The distribution of circumferential stresses is shown in Fig. 12 and similar patterns can also observe in this figure. Fig. 13 illustrates radial electrical displacement distributions with various m for actuator layer. It can easily be seen from this figure which radial electrical displacements decrease as the graded index m decreases at the same radial point. Fig. 14 indicates the distribution of electric potential in the actuator along the thickness. Fig. 15 shows the distribution of electric potential in the sensor layer that increases as the graded index m decreases at the same radial point. According to the given mechanical boundary conditions, radial electrical displacement at the outside and electric potential at the inside surfaces of the actuator layer are zero and the electric potential in sensor layer satisfies the prescribed electrical boundary conditions.

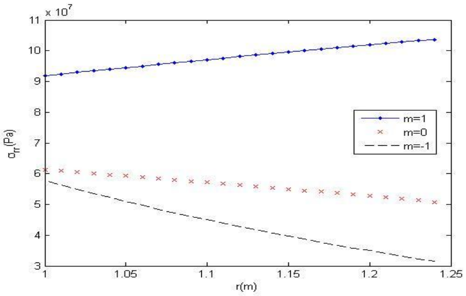

Fig.11
Radial stress in the FGM layer along the thickness.

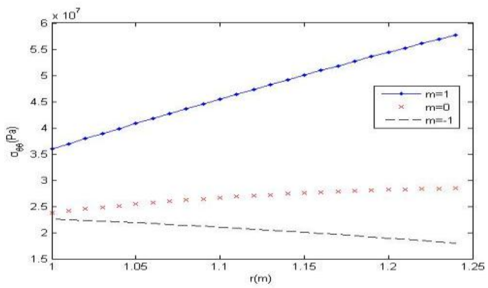

Fig.12
Circumferential stress in the FGM layer along the thickness.

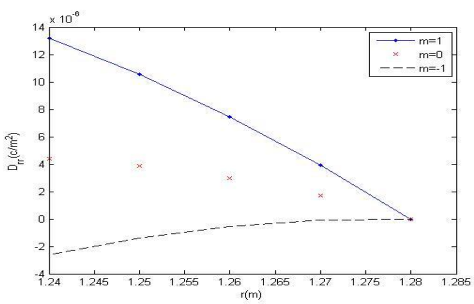

Fig.13
Radial electrical displacement in the actuator layer along the thickness.

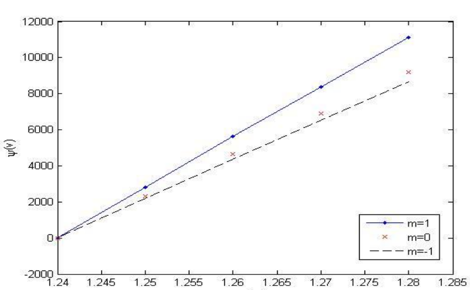

Fig.14
Electric potential distribution in the actuator along the thickness.

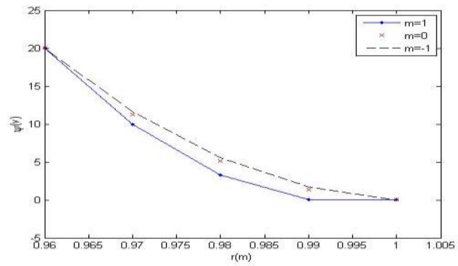

Fig.15
Electric potential distribution in the sensor layer along the thickness.

To verify the proposed method, assuming that the electric potential to the piezoelectric layers are zero consider the functionally graded hollow cylinder with inner radius $b = 0.02m$ and outer radius $c = 0.024m$. The comparison of the present method and the method in Ref. [12] for $m_1 = -11.9839, m_2 = 1.3103, m_3 = -1.4937$ and the initial temperature of the cylinder $T(r,0) = 50\Gamma(100r)^{\circ}C$, where Γ is the mathematical Gamma function such that the cylinder is heated by $R(r,t) = 6\times10^6 \times \frac{1}{r}\sin(5t)\frac{W}{m^3}$, are illustrated in Fig. 16. The results are in good agreement with obtained results from Ref. [12].

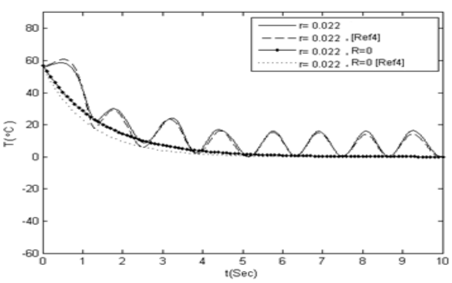

Fig.16
The comparison of present results and method in Ref. [12].

5 CONCLUSIONS

This paper presents an analytical study of piezo thermoelastic behavior of an FGM hollow cylinder with piezoelectric layers subjected to radially symmetric loadings with heat source. The method of solution is based on the direct method, rather than other methods. This method does not have the limitations of the potential function or numerical methods and more various thermal boundary conditions may be handled using the proposed method. The potential function method requires a wide experience of particular solutions and even then is not guaranteed to be successful and requires examination of the assumed solutions with a view toward finding one that will satisfy the governing equations and boundary conditions. In addition, analytical solution is the solution for multitude of particular cases, while the numerical solution has to be obtained anew for each such case separately. Moreover, numerical method always works with iteration, while analytical methods the final answer is straight forward.

By using this method and considering the special boundary conditions and material properties for piezoelectric-FGM-piezoelectric hollow cylinder, the mechanical and electrical displacements and stresses can be controlled and optimized to design and use this kind of structures.

APPENDIX A

$$\begin{aligned} &x_1 = \frac{2k_1}{k_2(c)}c - \frac{k_1(X_{11}d^2 + 2X_{12}d)}{k_2(c)(X_{11}d + X_{12})}, \qquad y_1 = 2b - 2c, \qquad z_1 = \frac{k_3(X_{21}a^2 + 2X_{22}a)}{k_2(b)(X_{21}a + X_{22})} - \frac{2k_3}{k_2(b)}b \\ &x_2 = c^2 - \frac{c(X_{11}d^2 + 2X_{12}d)}{X_{11}d + X_{12}} - \frac{2k_1}{k_2(c)}c^2 + \frac{k_1c\,(X_{11}d^2 + 2X_{12}d)}{k_2(c)\left(X_{11}d + X_{12}\right)}, \qquad y_2 = c^2, \qquad z_2 = 0 \end{aligned} \tag{A.1}$$

$$x_3=\frac{2k_1}{k_2(c)}bc-\frac{k_1b(X_{11}d^2+2X_{12}d)}{k_2(c)(X_{11}d+X_{12})},\qquad y_3=b^2-2bc\qquad z_3=\frac{b(X_{21}a^2+2X_{22}a)}{X_{21}a+X_{22}}-b^2$$

$$h_1=\frac{k_3}{k_2(b)}\frac{g_4}{X_{21}a+X_{22}}-\frac{k_1g_1}{k_2(c)(X_{11}d+X_{12})}-Y_{2,r}(b,\mathrm{t})-\frac{\mathrm{k}_1}{\mathrm{k}_2(c)}Y_{1,r}(c,\mathrm{t}) \tag{A.1}$$

$$h_2=Y_2(c,\mathrm{t})-\frac{\mathrm{g}_1c}{X_{11}d+X_{12}}+\frac{k_1c}{k_2(c)}Y_{1,r}(c,\mathrm{t})+\frac{\mathrm{k}_1c}{k_2(c)}\frac{g_1}{X_{11}d+X_{12}}$$

$$h_3=Y_3(b,\mathrm{t})+\frac{\mathrm{g}_4b}{X_{21}a+X_{22}}-\frac{k_1b}{k_2(c)}Y_{1,r}(c,\mathrm{t})-\frac{\mathrm{k}_1bg_1}{k_2(c)(X_{11}d+X_{12})}$$

$$A_1=\frac{\begin{vmatrix}h_1&y_1&z_1\\h_2&y_2&z_2\\h_3&y_3&z_3\end{vmatrix}}{\begin{vmatrix}x_1&y_1&z_1\\x_2&y_2&z_2\\x_3&y_3&z_3\end{vmatrix}}\qquad A_2=\frac{\begin{vmatrix}x_1&y_1&h_1\\x_2&y_2&h_2\\x_3&y_3&h_3\end{vmatrix}}{\begin{vmatrix}x_1&y_1&z_1\\x_2&y_2&z_2\\x_3&y_3&z_3\end{vmatrix}}\qquad A_3=\frac{\begin{vmatrix}x_1&y_1&h_1\\x_2&y_2&h_2\\x_3&y_3&h_3\end{vmatrix}}{\begin{vmatrix}x_1&y_1&z_1\\x_2&y_2&z_2\\x_3&y_3&z_3\end{vmatrix}} \tag{A.2}$$

$$B_1=\frac{g_1-A_1(X_{11}d^2+2X_{12}d)}{X_{11}d+X_{12}},\quad B_2=\frac{k_1}{k_2(c)}\left[Y_{1,r}(c,t)+2A_1c+\frac{g_1-A_1(X_{11}d^2+2X_{12}d)}{X_{11}d+X_{12}}\right]-2A_2c$$

$$B_3=\frac{g_4-A_3(X_{21}a^2+2X_{22}a)}{X_{21}a+X_{22}} \tag{A.3}$$

$$a_k=\left[\eta_k^2-\frac{C_{22}}{C_{11}}\right],\qquad b_k=\left[\frac{e_{11}}{C_{11}}\eta_k^2-\frac{e_{21}}{C_{11}}\eta_k\right]$$

$$a'_k=\left[\eta_k^2+\frac{e_{21}}{e_{11}}\eta_k\right],\qquad b'_k=\left[-\frac{\eta_{11}}{e_{11}}\eta_k^2\right] \tag{A.4}$$

$$\begin{cases}D_3=D_4=D_7=D_8=0\\D_{11}=D_{12}=D_{15}=D_{16}=0\end{cases} \tag{A.5}$$

$$x_1=\left[2k\,(2k+1)+6k+\left(1-\frac{C_{22}}{C_{11}}\right)\right],\qquad x_2=\left[\frac{e_{11}}{C_{11}}2k\,(2k+1)+2k\left(\frac{3e_{11}-e_{21}}{C_{11}}\right)+\left(\frac{e_{11}-e_{21}}{C_{11}}\right)\right]$$

$$x_3=\left[2k\,\frac{\alpha_r}{C_{11}}+\frac{(\alpha_r-\alpha_\theta)}{C_{11}}\right],\qquad x_4=\left[4-\frac{C_{22}}{C_{11}}\right],\qquad x_5=\left[\frac{4e_{11}-2e_{21}}{C_{11}}\right],\qquad x_6=\left[9-\frac{C_{22}}{C_{11}}\right]$$

$$x_7=\left[\frac{9e_{11}-3e_{21}}{C_{11}}\right],\qquad x_8=\left[2k\,(2k-1)+2k\left(3+\frac{e_{21}}{e_{11}}\right)+\left(\frac{e_{11}+e_{21}}{e_{11}}\right)\right] \tag{A.6}$$

$$x_9=\left[-2k\,(2k-1)\left(\frac{\eta_{11}}{e_{11}}\right)-6k\left(\frac{\eta_{11}}{e_{11}}\right)-\left(\frac{\eta_{11}}{e_{11}}\right)\right],\qquad x_{10}=\left[-2k\,\frac{P_r}{e_{11}}-\frac{P_r}{e_{11}}\right]$$

$$x_{11}=\left[\frac{4e_{11}+2e_{21}}{e_{11}}\right],\qquad x_{12}=\left[-\frac{4\eta_{11}}{e_{11}}\right],\qquad x_{13}=\left[\frac{9e_{11}+3e_{21}}{e_{11}}\right],\qquad x_{14}=\left[-\frac{9\eta_{11}}{e_{11}}\right]$$

$$\begin{Bmatrix} D_1 \\ D_9 \end{Bmatrix} = \frac{\begin{vmatrix} x_3 & x_2 \\ x_{10} & x_9 \end{vmatrix}}{\begin{vmatrix} x_1 & x_2 \\ x_8 & x_9 \end{vmatrix}}, \qquad \begin{Bmatrix} D_2 \\ D_{10} \end{Bmatrix} = \frac{\begin{vmatrix} \begin{Bmatrix} c_1 \\ c_3 \end{Bmatrix} x_3 & x_2 \\ \begin{Bmatrix} c_1 \\ c_3 \end{Bmatrix} x_{10} & x_9 \end{vmatrix}}{\begin{vmatrix} x_1 & x_2 \\ x_8 & x_9 \end{vmatrix}}, \qquad \begin{Bmatrix} D_5 \\ D_{13} \end{Bmatrix} = \frac{\begin{vmatrix} x_1 & x_3 \\ x_8 & x_{10} \end{vmatrix}}{\begin{vmatrix} x_1 & x_2 \\ x_8 & x_9 \end{vmatrix}}, \qquad \begin{Bmatrix} D_6 \\ D_{14} \end{Bmatrix} = \frac{\begin{vmatrix} x_1 & \begin{Bmatrix} c_1 \\ c_3 \end{Bmatrix} x_3 \\ x_8 & \begin{Bmatrix} c_1 \\ c_3 \end{Bmatrix} x_{10} \end{vmatrix}}{\begin{vmatrix} x_1 & x_2 \\ x_8 & x_9 \end{vmatrix}}$$

$$\begin{aligned}
y_1 &= f\,(2k+P)\big(f\,(2k+P)-1\big)+\left(2m_2+m_1-2\beta+3\right)f\,(2k+P)+\left(m_2-\beta+1\right)\left(m_2-\beta\right) \\
&\quad +\left(m_1+1\right)\left(m_2-\beta+1\right)+\left(\frac{\upsilon m_1}{1-\upsilon}-1-\frac{(1-2\upsilon)n^2}{2-2\upsilon}\right) \\
y_2 &= \frac{(1+\upsilon)\alpha_\circ}{(1-\upsilon)}\left(m_1+m_2-\beta+f\,(2k+P)\right) \\
y_3 &= f\,(2k-P)\big(f\,(2k-P)-1\big)+\left(2m_2+m_1-2\beta+3\right)f\,(2k-P)+\left(m_2-\beta+1\right)\left(m_2-\beta\right) \\
&\quad +\left(m_1+1\right)\left(m_2-\beta+1\right)+\left(\frac{\upsilon m_1}{1-\upsilon}-1-\frac{(1-2\upsilon)n^2}{2-2\upsilon}\right) \\
y_4 &= \frac{(1+\upsilon)\alpha_\circ}{(1-\upsilon)}c_{2n}\left(m_1+m_2-\beta+f\,(2k+P)\right) \\
y_5 &= \left(m_2+2\right)\left(m_2+1\right)+\left(m_1+1\right)\left(m_2+2\right)+\left(\frac{\upsilon m_1}{1-\upsilon}-1\right) \\
y_6 &= \frac{(1+\upsilon)\alpha_\circ}{(1-\upsilon)}\left(m_1+m_2+1\right)B_2 \\
y_7 &= \left(m_2+3\right)\left(m_2+2\right)+\left(m_2+3\right)\left(m_1+1\right)+\left(\frac{\upsilon m_1}{1-\upsilon}-1\right) \\
y_8 &= \frac{(1+\upsilon)\alpha_\circ}{(1-\upsilon)}\left(m_1+m_2+2\right)A_2
\end{aligned} \tag{A.7}$$

$$D_{17}=\frac{y_2}{y_1} \qquad D_{18}=\frac{y_4}{y_3} \qquad D_{19}=\frac{y_6}{y_5} \qquad D_{20}=\frac{y_8}{y_7} \tag{A.8}$$

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