Generalized Thermoelastic Problem of a Thick Circular Plate Subjected to Axisymmetric Heat Supply

J.J. Tripathi [1, *], G.D. Kedar [2], K.C. Deshmukh [2]

[1]*Department of Mathematics, Dr. Ambedkar College, Deekshabhoomi, Nagpur -440010, Maharashtra, India*
[2]*Department of Mathematics, R.T.M. Nagpur University, Nagpur-440033, Maharashtra ,India*

ABSTRACT

The present work is aimed at analyzing the thermoelastic disturbances in a circular plate of finite thickness and infinite extent subjected to constant initial temperature and axisymmetric heat supply. Integral transform technique is used. Analytic solutions for temperature, displacement and stresses are derived within the context of unified system of equations in generalized thermoelasticity in the Laplace transform domain using potential functions. Inversion of Laplace transforms is done by employing a numerical scheme. Temperature, displacement and stresses developed in the thick circular plate are obtained and illustrated graphically for copper (pure) material.   © 2016 IAU, Arak Branch.All rights reserved.
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1 INTRODUCTION

THE theory of thermoelasticity has seen a swift development in the past decade, affected by many different engineering sciences. The theory of thermoelasticity can be broadly classified as uncoupled and coupled thermoelasticity. The heat conduction equation in classical uncoupled theory is independent of elastic terms which contradict the fact that elastic changes and heat changes affect each other. The heat conduction equation is parabolic thus predicting infinite speeds of propagation for heat waves. Thus, the Classical uncoupled theory is found to be incompatible with physical observations. The theory of coupled thermoelasticity was introduced by Biot [1] by introducing the elastic terms in the heat conduction equation, eliminating the first paradox of the classical theory. This theory still predicted infinite speed of heat propagation which is contrary to the actual phenomena.

Modifications to the coupled theory were later introduced, thus paving way for the generalized thermoelasticity theories. The first modification was due to Lord and Shulman [2]. They introduced one relaxation parameter in the Fourier's heat equation to obtain a hyperbolic heat conduction equation. The heat equation of Lord-Shulman theory is of wave type, hyperbolic in nature, thus predicting finite speeds of heat and elastic wave propagation. The second generalization was due to Green and Lindsay [3] who obtained the equations of generalized thermoelasticity with two relaxation parameters. Later on, Şuhubi [4] also obtained these equations independently. Dhaliwal and Sherief [5] extended the Lord-Shulman theory of generalized thermoelasticity to homogeneous anisotropic materials. Chandrasekariah [6] studied the thermal disturbances with second sound. A review article comparing various theories of generalized thermoelasticity was given by Hetnarski and Ignaczak [7].

Sherief and Anwar [8] have discussed a problem of heat conduction in Lord-Shulman theory for a thermoelastic cylindrical medium composed of two different materials. Maghraby and Abdel Halim [9] studied a problem of generalized thermoelasticity in Lord-Shulman theory for a half space subjected to a known axisymmetric

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*Corresponding author. Tel.: +91 9823888479.
E-mail address: tripathi.jitesh@gmail.com (J.J. Tripathi).

temperature distribution. Othman and Abbas [10] discussed a thermal shock problem of generalized thermoelasticity for a non homogeneous isotropic hollow cylinder using finite element method. Youssef [11] studied two dimensional generalized thermoelastic problem for a half space subjected to ramp type heating. Awad [12] wrote a note on the spatial decay estimates in non-classical linear thermoelastic semi-cylindrical bounded domains. Mukhopadhyay and Kumar [13] discussed thermoelastic interactions on two-temperature generalized thermoelasticity in an infinite medium with a cylindrical cavity. Tripathi et.al [14] solved a problem of semi-infinite cylinder with heat sources in the context of Lord-Shulman theory and Classical coupled thermoelasticity. Mallik and Kanoria [15] studied the two dimensional problem in generalized thermoelasticity of thermoelastic interaction for a transversely isotropic thick plate having heat source. These problems are solved using eigenvalue approach. Bagri and Eslami [16] have got the unified generalized thermoelastic solution for cylinders and spheres. A two dimensional problem for a half space and for a thick circular plate with heat sources have been solved by El-Maghraby [17, 18]. Youssef [19] studied the problem of a generalized thermoelastic medium subjected to moving heat source and ramp type heating using state –space approach. Recently, Tripathi et al. [20] studied a problem of generalized thermoelastic diffusion in thick circular plate.

The present work is aimed at analyzing the effects of phase lags on wave propagation under axisymmetric distributions and to investigate the nature of distributions of different fields in a thick circular plate under axisymmetric temperature distribution in the context of unified system of theories of generalized thermoelasticity (i.e. L-S and G-L). The classical coupled (CT) theory is recovered as a special case. The exact expressions for temperature distribution, displacement components and the stress are obtained in the Laplace transform domain. Numerical inversion of Laplace transforms are performed using Gaver-Stehfast algorithm [21, 22, 23] which is considerably more stable and computationally efficient than inversion using the discrete Fourier transform [24] and all integrals were evaluated using Romberg's integration technique [25] with variable step size. The results presented here will be useful in many engineering problems like thick-walled pressure vessels, such as a nuclear containment vessel, a cylindrical roller, such as a roller bearing, etc.

2   FORMULATION OF THE PROBLEM

The present paper deals with the thick plate of thickness $2b$ occupying the space D defined by $0 \leq r < \infty, -b \leq z \leq b$. Let the plate be subjected to a transient axisymmetric temperature field dependent on the radial and axial directions of the cylindrical co-ordinate system. The thick circular plate is initially held a constant temperature T_0 and a heat flux $QF(r)$ is prescribed on the upper and lower boundary surfaces. Under these conditions, the thermoelastic quantities in a thick circular plate of infinite extent are required to be determined. We also assume that the heat source and body forces are absent.

All the field equations for isotropic media in the absence of heat source and body forces are formulated in a unified system [11] as,

(i)   Equation of motion

$$\mu \ddot{u}_{i,jj} + (\lambda + \mu)\, u_{j,ji} - \gamma \left(1 + \tau_1 \frac{\partial}{\partial t}\right) T,_i = \rho \ddot{u}_i \tag{1}$$

(ii) Equation of heat conduction

$$\mathbf{K} T,_{ii} = \rho \mathbf{C}_{\mathrm{E}} \left(\frac{\partial}{\partial t} + \tau_0 \frac{\partial^2}{\partial t^2}\right) \dot{T} + \left(1 + \upsilon \tau_0 \frac{\partial}{\partial t}\right) T_0\, \gamma\, \dot{u}_{j,j} \tag{2}$$

(iii) The constitutive relations

$$\sigma_{ij} = \mu \left(u_{i,j} + u_{j,i}\right) + \left(\lambda u_{i,i} - \gamma \left(T + \tau_1 \dot{T}\right)\right) \delta_{ij} \tag{3}$$

Eqs. (8)-(10) reduce to Eqs. (1)-(3) (CCTE) when $\upsilon = 1, \tau_0 = \tau_1 = 0$. Putting $\upsilon = 1, \tau_1 = 0$ and $\tau_0 > 0$, the equations reduce to (1), (4) and (3) for the ETE model, while when $\upsilon = 0$, $\tau_1 > \tau_0 > 0$, the equations reduce to (5)-

(7) for the TRDTE model.

The dilatation e is given by

$$e_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i} \right) \text{ and } e = e_{ii} \tag{4}$$

where λ and μ are Lame's constants, τ_0 and τ_1 are relaxation times, γ is a material constant given by $\gamma = (3\lambda + 2\mu)\alpha_t$, α_t is the coefficient of linear thermal expansion, T_0 is the reference temperature chosen such that $\left| (T - T_0)/T_0 \right| << 1$.

Eqs. (1)-(4) give the complete set of unified system of field equations in the context of generalized thermoelasticity. We take the axis of symmetry as the z axis and the origin of the system of co-ordinates at the middle plane between the upper and lower faces of the plate. The problem is studied using the cylindrical polar co-ordinates (r, φ, z). Due to the rotational symmetry about the z axis of the problem, all quantities are independent of the co-ordinate φ. The displacement vector, thus, has the form $\vec{u} = (u, 0, w)$.

The equations of motion can be written as:

$$\mu \nabla^2 u - \frac{\mu}{r^2} u + (\lambda + \mu)\frac{\partial e}{\partial r} - \gamma\left(1 + \tau_1 \frac{\partial}{\partial t} \right)\frac{\partial T}{\partial r} = \rho\frac{\partial^2 u}{\partial t^2} \tag{5}$$

$$\mu \nabla^2 w + (\lambda + \mu)\frac{\partial e}{\partial z} - \gamma\left(1 + \tau_1 \frac{\partial}{\partial t} \right)\frac{\partial T}{\partial z} = \rho\frac{\partial^2 w}{\partial t^2} \tag{6}$$

The generalized equation of heat conduction has the form

$$k \nabla^2 T = \rho C_E \left(\frac{\partial}{\partial t} + \tau_0 \frac{\partial^2}{\partial t^2} \right)T + T_0\gamma\left(\frac{\partial}{\partial t} + \upsilon\tau_0 \frac{\partial^2}{\partial t^2} \right)e \tag{7}$$

where T is the absolute temperature and e is the dilatation given by the relation

$$e = \frac{u}{r} + \frac{\partial u}{\partial r} + \frac{\partial w}{\partial z} = \frac{1}{r}\frac{\partial}{\partial r} ru + \frac{\partial w}{\partial z} \tag{8}$$

where the Laplacian operator is given by

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$$

The following constitutive relations supplement the above equations

$$\sigma_{\phi\phi} = 2\mu\frac{u}{r} + \lambda e - \gamma\left(T - T_0 + \tau_1 \frac{\partial T}{\partial t} \right) \tag{9}$$

$$\sigma_{rr} = 2\mu\frac{\partial u}{\partial r} + \lambda e - \gamma\left(T - T_0 + \tau_1 \frac{\partial T}{\partial t} \right) \tag{10}$$

$$\sigma_{zz} = 2\mu\frac{\partial w}{\partial z} + \lambda e - \gamma\left(T - T_0 + \tau_1 \frac{\partial T}{\partial t} \right) \tag{11}$$

$$\sigma_{rz} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right) \tag{12}$$

$$\sigma_{r\phi} = \sigma_{z\phi} = 0 \tag{13}$$

We shall use the following non-dimensional variables

$$\mathrm{r}' = \upsilon\mathrm{r}\ ,\ \mathrm{z}' = \upsilon\mathrm{z}\ ,\ \mathrm{u}' = \upsilon\mathrm{u}\ ,\ \mathrm{w}' = \upsilon\mathrm{w}\ ,\ \mathrm{t}' = \mathrm{c}_1\upsilon\mathrm{t}\ ,\ \tau_0' = \mathrm{c}_1\upsilon\tau_0\ ,\ \tau_1' = \mathrm{c}_1\upsilon\tau_1,\ \sigma_{ij}' = \frac{\sigma_{ij}}{\mu}\ ,\ \theta = \frac{\gamma(T-T_0)}{(\lambda+2\mu)}$$

where $\upsilon = \dfrac{\rho c_1 C_E}{k}$ is the dimensionless characteristic length, $c_1 = \sqrt{\dfrac{\lambda+2\mu}{\rho}}$, c_1 is the speed of propagation of longitudinal wave.

Using the above non-dimensional variables, the governing Eqs. (5)-(13) take the form,

$$\nabla^2 u - \frac{1}{r^2}u + (\xi^2-1)\frac{\partial e}{\partial r} - \xi^2\left(1+\tau_1\frac{\partial}{\partial t}\right)\frac{\partial\theta}{\partial r} = \xi^2\frac{\partial^2 u}{\partial t^2} \tag{14}$$

$$\nabla^2 w + (\xi^2-1)\frac{\partial e}{\partial z} - \xi^2\left(1+\tau_1\frac{\partial}{\partial t}\right)\frac{\partial\theta}{\partial z} = \xi^2\frac{\partial^2 w}{\partial t^2} \tag{15}$$

$$\nabla^2\theta = \left(\frac{\partial}{\partial t} + \tau_0\frac{\partial^2}{\partial t^2}\right)\theta + \varepsilon\left(\frac{\partial}{\partial t} + \nu\tau_0\frac{\partial^2}{\partial t^2}\right)e \tag{16}$$

$$\sigma_{rr} = 2\frac{\partial u}{\partial r} + \left(\xi^2-2\right)e - \xi^2\left(1+\tau_1\frac{\partial}{\partial t}\right)\theta \tag{17}$$

$$\sigma_{\varphi\varphi} = 2\frac{u}{r} + \left(\xi^2-2\right)e - \xi^2\left(1+\tau_1\frac{\partial}{\partial t}\right)\theta \tag{18}$$

$$\sigma_{zz} = 2\frac{\partial w}{\partial z} + \left(\xi^2-2\right)e - \xi^2\left(1+\tau_1\frac{\partial}{\partial t}\right)\theta \tag{19}$$

$$\sigma_{rz} = 2\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right) \tag{20}$$

where $\xi^2 = \dfrac{(\lambda+2\mu)}{\mu}, \varepsilon = \dfrac{T_0\beta^2}{\rho C_E(\lambda+2\mu)}$.

Using Helmholtz decomposition theorem, we seek the displacement components u and w in the form,

$$u = \frac{\partial\phi}{\partial r} + \frac{\partial^2\psi}{\partial r\partial z} \tag{21}$$

$$w = \frac{\partial\phi}{\partial z} - \left(\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r}\right) \tag{22}$$

where the potential functions ϕ and ψ are the Lame's potentials representing irrotational and rotational parts of the

displacement vector $\vec{u}$ respectively.

From Eqs. (4), (21) and (22), we obtain

$$e = \nabla^2 \phi \tag{23}$$

Using Eq. (21)-(23), the Eqs. (14)-(16) become,

$$\left(\nabla^2 - \frac{\partial^2}{\partial t^2} \right) \phi - \left(1 + \tau_1 \frac{\partial}{\partial t} \right) \theta = 0 \tag{24}$$

$$\left(\nabla^2 - \xi^2 \frac{\partial^2}{\partial t^2} \right) \psi = 0 \tag{25}$$

$$\left(\nabla^2 - \frac{\partial}{\partial t} - \tau_0 \frac{\partial^2}{\partial t^2} \right) \theta - \varepsilon \left(\frac{\partial}{\partial t} + v\tau_0 \frac{\partial^2}{\partial t^2} \right) \nabla^2 \phi = 0 \tag{26}$$

Eq. (25) for the function ψ represents a wave equation with wave velocity $v = 1/\xi$. This represents a transverse wave and it has no effect on temperature. Eq.(26) represents a longitudinal thermal wave moving with velocity $v_L = 1/\sqrt{\tau_0}$.

We shall assume that the initial state of the medium is quiescent. The boundary conditions of the problem are taken as:

$$\frac{\partial \theta}{\partial z} = \pm Q \, F(r,z) \,, \quad z = \pm b \tag{27}$$

and the traction free surface stress functions,

$$\sigma_{zz} = \sigma_{rz} = 0 \,, \quad z = \pm b \tag{28}$$

3   ANALYTICAL SOLUTION

Applying the Laplace transform defined by the relation,

$$\bar{f}(r,z,p) = L\left[f(r,z,t) \right] = \int_0^\infty e^{-st} f(r,z,t) dt$$

and the Hankel transform of order zero with respect to r of a function $\bar{f}(r,z,s)$ defined by the relation,

$$\bar{f}^*(\eta,z,p) = H\left[\bar{f}(r,z,p) \right] = \int_0^\infty \bar{f}(r,z,p) r J_0(\eta r) dr$$

where J_0 is the Bessel function of the first kind of order zero.

The inverse Hankel transform is given by the relation

$$\bar{f}(r,z,p) = H^{-1}\left[\bar{f}^*(\eta,z,p) \right] = \int_0^\infty \bar{f}^*(\eta,z,p) \, \eta J_0(\eta r) d\eta$$

Taking the Laplace and Hankel transform of both sides of Eqs. (24)-(26), we get,

$$\left(D^2 - \eta^2 - p^2\right)\bar{\phi}^* - \left(1 + \tau_1 p\right)\bar{\theta}^* = 0 \tag{29}$$

$$\left(D^2 - \eta^2 - \xi^2 p^2\right)\bar{\psi}^* = 0 \tag{30}$$

$$\left(D^2 - \eta^2 - p - \tau_0 p^2\right)\bar{\theta}^* - \varepsilon p\left(1 + v\tau_0 p\right)\left(D^2 - \eta^2\right)\bar{\phi}^* = 0 \tag{31}$$

where $D = \partial/\partial z$. On eliminating $\bar{\theta}^*$ between Eqs. (29) and (31), we get,

$$\left(D^2 - \alpha_1^2\right)\left(D^2 - \alpha_2^2\right)\bar{\phi}^* = 0 \tag{32}$$

where $\pm\alpha_1$ and $\pm\alpha_2$ are the roots of the characteristic equation given by,

$$\alpha^4 - \left(f(p) + 2\eta^2\right)\alpha^2 + \eta^2(f(p) + \eta^2) + p^3(1 + \tau_0 p) = 0 \tag{33}$$

where $f(p) = p^2 + p(1 + \tau_0 p) + \varepsilon p(1 + v\tau_0 p)(1 + \tau_1 p)$. The solution of Eq. (32) can be written in the form,

$$\bar{\phi}^* = \sum_{i=1}^{2}\bar{\phi}_i^* \ , \ i = 1,2 \tag{34}$$

where $\bar{\phi}_i^*$ is a solution of the Eq. (35).

$$\left(D^2 - \alpha_i^2\right)\bar{\phi}_i^* = 0 \ , \ i = 1,2. \tag{35}$$

The solution of the above equation can be written in the form

$$\bar{\phi}_i^* = C_i(\eta, p)\cosh(\alpha_i z) \ , \ i = 1,2 \tag{36}$$

Using Eq. (36) in Eq. (34), we get,

$$\bar{\phi}^* = \sum_{i=1}^{2}C_i(\eta, p)\cosh(\alpha_i z) \tag{37}$$

Solving Eqs. (29) and (30), we get the solutions $\bar{\psi}^*$ and $\bar{\theta}^*$ in the form,

$$\bar{\psi}^* = D(\eta, p)\sinh(az) \tag{38}$$

where $a^2 = \eta^2 + \xi^2 p^2$.

$$\bar{\theta}^* = \sum_{i=1}^{2}\frac{\alpha_i^2 - \eta^2 - p^2}{1 + \tau_1 p}C_i(\eta, p)\cosh(\alpha_i z) \tag{39}$$

On using Eqs. (21), (22) in Eqs. (12) and (13), the stress components $\bar{\sigma}_{zz}^*$, $\bar{\sigma}_{rz}^*$ become,

$$\bar{\sigma}_{zz}^* = \left(\xi^2 p^2 + 2\eta^2\right)\bar{\phi}^* + 2\eta^2\frac{\partial\bar{\psi}^*}{\partial z} \tag{40}$$

$$\bar{\sigma}^*_{rz} = H\left\{\frac{\partial}{\partial r}\left[2D\bar{\phi} + \left(2D^2 - \xi^2 p^2\right)\bar{\psi}\right]\right\} \tag{41}$$

where H represents the Hankel transform.

Applying the Laplace and Hankel transforms to the boundary conditions (27) and (28), we get,

$$\frac{\partial\bar{\theta}^*}{\partial z} = \pm Q\,\bar{F}^*(\eta,z) \qquad , z = \pm b \tag{42}$$

$$\bar{\sigma}^*_{zz} = \bar{\sigma}^*_{rz} = 0 \qquad , z = \pm b \tag{43}$$

Here we have considered the function $F(r,z)$ which falls off exponentially as one moves away from the centre of the plate in the radial direction and increases symmetrically along the axial direction given by,

$$F(r,z) = z^2 e^{-\omega r} \qquad , \omega > 0$$

Thus on applying Laplace and Hankel transforms, we get,

$$\bar{F}^*(\eta,z) = \frac{z^2\omega}{p\left(\omega^2 + \eta^2\right)^{3/2}} \tag{44}$$

Making use of the values of $\bar{\theta}, \bar{\sigma}_{zz}$ and $\bar{\sigma}_{rz}$ in the boundary conditions (42), (43) and with the aid of Eq. (44), we get,

$$\sum_{i=1}^{2}\frac{C_i\left(\alpha_i^2 - \eta^2 - p^2\right)\alpha_i\sinh(\alpha_i b)}{(1+\tau_1 p)} = Q\frac{b^2\omega}{p\left(\omega^2 + \eta^2\right)^{3/2}} \tag{45}$$

$$\left(\xi^2 p^2 + 2\eta^2\right)\sum_{i=1}^{2}C_i\cosh(\alpha_i b) + 2\eta^2 a\,D\cosh(ab) = 0 \tag{46}$$

$$2\sum_{i=1}^{2}\alpha_i C_i\sinh(\alpha_i b) + \left(2a^2 - \eta^2 p^2\right)D\cosh(ab) = 0 \tag{47}$$

Eqs. (45)-(47) is a system of linear equations with C_1, C_2 and D as unknown parameters. The solution of this system of linear equations is given by

$$C_1 = -\frac{(1+\tau_1 p)\left(4\alpha_2 a\eta^2\tanh(b\alpha_2) + x\,(\eta^2 p^2 - 2)\right)}{p\cosh(b\alpha_1)Y}\theta_0^*$$

$$C_2 = \frac{(1+\tau_1 p)\left(4\alpha_1 a\eta^2\tanh(b\alpha_1) + x\,(\eta^2 p^2 - 2)\right)}{p\cosh(b\alpha_2)Y}\theta_0^*$$

$$D = -\frac{2x\,(1+\tau_1 p)\left(\alpha_1\tanh(b\alpha_1) - \alpha_2\tanh(b\alpha_2)\right)}{p\cosh(ab)Y}\theta_0^*$$

where $\theta_0^* = Q\dfrac{b^2\omega}{\left(\omega^2 + \eta^2\right)^{3/2}}, x = \left(\xi^2 p^2 + 2\eta^2\right), k = (2a^2 + \eta^2 p^2)$ and $m_i = \left(\alpha_i^2 - \eta^2 - p^2\right), i = 1,2$.

$$Y = \left\{xk\left(m_1\alpha_1\tanh(b\alpha_1) - m_2\alpha_2\tanh(b\alpha_2)\right) - 4a\alpha_1\alpha_2\eta^2\tanh(b\alpha_1)\tanh(b\alpha_2)(m_2 - m_1)\right\}$$

3.1 Inversion of Hankel transform

Taking inverse Hankel transform of Eq. (37), we obtain,

$$\bar{\theta}(r,z,p) = \int_0^\infty \left\{ \sum_{i=1}^2 \frac{\alpha_i{}^2 - \eta^2 - p^2}{1 + \tau_1 p} C_i(\eta,p)\cosh(\alpha_i z) \right\} \eta J_0(\eta r) d\eta \tag{48}$$

Using Eqs. (37)-(39) in (21)-(22) and taking inverse Hankel transform, yields the solution for displacement components in Laplace transform domain,

$$\bar{u}(r,z,p) = \int_0^\infty -\eta^2 \left[\sum_{i=1}^2 C_i(\eta,p)\cosh(\alpha_i z) + D(\eta,p)\cosh(a z) \right] J_1(\eta r) d\eta \tag{49}$$

$$\bar{w}(r,z,p) = \int_0^\infty \left[\sum_{i=1}^2 \alpha_i C_i(\eta,p)\sinh(\alpha_i z) + D(\eta,p)\eta^2 \sinh(a z) \right] \eta J_0(\eta r) d\eta \tag{50}$$

Applying Laplace transform to both sides of Eqs. (17)-(19) and using the solutions given in Eqs. (48)-(50), we obtain the stress components in the Laplace transform domain,

$$\bar{\sigma}_{rr}(r,z,p) = \int_0^\infty \left\{ \begin{array}{l} \eta J_0(\eta r) \sum_{i=1}^2 (\xi^2 p^2 + 2\eta^2 - 2\alpha_i{}^2) C_i(\eta,p)\cosh_i(\alpha_i z) \\ +2\eta^3 \left(\frac{1}{\eta r} J_1(\eta r) - J_0(\eta r) \right) \left[\sum_{i=1}^2 C_i(\eta,p)\cosh\alpha_i z + D(\eta,p)a\cosh(a z) \right] \end{array} \right\} d\eta \tag{51}$$

$$\bar{\sigma}_{\varphi\varphi}(r,z,p) = \int_0^\infty \left\{ \begin{array}{l} \eta J_0(\eta r) \left[\sum_{i=1}^2 (\xi^2 p^2 + 2\eta^2 - 2\alpha_i{}^2) C_i(\eta,p)\cosh_i(\alpha_i z) \right] \\ -\frac{2}{r} \left[\sum_{i=1}^2 C_i(\eta,p)\cosh(\alpha_i z) + D(\eta,p)a\cosh(a z) \right] \eta^2 J_1(\eta r) \end{array} \right\} d\eta \tag{52}$$

$$\bar{\sigma}_{zz}(r,z,p) = \int_0^\infty (\xi^2 p^2 + 2\eta^2) \left[\sum_{i=1}^2 C_i(\eta,p)\cosh_i(\alpha_i z) + 2\eta^2 D(\eta,p)\cosh(a z) \right] \eta J_0(\eta r) d\eta \tag{53}$$

Eqs. (48)-(53) present the complete set of variables in the Laplace transform domain.

4   INVERSION OF DOUBLE TRANSFORMS

Due to the complexity of the solution in the Laplace transform domain, the inverse of the Laplace transform is obtained using the Gaver-Stehfast algorithm. Gaver [21] and Stehfast [22, 23] derived the formula given below. By this method the inverse $f(t)$ of the Laplace transform $\bar{f}(s)$ is approximated by,

$$f(t) = \frac{\ln 2}{t} \sum_{j=1}^K D(j,K) F\left(j \frac{\ln 2}{t} \right) \tag{54}$$

with

$$D(j,K) = (-1)^{j+M} \sum_{n=m}^{\min(j,M)} \frac{n^M (2n)!}{(M-n)! n! (n-1)! (j-n)! (2n-j)!} \tag{55}$$

where K is an even integer, whose value depends on the word length of the computer used. $M = K/2$ and m is the

integer part of the $(j+1)/2$. The optimal value of K was chosen as described in Gaver-Stehfast algorithm, for the fast convergence of results with the desired accuracy. The Romberg numerical integration technique with variable step size was used to evaluate the integrals involved. All the programs were made in mathematical software Matlab.

5   NUMERICAL RESULTS AND DISCUSSION

Mathematical model is prepared with Copper material for purposes of numerical computations. The material constants of the problem are given below [11]

$$T_0 = 293\,K \;,\; \rho = 8954\ kg\,m^{-3}\;,\; \tau_0 = 0.02\;,\; \tau_1 = 0.08\;,\; k = 386 JK^{-1}m^{-1}s^{-1}, \alpha_t = 1.78 \times 10^{-5} K^{-1}, C_E = 383.1 m^2 K \;,$$
$$\mu = 3.86 \times 10^{10}\, Nm^{-2}\;,\; \lambda = 7.76 \times 10^{10}\, Nm^{-2}$$

Using these values, it was found that $\eta = 8886.73\,m\,s^{-2}, \varepsilon = 0.0168\,, \beta^2 = 4$. It should be noted that a unit of non-dimensional time corresponds to $6.5 \times 10^{-12}\,s$, while a unit of non-dimensional length is equal to $2.7 \times 10^{-8}\,m$. We consider a thick circular plate of height $b = 1m$ and the constants in the problem are taken as $b = 1m$ and $Q = 1$. The computations were carried out for three values of time $t = \{0.1, 0.5, 1.2\}$.
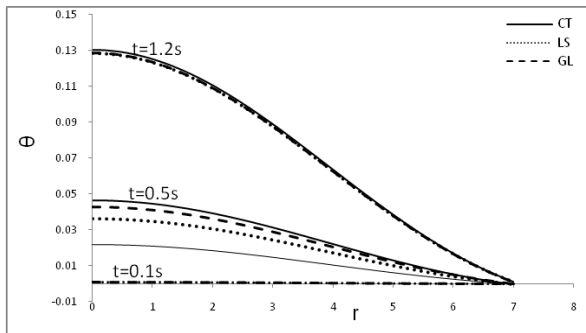

Fig.1
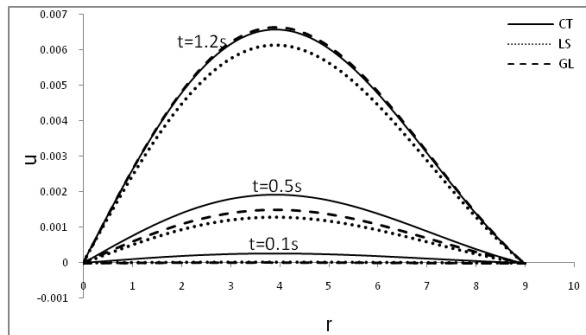Temperature distribution θ in the middle plane.

Fig.2
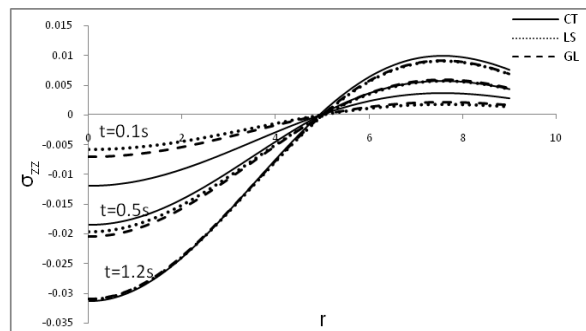Radial displacement u distribution in the middle plane.

Fig.3
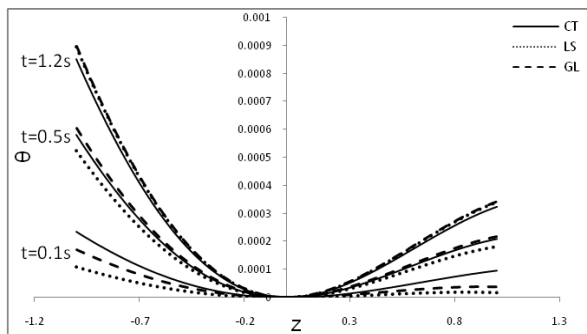Axial stress component σ_{zz} along the radial direction in the middle plane.

Fig.4
Temperature distribution θ in axial direction at $r = 0.1m$.
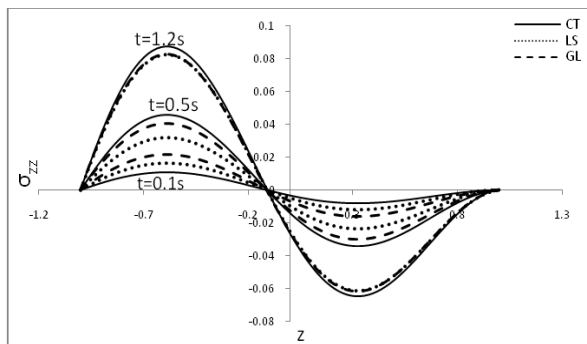

Fig.5
Axial stress component σ_{zz} along the axial direction
at $r = 0.1m$.

The numerical values for temperature θ, the radial displacement component u, and the axial stress component σ_{zz} have been calculated at the middle of the plane ($z = 0$) for different time instants $t = 0.1, 0.5, 1.2s$ along the radial direction and are displayed graphically for Classical Coupled Thermoelasticity (CT theory) , Lord-Shulman theory (L-S theory) and Green-Lindsay theory of generalized thermoelasticity and as shown in Figs. 1, 2 and 3 respectively. The graphs of the stress component σ_{rr} and $\sigma_{\phi\phi}$ are very much similar to the axial stress component and hence are not taken up for discussion. Since the displacement function w is an odd function of z, its value on the middle plane is always zero and it is omitted here.

Fig.1 depicts the non-dimensional temperature distribution θ, Fig.2 depicts the radial component of displacement u and Fig.3 represents axial stress component σ_{zz} along the radial direction at the middle plane ($z = 0$) at different time instants $t = 0.1, 0.5, 1.2s$ respectively. Classical Coupled Theory of thermoelasticity (CT) predicts an infinite speed of wave propagation, whereas the Lord-Shulman (LS) model and Green-Lindsay (GL) model of generalized thermoelasticity involves the introduction of one relaxation time τ_0 and two relaxation times τ_0, τ_1 respectively. The heat conduction equation is hyperbolic in nature thus the heat wave assumes finite propagation speeds. It is clearly observed from Fig.1, Fig.2 and Fig.3 that at small times the CT, LS and GL theory show different results. In CT, the heat is transmitted throughout the medium instantaneously and hence the solution is not zero. On the other hand, LS and GL theories predict finite speed of heat propagation. Hence, for short time the heating effect is not seen in the plate, the radial displacement is identically zero and the stresses are negligible. For large time, the heat wave reaches all points in the medium even with finite speed of propagation. Hence, for large times the three theories are in somewhat agreement. It is also observed that the non-dimensional temperature θ drops gradually along the radial direction. It is further observed from Fig.3 that the axial stress σ_{zz} is compressive till $r = 5m$ and after that becomes tensile in nature.

Fig.4 shows the plot of non-dimensional temperature θ in the axial direction i.e. along the z axis at $r = 0.1m$. From the graphs, it can be observed that the temperature at the upper and lower face is more as compared to the rest of the thick plate. Thus, the heating takes place at the faces of the plate. It is also seen that at small times CT, LS and GL theories predict different results whereas at large times the three theories predict a similar result.

Fig. 5 depicts the plot of axial stress σ_{zz} in the axial direction i.e. along the z axis at $r = 0.1m$. It is observed that the stress is tensile at the lower edge of the plate till $z \approx -0.1m$ and then becomes compressive thereafter.

6 CONCLUSIONS

In this problem, we have used the generalized theories of thermoelasticity LS and GL models to solve the problem for thick circular plate of infinite extent with axisymmetric heat supply and compared the models with CT. From the graphs, it can be clearly seen that for LS and GL theories, the temperature, displacement and stress distributions are identically zero if not absolutely zero for small time whereas for CT theory the temperature, displacement and stress distributions are non zero. This may be because the heat wave is transmitted throughout the medium instantaneously for CT theory for small time. As generalized theories (i.e. LS and GL models) involve a hyperbolic wave equation, predicting finite speeds of heat propagation whereas the CT model involves a parabolic heat equation, thus predicting infinite speeds of heat wave propagation. As a special case, we have constructed a mathematical model for copper plate with axisymmetric heat supply and the results are compared in CT, LS and GL theories. We may also conclude that the system of equations in this paper may prove to be useful in studying the thermal characteristics of various bodies in important engineering problems using the more realistic generalized models of thermoelasticity instead of Classical Coupled theory.

REFERENCES

[1]   Biot M., 1956, Thermoelasticity and irreversible thermodynamics, *Journal of Applied Physics* **27**: 240-253.
[2]   Lord H., Shulman Y., 1967, A generalized dynamical theory of thermo-elasticity, *Journal of the Mechanics and Physics of Solids* **15**: 299-309.
[3]   Green A. E., Lindsay K. A., 1972, Thermoelasticity, *Journal of Elasticity* **2**: 1-7.
[4]   Suhubi E., 1975, *Thermoelastic Solids*, Academic Press, New York.
[5]   Dhaliwal R. S., Sherief H., 1980, Generalized thermoelasticity for anisotropic media, *Quarterly of Applied Mathematics* **33**: 1-8.
[6]   Chandrasekariah D. S., 1986, Thermoelasticity with second sound: a review, *Applied Mechanics Review* **39**: 355-376.
[7]   Hetnarski R. B., Ignaczak J., 1999, Generalized thermoelasticity, *Journal of Thermal Stresses* **22**: 451-476.
[8]   Sherief H. H., Anwar M. N., 1989, A problem in generalized thermoelasticity for an infinitely long annular cylinder composed of two different materials, *Acta Mechanica* **80**: 137-149.
[9]   Maghraby N. M., Abdel Halim A. A., 2010, A generalized thermoelastic problem for a half space with heat sources under axisymmetric distribution, *Australian Journal of Basic and Applied Science* **4**(8): 3803-3814.
[10]   Othman M. I. A., Abbas I. A., 2012, Generalized thermoelasticity of thermal shock problem in a non homogeneous isotropic hollow cylinder with energy dissipation, *International Journal of Thermophysics* **33**: 913-923.
[11]   Youssef H. M., 2006, Two-dimensional generalized thermoelasticity problem for a half- space subjected to ramp-type heating, *European Journal of Mechanics A/Solids* **25**: 745-763.
[12]   Awad E. S., 2010, A note on the spatial decay estimates in non-classical linear thermoelastic semi- cylindrical bounded domains, *Journal of Thermal Stresses* **33**: 187-201.
[13]   Mukhopadhyay S., Kumar R., 2009,Thermoelastic interactions on two-temperature generalized thermoelasticity in an infinite medium with a cylindrical cavity, *Journal of Thermal Stresses* **32**: 341-360.
[14]   Tripathi J. J., Kedar G. D., Deshmukh K. C., 2014, Dynamic problem of generalized thermoelasticity for a semi-infinite cylinder with heat sources, *Journal of Thermoelasticity* **2**(1): 01-08.
[15]   Mallik S. H., Kanoria M., 2008, A two dimensional problem for a transversely isotropic generalized thermoelastic thick plate with spatially varying heat source, *European Journal of Mechanics A/Solids* **27**: 607-621.
[16]   Bagri A., Eslami M.R., 2007, A unified generalized thermoelasticity; solution for cylinders and spheres, *International Journal of Mechanical Sciences* **49**: 1325-1335.
[17]   El-Maghraby N. M., 2004, A two dimensional problem in generalized thermoelasticity with heat sources, *Journal of Thermal Stresses* **27**: 227-240.
[18]   El-Maghraby N. M., 2005, A two dimensional problem for a thick plate and heat sources in generalized thermoelasticity, *Journal of Thermal Stresses* **28**: 1227-1241.
[19]   Youssef H. M., 2009, State-space approach of two-temperature generalized thermoelastic medium subjected to moving heat source and ramp-type heating, *Journal of Mechanics of Materials and Structures* **4**(9): 1637-1649.
[20]   Tripathi J. J, Kedar G. D., Deshmukh K. C., 2015, Generalized thermoelastic diffusion problem in a thick circular plate with axisymmetric heat supply, *Acta Mechanica* **226**: 2121-2134.

[21]   Gaver D. P., 1966, Observing stochastic processes and approximate transform inversion, *Operations Research* **14**: 444-459.

[22]   Stehfast H., 1970, Algorithm 368: Numerical inversion of Laplace transforms, *Association for Computing Machinery* **13**: 47-49.

[23]   Stehfast H., 1970, Remark on algorithm 368, Numerical inversion of Laplace transforms*, Association for Computing Machinery* **13**: 624.

[24]   Knight J. H., Raiche A. D., 1982, Transient electromagnetic calculations using gaver-stehfast inverse Laplace transform method, *Geophysics* **47**: 47-50.

[25]   Press W. H., Flannery B. P., Teukolsky S. A., Vetterling W. T., 1986, *Numerical Recipes*, Cambridge University Press, Cambridge, The art of Scientific Computing.