A Contact Problem of an Elastic Layer Compressed by Two Punches of Different Radii

K. Seghir, M. Bendaoui, R. Benbouta [*]

Department of Mechanical Engineering, Faculty of Technology, University of Batna2, Algeria

ABSTRACT

The elasticity mixed boundary values problems dealing with half-space contact are generally well resolved. A large number of these solutions are obtained by using the integral transformation method and methods based the integral equations. However, the problems of finite layer thicknesses are less investigated, despite their practical interests. This study resolves a quasi-stationary problem of an isotropic elastic layer compressed by two rigid cylinders with flat ends. Hankel transformation and auxiliary functions with boundary conditions reduce the differential equation to an algebraic equations system, which can be solved in a numerical way. The contact efforts equations are established. From the general method, solutions of particular cases are also resolved. A particular case is studied, the contact zone pressure and stresses distribution curves are presented.

© 2018 IAU, Arak Branch. All rights reserved.

Keywords: Contact problem, Elastic layer, Cylindrical punches, Flat ends.

1 INTRODUCTION

WE consider an axisymmetric problem of an elastic layer thickness (H) indentation by two rigid coaxial cylindrical punches with different radii on its boundary surfaces (Fig.1). A calculation method of the contact stresses is developed using the Hankel transform reducing the differential equation linking the stresses and the displacement components to a system of integral equations. The approach in this work takes its origin from previous studies carried out by Harding and Sneddon [1] which seems to be the first to use the Hankel transformation to reduce the Boussinesq problem for a circular solid punch to the resolution of a pair of double integral equations. Ufliand [2] presents in his book a synthesis of early work concerning the integral transformations in elasticity problems of contacts. Kuz'min and Ufliand [3] use the same technique to solve the problem of a compressed layer by two identical punches under equal pressures and lead to approximations of stresses and displacements to the interfaces in the forms of power series. Later Zakorko [4], by the same approach finished to approximate the distribution of the layer-flat ended punches contacts efforts to a convergent series for small parameters smaller indenter radius. Dhaliwal and Sing [5] studied an interesting alternative of shear punching in an elastic layer and reduced the problem by the Hankel transform to the solution of a pair of simultaneous Fredholm integrals of the second kind. Many other works include detailed descriptions of applications of the integral transformations to contact problems [6-8]. In recent works authors return to the use of the Hankel transform to solve loading of the contact finite layers' thicknesses. We refer to Keer and Kuo [9], using the Hankel integral transformation reducing the problem to an integral equation they solved numerically. Matnyak [10] studied the stress distribution under the

---

[*]Corresponding author. Tel.: +213 33812143.
E-mail address: r_benbouta@yahoo.fr (R.Benbouta).

punch moving on a prestressed elastic layer and also used the Hankel transformation. This method seems to suit perfectly thermo elastic problems solving, and in fact Shelestovskii and Gabrusev [11] used this method in their work and analyzed the effect of the punches geometry on the distribution of thermal and mechanical components of the contact forces. The same method is applied by Ruiny and Dahan [12] under the assumption that stresses and displacements are supposed to tend toward zero for (r, z) tending to infinity. In our present contribution, we assume that at points far from the punch, the components of stresses and displacements are negligible and that perturbations of the planarity of the free faces are rather weak. In a first time, we built up a system of integral equations with boundary conditions similar to the ones in previous works [3,4]. The difference is that, after applying the inverse Hankel transform to differential equations and writing the general expressions of stresses and displacements, we introduce two auxiliary functions in the boundary conditions and a suitable change in the variables. This facilitates the integral equations resolution and lead to a system of numerical integral equations.

In this work, the aim is to get a solution to the problem, in a simpler way, for which the classical results formulated as power series do not converge (at least rapidly) for $R \geq H$. Expressions giving the stresses distribution under the punches are defined and the corresponding practical curves are plotted.

2   PROBLEM FORMULATIONS

A layer thickness (H) of an elastic material with isotropic parallel faces is statically compressed by two flat ended rigid cylinders at its boundaries (Fig. 1). We assume that the line of action of the two forces coincide with the symmetric axis of the two punches. This is a symmetrical axial problem and can be represented in a cylindrical coordinates system (r, θ, z) coinciding with the upper surface layer.

Considering the following boundary conditions:

$$0 \leq r < \infty \qquad , \ z = 0 \quad : \quad \tau_{rz} = 0 \tag{1}$$

$$0 \leq r < \infty \qquad , \ z = -H \quad : \quad \tau_{rz} = 0 \tag{2}$$

$$R \leq r < \infty \qquad , \ z = 0 \quad : \quad \sigma_z = 0 \tag{3}$$

$$R_1 \leq r < \infty \qquad , \ z = -H \quad : \quad \sigma_z = 0 \tag{4}$$

$$0 \leq r < R \qquad , \ z = 0 \quad : \quad U_z = -\varepsilon \tag{5}$$

$$0 \leq r < R_1 \qquad , \ z = -H \quad : \quad U_z = +\varepsilon_1 \tag{6}$$

$\varepsilon, \varepsilon_1$: Penetration depth of the two punches in the elastic layer.
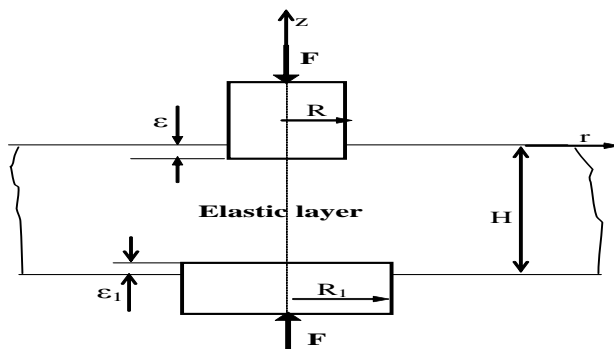

Fig.1
The problem schematic representation.

3   GENERAL SOLUTION OF THE ELASTICITY PROBLEM IN AXIAL SYMMETRY

The differential equation governing the problem with axial symmetry in cylindrical coordinates is the b_i harmonic equation [13,14].

$$\nabla^4 \phi(r,z) = 0 \tag{7}$$

where: $\nabla^2 = \dfrac{\partial^2}{\partial^2 r} + \dfrac{1}{r}\dfrac{\partial}{\partial r} + \dfrac{\partial^2}{\partial^2 z}$

Using for the equation solution the Hankel inverse transform of zero order [2,15], the function $\phi(r,z)$ is defined by:

$$\phi(r,z) = \int_0^\infty \xi \bar{\phi}(\xi,z) J_0(\xi r) d\xi \tag{8}$$

J_0 : The Bessel function of order zero.

$\phi(r,z)$: The stress function defined by the following relations.

$$U_z = \frac{\lambda+2G}{G}\nabla^2\phi(r,z) - \frac{\lambda+G}{G}\frac{\partial^2\phi(r,z)}{\partial z^2} \tag{9}$$

$$U_r = \frac{\lambda+G}{G}\frac{\partial^2\phi(r,z)}{\partial r\,\partial z} \tag{10}$$

λ and G are elastic coefficients. After substitution the general solution is:

$$\bar{\phi}(\xi,z) = \left[B_1(\xi)+B_2(\xi)z\right]e^{-\xi z} + \left[B_3(\xi)+B_4(\xi)z\right]e^{\xi z} \tag{11}$$

B_1, B_2, B_3, B_4 are functions of the variable (ξ) determined by the boundary conditions. Introducing (11) into (8) and in the expressions for the displacements and the stresses given by [3]. General expressions of stresses and displacements can be written in the form of integral expressions.

$$Uz = -\int_0^\infty \frac{1}{\eta}\left\{\begin{array}{l}\left[\dfrac{\eta}{b_1 R}F_1(\eta)+(2+\dfrac{\eta\zeta}{b_1})F_2(\eta)\right]e^{-\eta\zeta} \\[2mm] +\left[\dfrac{\eta}{b_1 R}F_3(\eta)-(2-\dfrac{\eta\zeta}{b_1})F_4(\eta)\right]e^{\eta\zeta}\end{array}\right\} J_0(\eta\rho)d\eta \tag{12}$$

$$\sigma_z = \frac{2b_3}{R}\int_0^\infty \left\{\begin{array}{l}\left[\dfrac{\eta}{R}F_1(\eta)+(b_1+\eta\zeta)F_2(\eta)\right]e^{-\eta\zeta} \\[2mm] +\left[-\dfrac{\eta}{R}F_3(\eta)+(b_1-\eta\zeta)F_4(\eta)\right]e^{\eta\zeta}\end{array}\right\} J_0(\eta\rho)d\eta \tag{13}$$

$$\tau_{rz} = \frac{2b_3}{R}\int_0^\infty \left\{\begin{array}{l}\left[\dfrac{\eta}{R}F_1(\eta)-(b_2-\eta\zeta)F_2(\eta)\right]e^{-\eta\zeta} \\[2mm] +\left[\dfrac{\eta}{R}F_3(\eta)+(b_2+\eta\zeta)F_4(\eta)\right]e^{\eta\zeta}\end{array}\right\} J_1(\eta\rho)d\eta \tag{14}$$

b_i (i = 1-3) are physical constants of the material $b_1 = \dfrac{G}{\lambda+G}$, $\qquad b_2 = \dfrac{\lambda}{\lambda+G}$ and $b_3 = \lambda+G$.

$F_i(\eta) = \dfrac{\eta^3}{R^3} B_i(\dfrac{\eta}{R})$: (i=1 to 4) are the unknown functions.

$(\rho = \dfrac{r}{R}$, $\eta = \xi.R$ *et* $\zeta = \dfrac{Z}{R})$ are dimensionless parameters.

4 SOLVING METHOD AT THE BOUNDARY VALUE PROBLEM

Satisfying the boundary conditions (1), (2) and introducing the functions $\phi_1(\eta), \phi_2(\eta)$ in the conditions (3) and (4) such that:

$$\frac{\eta}{R}F_1(\eta) - b_2 F_2(\eta) + \frac{\eta}{R}F_3(\eta) + b_2 F_4(\eta) = 0$$

$$[\frac{\eta}{R}F_1(\eta) - (b_2+\eta\ell)F_2(\eta)]e^{\eta\ell} + [\frac{\eta}{R}F_3(\eta) + (b_2-\eta\ell)F_4(\eta)]e^{-\eta\ell} = 0$$

$$\frac{\eta}{R}F_1(\eta) + b_1 F_2(\eta) - \frac{\eta}{R}F_3(\eta) + b_1 F_4(\eta) = \eta\phi_1(\eta)R \tag{15}$$

$$[\frac{\eta}{R}F_1(\eta) + (b_1-\eta\ell)F_2(\eta)]e^{\eta\ell} + [\frac{-\eta}{R}F_3(\eta) - (b_1+\eta\ell)F_4(\eta)]e^{-\eta\ell} = \eta\phi_2(\eta)R$$

with: $\ell = H/R$. Solving the system of algebraic Eqs. (15) and expressing the functions F_i (i =1−4) by ϕ_j (j =1,2).

We get, after substitution F_i (i =1−4) in the conditions (3), (4), (5) and (6), the following system of integral equations:

$$\int_0^\infty \phi_1(\eta)J_0(\eta\rho)d\eta = \frac{-\varepsilon b_1}{R(1+b_1)} + \int_0^\infty (\phi_2(\eta)P_2(\eta\ell) - \phi_1(\eta)P_1(\eta\ell))J_0(\eta\rho)d\eta \quad ; \ \rho \prec 1 \tag{16}$$

$$\int_0^\infty \eta\phi_1(\eta)J_0(\eta\rho)d\eta = 0 \quad ; \ \rho \succ 1 \tag{17}$$

$$\int_0^\infty \phi_2(\eta)J_0(\eta\rho)d\eta = \frac{-\varepsilon_1 b_1}{R(1+b_1)} + \int_0^\infty (\phi_1(\eta)P_2(\eta\ell) - \phi_2(\eta)P_1(\eta\ell))J_0(\eta\rho)d\eta \quad ; \rho \prec \rho_1 \tag{18}$$

$$\int_0^\infty \eta\phi_2(\eta)J_0(\eta\rho)d\eta = 0 \quad ; \rho \succ \rho_1 \tag{19}$$

where: $\rho_1 = \dfrac{R_1}{R}$; $P_2(\eta\ell) = \dfrac{sh\eta\ell + \eta\ell ch\eta\ell}{sh^2\eta\ell - \eta^2\ell^2}$; $P_1(\eta\ell) = \dfrac{\eta\ell + \eta^2\ell^2 + e^{-\eta\ell}sh\eta\ell}{sh^2\eta\ell - \eta^2\ell^2}$

4.1 Case of punches with similar radius

When the layer is compressed by two punches having the same radii $R=R_1$ with two equal forces: $\varepsilon_1 = \varepsilon$ and $\phi_1(\eta) = \phi_2(\eta) = \phi(\eta)$, the previous equations system (16) - (19) is reduced to the following system of two integral equations:

$$\int_0^\infty \phi(\eta)J_0(\eta\rho)d\eta = \frac{-\varepsilon b_1}{R(1+b_1)} + \int_0^\infty G(\eta\ell)\phi(\eta\ell)J_0(\eta\rho)d\eta\ell)\phi(\eta\ell)J_0(\eta\rho)d\eta \quad ; \ \rho \prec 1$$

$$\int_0^\infty \eta\phi(\eta)J_0(\eta\rho)d\eta = 0 \quad ; \ \rho \succ 1 \tag{20}$$

where: $G(\eta\ell) = P_1(\eta\ell) - P_2(\eta\ell)$

4.2 Case of a single punch

Considering the case where R_1 tends to infinity (or ε_1 equal to zero). In either assumption, the underneath punch becomes a support plan. Taking $\varepsilon_1 = 0$, from Eq. (18), we deduce $\phi_2(\eta)$ function of $\phi_1(\eta)$ so that:

$$\phi_2(\eta) = \frac{sh\eta\ell + \eta\ell ch\eta\ell}{\eta\ell + sh\eta\ell ch\eta\ell}\phi_1(\eta) \tag{21}$$

Substituting in Eq. (16) $\phi_2(\eta)$ by its new expression (21), we obtain another system of two integral equations, which is written as:

$$\int_0^\infty \phi_1(\eta)J_0(\eta\rho)d\eta = -\frac{\varepsilon}{R}\frac{b_1}{1+b_1} + \int_0^\infty \phi_1(\eta)G_1(2\eta\ell)J_0(\eta\rho)d\eta \quad ; \ \rho \prec 1$$

$$\int_0^\infty \phi_1(\eta)J_0(\eta\rho)d\eta = 0 \quad ; \ \rho \succ 1 \tag{22}$$

where: $G_1(2\eta l) = \dfrac{1+2\eta\ell - e^{-2\eta\ell}}{sh^2 2\eta\ell + 2\eta\ell}$

In both cases (15) and (16) it's only necessary to solve a system with two integral equations. The only difference lies in the expression of the functions $G(\eta\ell)$ and $G_1(2\eta\ell)$ in Eqs. (20) and (22) to get to an expression of constraints in the following form:

$$\sigma_z(\rho,0) = \frac{-F}{2\pi R^2}\left[\frac{\psi(1)}{\sqrt{1-\rho^2}} - \int_\rho^1 \frac{\psi'(t)}{\sqrt{t^2-\rho^2}}\right] \tag{23}$$

The function $\psi(t)$ is the one to be determined numerically from the following equation; concerning the methodology see [16].

$$\psi(t) - \frac{2}{\pi}\int_0^1 \psi(x)dx \int_0^\infty Q(\eta\ell)\cos\eta x\left[\cos\eta t - \frac{\sin\eta}{\eta}\right]d\eta = 1 \quad :0 \le t \le 1 \tag{24}$$

where:

$Q(\eta\ell) = G(\eta\ell)$ for the system (15)

$Q(\eta\ell) = G_1(2\eta\ell)$ for the system (16)

4.3 Case of a half space

If the relative thickness ($\ell = \frac{H}{R}$) tend towards infinity in (*4.2 Case of a single punch*) the solution will be that of an infinite semi space. The function $G(2\eta\ell)$ will take the value zero and consequently the function $\psi(t)$ will be unity according to (24). The interface pressure (23) can be written as:

$$\sigma_z(\rho,0) = \frac{-F}{2\pi R^2} \frac{1}{\sqrt{1-\rho^2}}$$

4.4 Study of the general case of two punches of different various radii

We propose to solve the case corresponding to punches with different radii ($R \neq R_1$) solicited by two similar forces (Fig.1).

Changing variables ($\rho = \rho_1\rho^*$ and $\eta^* = \rho_1\eta$) and the following functions representation $\phi_1(\eta)$ and $\phi_2(\eta)$ [8,15,17] ;

$$\phi_1(\eta) = \frac{b_1}{1+b_1} \int_0^1 f_1(t)\cos(\eta t)dt \tag{25}$$

$$\phi_2(\eta) = \phi_2(\frac{\eta^*}{\rho_1}) = \rho_1^2 \phi_2^*(\eta^*) = \rho_1^2 \frac{b_1}{1+b_1} \int_0^1 f_2(t)\cos(\eta^* t)dt \tag{26}$$

Eqs. (17) and (19) are satisfied.

$$\int_0^\infty \eta\phi_1(\eta)J_0(\eta\rho)d\eta = \begin{cases} 0 & ; \quad \rho \succ 1 \\[2mm] \dfrac{f_1(1)}{\sqrt{1-\rho^2}} - \int_\rho^0 \dfrac{f_1'(t)dt}{\sqrt{t^2-\rho^2}} & ; \quad \rho \prec 1 \end{cases} \tag{27}$$

$$\int_0^\infty \eta\phi_2(\eta)J_0(\eta\rho)d\eta = \int_0^\infty \eta^*\phi_2^*(\eta^*)J_0(\eta^*\rho^*)d\eta^* = \begin{cases} 0 & ; \quad \rho^* \succ 1 \\[2mm] \dfrac{f_2(1)}{\sqrt{1-\rho^{*2}}} - \int_{\rho^*}^0 \dfrac{f_2'(t)dt}{\sqrt{t^2-\rho^{*2}}} & ; \quad \rho^* \prec 1 \end{cases} \tag{28}$$

and the stresses at the contact surfaces *(z = 0, z = -H)* are determined by:

$$\sigma_z(\rho,0) = \frac{2b_1b_3}{1+b_1} \left(\frac{f_1(1)}{\sqrt{1-\rho^2}} - \int_\rho^1 \frac{f_1'(t)dt}{\sqrt{t^2-\rho^2}} \right)$$

$$\sigma_z(\rho^*,-\ell) = \frac{2b_1b_3}{1+b_1}\left(\frac{f_2(1)}{\sqrt{1-\rho^{*2}}} - \int_{\rho*}^{1}\frac{f_2'(t)dt}{\sqrt{t^2-\rho^{*2}}}\right)$$

Eqs. (16) and (18) are transformed to Abel integrals,

$$\int_0^{\rho}\frac{f_1(t)dt}{\sqrt{\rho^2-t^2}} = g_1(\rho) \quad \text{and} \quad \int_0^{\rho^*}\frac{f_2(t)dt}{\sqrt{\rho^{*2}-t^2}} = g_2(\rho^*) \tag{29}$$

which can be easily solved in the way [15,17] ;

$$f_1(t) = \frac{2}{\pi}\frac{d}{dt}\int_0^{t}\frac{\rho g_1(\rho)d\rho}{\sqrt{t^2-\rho^2}} \quad \text{and} \quad f_2(t) = \frac{2}{\pi}\frac{d}{dt}\int_0^{t}\frac{\rho^* g_2(\rho^*)d\rho^*}{\sqrt{t^2-\rho^{*2}}} \tag{30}$$

with:

$$g_1(\rho) = \frac{-\varepsilon}{R} + \rho_1^2\int_0^{\infty}P_2(\eta\ell)J_0(\eta\rho)d\eta\int_0^{1}f_2(t)\cos\rho_1\eta t dt - \int_0^{\infty}P_1(\eta\ell)J_0(\eta\rho)d\eta\int_0^{1}f_1(t)\cos\eta t dt \tag{31}$$

$$g_2(\rho^*) = \frac{-\varepsilon_1}{\rho_1 R} - \int_0^{\infty}P_1(\eta^*\ell^*)J_0(\eta^*\rho^*)d\eta^*\int_0^{1}f_2(t)\cos\eta^* t dt + \frac{1}{\rho_1^2}\int_0^{\infty}P_2(\eta^*\ell^*)J_0(\eta^*\rho^*)d\eta^*\int_0^{1}f_1(t)\cos\frac{\eta^*}{\rho_1}t dt \tag{32}$$

Substituting (31) and (32) in each expression (30) and taking into account (25) and (26), we get two Fredholm integral equations as functions $f_1(t)$ and $f_2(t)$:

$$f_1(t) = -\frac{2\varepsilon}{\pi R} - \frac{2}{\pi}\int_0^{1}f_1(x)dx\int_0^{\infty}P_1(\eta\ell)\cos\eta x\,\cos\eta t d\eta + \frac{2\rho_1^2}{\pi}\int_0^{1}f_2(x)dx\int_0^{\infty}P_2(\eta\ell)\cos\rho_1\eta x\cos\eta t d\eta \quad :(0\le t\le 1) \tag{33}$$

$$f_2(t) = -\frac{2\varepsilon_1}{\pi\rho_1 R} - \frac{2}{\pi}\int_0^{1}f_2(x)dx\int_0^{\infty}P_1(\eta^*\ell^*)\cos(\eta^* x)\cos(\eta^* t)d\eta^* + \frac{2}{\pi\rho_1^2}\int_0^{1}f_1(x)dx\int_0^{\infty}P_2(\eta^*\ell^*)\cos\frac{\eta^*}{\rho_1}x.\cos\eta^* t d\eta^* \quad :(0\le t\le 1) \tag{34}$$

Using the static equilibrium conditions at the interfaces ($z = 0$; $z = -H$).

$$F = -2\pi R^2\int_0^{1}\rho\sigma_z(\rho,0)d\rho \quad ; \ 0\le\rho\le 1 \tag{35}$$

$$F = -2\pi R^2\rho_1^2\int_0^{1}\rho^*\sigma_z(\rho^*,-\ell^*)d\rho^* \quad ; \ 0\le\rho^*\le 1 \tag{36}$$

The stress expressions at contact surfaces ($z = 0$; $z = -H$) can be also writing in the form:

$$\sigma_z(\rho,0) = \chi_0 \int\limits_0^1 f_1(t)\cos\eta t\, dt \int\limits_0^\infty \eta J_0(\eta\rho)d\eta \quad ; \quad \rho \prec 1 \tag{37}$$

$$\sigma_z(\rho^*,-\ell^*) = \chi_0 \int\limits_0^1 f_2(t)\cos\eta^* t\, dt \int\limits_0^\infty \eta^* J_0(\eta^*\rho^*)d\eta^* \quad ; \quad \rho^* \prec 1 \tag{38}$$

with: $\chi_0 = \dfrac{2b_3 b_1}{1+b_1}$

Taking (37) in (35) and (38) in (36), taking into account (34) and (33). Integrating with respect to variable (*t*), we get:

$$f_1(t) = -\frac{F}{2\pi R^2 \chi_0}\psi_1(t) \tag{39}$$

$$f_2(t) = -\frac{F}{2\pi R^2 \rho^2_1 \chi_0}\psi_2(t) \tag{40}$$

with:

$$\psi_1(t) + \frac{2}{\pi}\int\limits_0^1 \psi_1(x)K_1(x,t)dx - \frac{2}{\pi}\int\limits_0^1 \psi_2(x).K_2(x,t)dx = 1$$

$$\psi_2(t) + \frac{2}{\pi}\int\limits_0^1 \psi_2(x).K_3(x,t)dx - \frac{2}{\pi}\int\limits_0^1 \psi_1(x).K_4(x,t)dx = 1 \tag{41}$$

where:

$$K_1(x,t) = \int\limits_0^\infty P_1(\eta\ell)\cos\eta x\,(\cos\eta t - \frac{\sin\eta}{\eta})d\eta$$

$$K_2(x,t) = \int\limits_0^\infty P_2(\eta\ell)\cos\eta\rho_1 x\,(\cos\eta t - \frac{\sin\eta}{\eta})d\eta$$

$$K_3(x,t) = \int\limits_0^\infty P_1(\eta^*\ell^*)\cos\eta^* x\,(\cos\eta^* t - \frac{\sin\eta^*}{\eta^*})d\eta^*$$

$$K_4(x,t) = \int\limits_0^\infty P_2(\eta^*\ell^*)\cos\frac{\eta^*}{\rho_1}x\,(\cos\eta^* t - \frac{\sin\eta^*}{\eta^*})d\eta^*$$

$\psi_1(t)$ and $\psi_2(t)$ are unknown functions to be determined from the resolution of the equations system (41). Finally the stresses at the interfaces punches-layer taking into account (27) and (28) can be determined by the following equations:

$$\sigma_z(\rho,0) = \frac{-F}{2\pi R^2}\sigma^*(\rho,0) \qquad \sigma_z(\rho^*,-\ell) = \frac{-F}{2\pi R^2 \rho^2_1}\sigma_z^*(\rho^*,-\ell) \tag{42}$$

where: $\sigma_z^*(\rho, 0)$ and $\sigma_z^*(\rho^*, -\ell)$ are dimensionless stresses

$$\sigma_z^*(\rho, 0) = \frac{\psi_1(1)}{\sqrt{1-\rho^2}} - \int_\rho^1 \frac{\psi_1'(t)dt}{\sqrt{t^2 - \rho^2}} \qquad\qquad \sigma_z^*(\rho^*, -\ell) = \frac{\psi_2(1)}{\sqrt{1-\rho^{*2}}} - \int_{\rho^*}^1 \frac{\psi_2'(t)dt}{\sqrt{t^2 - \rho^{*2}}} \tag{43}$$

5   NUMERICAL COMPUTATIONS

The use of the finite sums method (numerical integration) to solve the integral Eqs. (41) leads to the system of linear algebraic equations with unknowns $\psi_1(t_i)$, $\psi_2(t_i)$, $(i = 1, n+1)$.

The approximation of $\psi_1(t)$ and $\psi_2(t)$ by polynomial types $\psi_1(t) = a_0 + \sum_{k=1}^n a_k t^k$ and $\psi_2(t) = b_0 + \sum_{k=1}^n b_k t^k$.

Allows the writing of (43) as:

$$\sigma_z^*(\rho, 0) = \frac{a_0}{\sqrt{1-\rho^2}} - \sum_{k=1}^n k a_k \int_\rho^1 \frac{t^{k-1}}{\sqrt{t^2 - \rho^2}} \, dt \tag{44}$$

$$\sigma_z^*(\rho^*, -\ell) = \frac{b_0}{\sqrt{1-\rho_1^{*2}}} - \sum_{k=1}^n k b_k \int_{\rho^*}^1 \frac{t^{k-1}}{\sqrt{t^2 - \rho^{*2}}} \, dt \tag{45}$$

5.1 Application

Taking the elastic layer with relative thickness ($\ell = H/R$) and taking for punches ($R_1/R = 1.5$). Discrediting the radiuses in $R = 20$ parts and $R_1 = 30$ parts. We use for the function $\psi_1(t_i)$ and $\psi_2(t_i)$ the polynomial approximation of degree ($k = 5$).The results of the stresses distribution in the contact zone $\sigma_z^*(\rho, 0)$ and $\sigma_z^*(\rho^*, -\ell)$ are represented by the curves in Fig. 2.

5.2 Graphical results

The curves intersection in the vicinity of $\rho = 0.8$ (Fig. 2) testifies of the effectiveness of the algorithm. In fact, the surface under the curve is constant. In order to preserve this value for any unspecified parameter variation, the corresponding curves must rotate round this point. The graphic result shows a critical situation in the case of punches with same radiuses $R_1 = R$. Knowing that, the sudden increase in the stresses on the interfaces follows the same circumferential line of action ($\rho = \rho_1 = 1$).
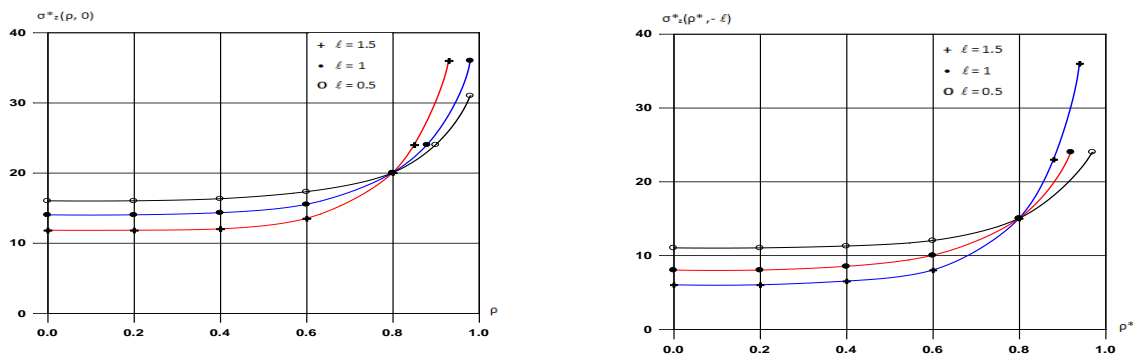

Fig.2
Stresses distribution in the contact zone.

6   CONCLUSIONS

We conclude from this work the following:
1. The semi analytical solution becomes very accurate with the development of numerical calculation and powerful computing means.
2. The solution to a problem of compressed layer by a single flat cylindrical punch is obtained from the general case with $\varepsilon_1 = 0$ (Eq.(18)) or by making the second punch dimension R_I tends to infinity.
3. Starting from the case of a single punch, we can deduct the solution of the infinite semi space requested by a flat punch (Boussinesq problem) by making H tends to infinity in (Eq. (24)).
4. To solve the problem of a layer compressed by two identical cylindrical punches, it is sufficient to take $R_{I} = R$ in Eqs. (16) -(19) which will lead to the resolution of a two equations system (Eq.(20)).
5. The solution of the case of the layer requested by two different flat cylindrical punches (Fig.1) summarizes in the solution of the algebraic system of Eqs. (41), the approximation of the functions ($\psi_1(t)$, $\psi_2(t)$) and the determination of the distribution of the stresses below each punch by Eq.(43).
6. Static balance implies the equality of resulting forces the pressures at the contact interfaces. Then the equality of the representative surfaces (the surface limited by the *x*-axis and the curves of stresses) of the stresses under each curves (*z=0, z=-H*) is an important result which marks the contribution of the calculation algorithm.
7. The static balance ensured by the equality of applied forces, permit to choose their intensities from the stresses expressions (42).
8. The developed algorithm is perfectly suitable for cases arising from the general case as it consists of the decomposition and the numerical resolution of Eq. (41) which leads to the computing time reduction making the analysis faster.

REFERENCES

[1] Harding J.W., Sneddon I.N., 1945, The elastic stresses produced by the indentation of the plane surface of a semi-infinite elastic solid by a rigid punch, *proceedings of the Cambridge Philosophical Society* **41**: 16-26.
[2] Ufliand Ja.S., 1965, *Survey of Articles on the Applications of Integral Transforms in the Theory of Elasticity*, Defense Technical Information Center, North Carolina State University.
[3] Kuz'min Iu.N., Ufliand Ia.S., 1967, The contact problem of an elastic layer compressed by two punches, *Journal of Applied Mathematics and Mechanics* **31**(4): 711-715.
[4] Zakorko V.N., 1978, Contact problem for a layer with two stamps, *Journal of Applied Mathematics and Mechanics* **42**(6): 1068-1073.
[5] Dhaliwal R.S., Sing B.M., 1977, Axisymmetric contact problem for an elastic layer on a rigid foundation with a cylindrical hole, *International Journal of Engineering Science* **15**: 421-428.
[6] Ufljand Ia.S., 1963, *Integral Transforms in Elasticity Problems*, Nauka. SSR.
[7] Sneddon I.N., 1959, A note on the axially symmetric punch problem, *Mathematika* **6**: 118-119.
[8] Sneddon I.N., 1977, *Application of Integral Transforms in the Theory of Elasticity,* CISM Courses and lectures, Springer-Verlag Vien, New York.
[9] Kuo C.H., Keer L., 1992, Contact stress analysis of a layered transversely isotropic half-space, *ASME Journal of Tribology* **114**: 253-262.
[10] Matnyac S.V., 2003, Contact stresses distribution under a rigid cylinder rolling over a pre-stressed strip, *International Applied Mechanics* **39**(7): 840-847.
[11] Shelestovshii B.G., Gabrusev G.V., 2004, Thermoelastic state of a transversely isotropic layer between two annular punches, *International Applied Mechanics* **40**(4): 67-77.
[12] Ruiny C., Dahan M., 2002, Chargements axisymétriques d'un bicouche transversalement isotrope, *Mécanique* **330** : 469-473.
[13] Grilitskii D.V., Kizima Y., 1981, *Theory of Deformation Elastic and Thermic, Axisymetric Contact Problem*, Nauka, Moscow.
[14] Argatov I.I., Dmitriev N.N., 2003, *Fundamentals of the Theory of Elastic Discrete Contact*, St. Polytechnics, St. Petersburg, Russian.
[15] Gradshteyn I.I., Ryzhik I.M., 1980, *Tables of Integrals, Series, and Products,* Academie press, New York.
[16] Seghir K., Benbouta R., Balbacha El., 2010, Analysis of stresses in the contact zone Rigid cylindrical indenter – transversely isotropic elastic layer, *Matériaux & Techniques* **98**: 227-232.
[17] Ditkin V.A., Prudnikov V.A., 1965, *Integral Transforms and Operational Calculus,* Pergamon Press, New York.