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Thermo-Viscoelastic Interaction Subjected to Fractional Fourier law with Three-Phase-Lag Effects

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ABSTRACT

In this paper, a new mathematical model of a Kelvin-Voigt type thermo-visco-elastic, infinite thermally conducting medium has been considered in the context of a new consideration of heat conduction having a non-local fractional order due to the presence of periodically varying heat sources. Three-phase-lag thermoelastic model, Green Naghdi models II and III (i.e., the models which predicts thermoelasticity without energy dissipation (TEWOED) and with energy dissipation (TEWED)) are employed to study the thermo-mechanical coupling, thermal and mechanical relaxation effects. In the absence of mechanical relaxations (viscous effect), the results for various generalized theories of thermoelasticity may be obtained as particular cases. The governing equations are expressed in Laplace-Fourier double transform domain. The inversion of the Fourier transform is carried out using residual calculus, where the poles of the integrand are obtained numerically in complex domain by using Laguerre's method and the inversion of the Laplace transform is done numerically using a method based on Fourier series expansion technique. Some comparisons have been shown in the form of the graphical representations to estimate the effect of the non-local fractional parameter and the effect of viscosity is also shown.

Keywords : Generalized thermoelasticity; Three-phase-lag model; Kelvin-Voigt model; Modified riemann-liouville fractional derivatives; Fractional Taylor’s series.

1 INTRODUCTION

LINEAR viscoelasticity has been an important area of research since the period of Maxwell, Boltzman, Voigt and Kelvin. Valuable information regarding linear viscoelasticity theory may be obtained in the books of Gross [1], Staverman and Schwarzl [2], Alfery and Gurnee [3], Ferry [4], Bland [5] and Lakes [6]. Many researchers like Biot [7, 8], Gurtin and Sternberg [9], Iiioushin and Pobedria [10], Tanner [11] have contributed notably on thermoviscoelasticity. Freudenthal [12] has pointed out that most of the solids, when subjected to dynamic loading, exhibit viscous effects.

The Kelvin-Voigt model is one of the macroscopic mechanical model often used to describe the viscoelastic behavior of a material. The model represents the delayed elastic response subjected to stress when the deformation is time dependent but recoverable. The dynamic interaction of thermal and mechanical fields in solids has great practical applications in modern aeronautics, astronautics, nuclear reactors and high-energy accelerators, for example.

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The modification of heat conduction equation from diffusive to a wave-type may be affected either by a microscopic consideration of the phenomenon of heat transport or in a phenomenological way by modifying the classical Fourier law of heat conduction. The first is due to Cattaneo [13], who obtained a wave-type heat equation by postulating a new law of heat conduction to replace the classical Fourier law. Puri and Kythe [14] investigated the effects of using the (Maxwell-Cattaneo) model in Stock's second problem for a viscous fluid and they note that, the non-dimensional thermal relaxation time to defined as to $=(CP_r)$ where C and P_r are the Cattaneo and Prandtl number, is of order $(10)^{-2}$.

Differential equations of fractional order have been the focus of many studies due to their frequent appearance in various applications in fluid mechanics, viscoelasticity, biology, physics and engineering. The most important advantage of using fractional differential equations in these and other applications is their non-local property. It is well known that the integer order differential operator is a local operator but the fractional order differential operator is non-local. This means that the next state of a system depends not only upon its current state but also upon all of its historical states. This is more realistic and it is one reason why fractional calculus has become more and more popular [15, 16].

Although the tools of fractional calculus have been available and applicable to various fields of study, the investigation of the theory of fractional differential equations has not only been started quite recently [15]. The differential equations involving Riemann-Liouville differential operators of fractional order $0<\alpha<1$, appear to be important in modeling several physical phenomena [17] and therefore seem to deserve an independent study of their theory parallel to the well-known theory of ordinary differential equations.

Fractional calculus has been used successfully to modify many existing models of physical processes. One can state that the whole theory of fractional derivatives and integrals was established in the 2nd half of the 19th century. The first application of fractional derivatives was given by Abel who applied fractional calculus in the solution of an integral equation that arises in the formulation of the tautochrone problem. The generalization of the concept of derivative and integral to a non-integer order has been subjected to several approaches and some various alternative definitions of fractional derivatives appeared [18–22]. One can refer Podlubny [16] for a survey of applications of fractional calculus. Some applications of fractional calculus to various problems of mechanics of viscoelastic fluid are reviewed in the literature [23–26]. In most of these investigations, the effect of the thermal state in the fluid is not considered.

Recently, a considerable research effort is expended to study anomalous diffusion, which is characterized by the time-fractional diffusion-wave equation by Kimmich [27] as follows

$$\rho c=\lambda I^{\xi}\nabla^2 c, \quad 0\le\xi\le 2, \tag{1}$$

where ρ is the mass density, c is the concentration, λ is the diffusion conductivity and the notation I^{ξ} is the Riemann-Liouville fractional integral, introduced as a natural generalization of the well-known $\xi-$fold repeated integral written in a convolution-type form as follows

$$\left.\begin{aligned} I^{\xi}f(t)&=\frac{1}{\Gamma(\xi)}\int_0^t (t-\tau)^{\xi-1}f(\tau)d\tau,\\ I^{0}f(t)&=f(t), \end{aligned}\right\} \quad 0<\xi\le 2 \tag{2}$$

where $\Gamma(\xi)$ is the gamma function.

According to Kimmich [27], Eq. (1) describes different cases of diffusion where $0<\xi<1$ corresponds to weak diffusion (subdiffusion), $\xi=1$ corresponds to normal diffusion, $1<\xi<2$ corresponds to strong diffusion (superdiffusion) and $\xi=2$ corresponds to ballistic diffusion.

It should be noted that the term diffusion is often used in a more generalized sense including various transport phenomena. Eq. (1) is a mathematical model of a wide range of important physical phenomena, for example, the subdiffusive transport occurs in widely different systems ranging from dielectrics and semiconductors through polymers to fractals, glasses, porous and random media. Superdiffusion is comparatively rare and has been observed in porous glasses, polymer chain, biological systems, transport of organic molecules and atomic clusters on surface. One might expect the anomalous heat conduction in media where the anomalous diffusion is observed.

Fujita [28] considered the heat wave equation for the case of $1 \le \xi \le 2$ as:

$$\rho c_v \theta = K I^{\xi} \theta_{,ii} , \tag{3}$$

where c_v is the specific heat, K is the thermal conductivity and the subscript "," means the derivative with respect to the coordinate x_i . Eq. (3) can be obtained as a consequence of the non local constitutive equation for the heat flux components q_i in the form

$$q_i = -K I^{\xi-1} \theta_{,i} \quad 1 < \xi \le 2 \tag{4}$$

Povstenko [29, 30] used the Caputo heat wave equation defined in the form

$$q_i = -K I^{\xi-1} \theta_{,i} \quad 0 < \xi \le 2 \tag{5}$$

Cattaneo introduced a law of heat conduction to replace the classical Fourier law in the form

$$q_i + \tau_0 \dot{q}_i = -K \nabla \theta. \tag{6}$$

Sherief et al. [31] introduced a formula of heat conduction as:

$$q_i + \tau_0 \frac{\partial^{\xi} q_i}{\partial t^{\xi}} = -K \frac{\partial \theta}{\partial t}. \quad 0 < \xi \le 1 \tag{7}$$

where

$$\frac{\partial^{\xi}}{\partial t^{\xi}} f(y,t) = \begin{cases} f(y,t) - f(y,0) & \xi \to 0, \\ I^{\xi-1} \dfrac{\partial f(y,t)}{\partial t} & 0 < \xi < 1, \\ \dfrac{\partial f(y,t)}{\partial t} & \xi = 1, \end{cases} \tag{8}$$

and proved a uniqueness theorem and derived a reciprocity relation and a variational principle.

In the limit, as ξ tends to one, Eq. (7) reduces to the well-known Cattaneo law used by Lord and Shulman [32] to derive the equation of the generalized theory of thermoelasticity with one relaxation time. It is known that Lebon et al. [33] and Jou et al. [34] showed that the classical entropy derived using this law instead of being monotonically increasing behaves in an oscillatory way. Strictly speaking, this result is not incompatible with the Clausius' formulation of the second law, which states that the entropy of the final equilibrium state must be higher than the entropy of the initial equilibrium state. However, the non-monotonic behavior of the entropy is in contradiction with the local equilibrium formulation of the second law, which requires that the entropy production must be positive everywhere at any time [33]. During the last two decades, this become the subject of many research papers and resulted in the introduction of what is known now as extended inversible thermodynamics.

Youssef [35] introduced another formula of heat conduction and taking into consideration (4)-(6)

$$q_i + \tau_0 \frac{\partial q_i}{\partial t} = -K I^{\xi-1} \nabla \theta. \quad 0 < \xi \le 2, \tag{9}$$

and the uniqueness theorem has been proved.

Ezzat established a new model of fractional heat conduction equation by using the new Taylor series expansion of time-fractional order, developed by Jumarie [36] as:

$$q_i + \frac{\tau_0^{\xi}}{\xi!}\frac{\partial^{\xi} q_i}{\partial t^{\xi}} = -K\nabla\theta. \quad 0<\xi\le 1, \tag{10}$$

El-Karamany and Ezzat [37] introduced two general models of fractional heat conduction law for a non-homogeneous anisotropic elastic solid. Uniqueness and reciprocal theorems are proved, and the convolutional variational principle is established and used to prove an uniqueness theorem with no restriction on the elasticity or thermal conductivity tensors except symmetry conditions. The two-temperature dynamic coupled, Lord-Shulman and fractional coupled thermoelasticity theories result as limit cases. For fractional thermoelasticity not involving two-temperatures, El-Karamany and Ezzat [38] established the uniqueness, reciprocal theorems and convolution variational principle. The dynamic coupled and Green-Naghdi thermoelasticity theories result as limit cases. The reciprocity relation in case of quiescent initial state is found to be independent of the order of differ-integration [37] and [38, 39]. Fractional order theory of a perfect conducting thermoelastic medium not involving two temperatures was investigated by Ezzat and El-Karamany [40]. The finite thermal wave propagation in an infinite half-space under this theory has been studied by Sur and Kanoria [41].

Recently, Roychoudhuri and Dutta [42] proposed a theory with a three-phase lag, which is able to contain all the previous theories at the same time. In this case, the Fourier law is replaced by an approximation of the equation $\vec{q}(P,t+\tau_q) = -\left[K\vec{\nabla}T(P,t+\tau_T) + K^{\phi}\vec{\nabla}\nu(P,t+\tau_\nu)\right]$, where $\vec{q}$ is the heat flux vector, ν is the thermal displacement that satisfies $\dot{\nu}=T$, K is the thermal conductivity of the thermoelastic material, $K^{\phi}\ (>0)$ is a material constant characteristic of the theory. This relation states that the heat flux vector at a point *P* in the material at time $t+\tau_q$ is related to the temperature at the same point at time $t+\tau_T$ and the thermal displacement at the same material point at time $t+\tau_\nu$. The delay times are caused by microstructural interactions, such as phonon scattering or phonon-electron interactions, and they are interpreted as the relaxation time due to fast transient effects of thermal inertia. The purpose of this work is to establish a mathematical model that includes three-phase lags in the heat flux vector, the temperature gradient and in the thermal displacement gradient. For this model, we can consider several kinds of Taylor approximations to recover the previously cited theories. In particular, the models of Green and Naghdi are recovered. This theory seems to be an extension of the one proposed by Tzou [43]. Recently Jumarie [36] and Ezzat [44] developed a new mathematical model using a new Taylor expansion on this model and retaining terms up to the $\alpha-$order terms in τ_T and τ_ν and terms up to the $2\alpha-$order terms in τ_q. One can obtain the generalized theory of heat conduction with time fractional order α valid at a point P at a time t as $\left(1+\frac{\tau_q^{\alpha}}{\alpha!}\frac{\partial^{\alpha}}{\partial t^{\alpha}}+\frac{\tau_q^{2\alpha}}{(2\alpha)!}\frac{\partial^{2\alpha}}{\partial t^{2\alpha}}\right)\vec{q} = -\left(\tau_\nu^{\phi}\vec{\nabla}T + K\frac{\tau_T^{\alpha}}{\alpha!}\frac{\partial^{\alpha}}{\partial t^{\alpha}}\vec{\nabla}T + K^{\phi}\vec{\nabla}\nu\right),\ 0<\alpha\le 1$, where $\tau_\nu^{\phi} = K + \frac{K^{\phi}\tau_\nu^{\alpha}}{\alpha!}\frac{\partial^{\alpha-1}}{\partial t^{\alpha-1}}$.

This present analysis shows the development of the studies of one dimensional thermoelastic disturbance in an infinite, isotropic, viscoelastic medium in the context of three-phase-lag thermoelastic model, GN model II (TEWOED) and GN model III (TEWED), in presence of the distributed periodically varying heat sources where the heat conduction equation consists of some non-local fractional order α. The governing equations for this problem are taken into Laplace-Fourier transform domain. The solutions for the displacement, temperature, thermal stress and strain in Laplace transform domain are obtained by Fourier inversion, which is carried out by using the residual calculus, where the poles of the integrand are obtained numerically in complex domain by using Laguerre's method. The numerical inversion of the Laplace transform are carried out by using a method based on Fourier series expansion technique [45]. Numerical results for the thermophysical quantities have been obtained for a copper like material and have been presented graphically to study the effect of the fractional order parameter on the studied fields. The effect of viscosity is also shown.

2 FORMULATION OF THE PROBLEM

We consider a isotropic infinitely extended Kelvin-Voigt-type thermo-viscoelastic medium at a uniform reference temperature θ_0 in presence of periodically varying heat sources distributed over a plane area.

The strain and stress tensors *E* and *T* are given by the following geometrical and constitutive relations respectively as:

$$E = \frac{1}{2}\left(\nabla\vec{u} + \nabla\vec{u}^{T}\right), \tag{11}$$

$$T = \lambda^{\phi}\left(\text{div }\vec{u}\right)I + \mu^{\phi}\left(\nabla\vec{u} + \nabla\vec{u}^{T}\right) - \gamma^{\phi}\theta I, \tag{12}$$

where $\vec{u}$ is the displacement vector, θ is the temperature increase with respect to the uniform reference temperature and I is a (3×3) unit matrix and the subscript T is the transpose of a matrix.

The governing field equations for the dynamic coupled generalized visco-thermoelasticity in absence of body forces based on the three-phase-lag thermoelasticity model are written as:

$$\mu^{\phi}\nabla^2\vec{u} + (\lambda^{\phi} + \mu^{\phi})\text{grad div }\vec{u} - \gamma^{\phi}\text{grad }\theta = \rho\ddot{\vec{u}}, \tag{13}$$

$$\left(1+\frac{\tau_q^{\alpha}}{\alpha!}\frac{\partial^{\alpha}}{\partial t^{\alpha}} + \frac{\tau_q^{2\alpha}}{(2\alpha)!}\frac{\partial^{2\alpha}}{\partial t^{2\alpha}}\right)\left(\rho c_v\ddot{\theta} + \gamma^{\phi}\theta_0\text{div }\ddot{\vec{u}} - \rho\dot{Q}\right) = \left(K^{\phi} + \tau_v^{\phi}\frac{\partial}{\partial t} + K\frac{\tau_T^{\alpha}}{\alpha!}\frac{\partial^{\alpha+1}}{\partial t^{\alpha+1}}\right)\nabla^2\theta, \tag{14}$$

where ρ is the density, c_v is the specific heat at constant strain and the parameters $\lambda^{\phi}, \mu^{\phi}$ and γ^{ϕ} are defined as:

$$\lambda^{\phi} = \lambda_e\left(1+\alpha_0\frac{\partial}{\partial t}\right), \quad \mu^{\phi} = \mu_e\left(1+\alpha_1\frac{\partial}{\partial t}\right), \quad \gamma^{\phi} = \gamma_e\left(1+\gamma_0\frac{\partial}{\partial t}\right)$$

where $\gamma_e = (3\lambda_e + 2\mu_e)\alpha_t, \gamma_0 = (3\lambda_e\alpha_0 + 2\mu_e\alpha_1)\alpha_t/\gamma_e; \lambda_e, \mu_e$ being Lame's constants, α_0, α_1 are the visco-thermoelastic relaxation times, α_t is the coefficient of linear thermal expansion, K^{ϕ} is an additional material constant, K is the thermal conductivity, Q is the rate of internal heat generation per unit mass, $\dot{v} = \theta, v$ being the thermal displacement, $\tau_v^{\phi} = K + K^{\phi}\frac{\tau_v^{\alpha}}{\alpha!}\frac{\partial^{\alpha-1}}{\partial t^{\alpha-1}}$, the delay time τ_v is called the phase-lag of thermal displacement gradient and τ_q is called the phase-lag of the heat flux. Here dot denotes derivatives with respect to time. GN theory type III and type II can be recovered from Eq. (14) by taking $\tau_q = \tau_T = \tau_v = 0$ and $\tau_q = \tau_T = \tau_v = 0, K << K^{\phi}$ respectively.

Introducing the following nondimensional variables

$$\begin{gathered} x' = \frac{x}{l}, \ t' = \frac{vt}{l}, \ \theta' = \frac{\theta-\theta_0}{\theta_0}, \ u' = \frac{(\lambda_e + 2\mu_e)u}{l\gamma_e\theta_0}, \tau'_{x'x'} = \frac{\tau_{xx}}{\gamma_0\theta_0}, \ e'_{x'x'} = e_{xx}, \\ \tau'_q = \frac{\tau_q v}{l}, \ \tau'_T = \frac{\tau_T v}{l}, \tau'_v = \frac{\tau_v v}{l}, \ \alpha'_0 = \frac{\alpha_0 v}{l}, \ \alpha'_1 = \frac{\alpha_1 v}{l}, \ \gamma'_0 = \frac{\gamma_0 v}{l}, \end{gathered} \tag{15}$$

where l is a standard length and v is a standard speed, then, after removing primes, Eqs. (11)-(14) can be written in non-dimensional form as follows

$$E = \frac{\gamma_e\theta_0}{\left(\lambda_e + 2\mu_e\right)}\left(\nabla\vec{u} + \nabla\vec{u}^{T}\right), \tag{16}$$

$$T = \left(1-\frac{2C_S^2}{C_P^2}\right)\left(1+\alpha_0\frac{\partial}{\partial t}\right)\left(\text{div }\vec{u}\right)I + \frac{C_S^2}{C_P^2}\left(1+\alpha_1\frac{\partial}{\partial t}\right)\left(\nabla\vec{u} + \nabla\vec{u}^{T}\right) - \left(1+\gamma_0\frac{\partial}{\partial t}\right)\theta I, \tag{17}$$

$$C_S^2\left(1+\alpha_1\frac{\partial}{\partial t}\right)\nabla^2\vec{u} + \left[\left(C_P^2 - C_S^2\right) + \left\{\alpha_0\left(C_P^2 - 2C_S^2\right) + \alpha_1 C_S^2\right\}\frac{\partial}{\partial t}\right]\text{grad div }\vec{u} - C_P^2\left(1+\gamma_0\frac{\partial}{\partial t}\right)\text{grad }\theta = \ddot{\vec{u}}, \tag{18}$$

$$\left(1+\frac{\tau_q^{\alpha}}{\alpha!}\frac{\partial^{\alpha}}{\partial t^{\alpha}}+\frac{\tau_q^{2\alpha}}{(2\alpha)!}\frac{\partial^{2\alpha}}{\partial t^{2\alpha}}\right)\left[\ddot{\theta}+\varepsilon\left(1+\gamma_0\frac{\partial}{\partial t}\right)\operatorname{div}\ddot{\vec{u}}-Q_0\right]=C_T^2\left(1+\frac{\tau_\nu^{\alpha}}{\alpha!}\frac{\partial^{\alpha}}{\partial t^{\alpha}}\right)\nabla^2\theta+C_K^2\left(\frac{\partial}{\partial t}+\frac{\tau_T^{\alpha}}{\alpha!}\frac{\partial^{\alpha+1}}{\partial t^{\alpha+1}}\right)\nabla^2\theta, \tag{19}$$

where

$$C_P=\frac{1}{v}\sqrt{\frac{\lambda_e+2\mu_e}{\rho}},\ C_S=\frac{1}{v}\sqrt{\frac{\mu_e}{\rho}},\ C_T=\frac{1}{v}\sqrt{\frac{K^{\phi}}{\rho c_v}},\ C_K=\sqrt{\frac{K}{\rho c_v l v}},\ Q_0=\frac{\dot{Q}l}{c_v\theta_0 v},\ \varepsilon=\frac{\gamma_e^2\theta_0}{\rho_0 c_v\left(\lambda_e+2\mu_e\right)},$$

and it is to be noted that GN model III and model II can be recovered from Eq. (19) by taking $\tau_q=\tau_T=\tau_\nu=0$ and $\tau_q=\tau_T=\tau_\nu=0,\ K<<K^{\phi}$ respectively. In the previous expressions, C_P, C_S, C_T represent non-dimensional dilatational, shear and thermal wave velocities respectively, C_K is the damping coefficient and ε is the thermoelastic coupling constant.

We now consider one dimensional disturbance of the medium so that the displacement vector $\vec{u}$ and the temperature θ can be taken in the following form

$$\vec{u}=\left(u(x,t),0,0\right),\qquad \theta=\theta(x,t). \tag{20}$$

Also, we assume that the medium is initially at rest and the undisturbed state is maintained at uniform reference temperature. Then we have

$$u(x,0)=\dot{u}(x,0)=\theta(x,0)=\dot{\theta}(x,0)=0. \tag{21}$$

In the present problem, Eqs. (16)-(19) reduce to

$$\tau_{xx}(x,t)=\left[1+\left\{\alpha_0+\left(\alpha_1-\alpha_0\right)\frac{2C_S^2}{C_P^2}\right\}\frac{\partial}{\partial t}\right]\frac{\partial u}{\partial x}-\left(1+\gamma_0\frac{\partial}{\partial t}\right)\theta, \tag{22}$$

$$e_{xx}(x,t)=\beta_1\frac{\partial u}{\partial x}\qquad \text{where}\quad \beta_1=\frac{\gamma_e\theta_0}{\lambda_e+2\mu_e}. \tag{23}$$

$$\left[C_P^2+\left\{\alpha_0\left(C_P^2-2C_S^2\right)+2\alpha_1 C_S^2\right\}\frac{\partial}{\partial t}\right]\frac{\partial^2 u}{\partial x^2}-C_P^2\left(1+\gamma_0\frac{\partial}{\partial t}\right)\frac{\partial\theta}{\partial x}=\frac{\partial^2 u}{\partial t^2}, \tag{24}$$

$$\left(1+\frac{\tau_q^{\alpha}}{\alpha!}\frac{\partial^{\alpha}}{\partial t^{\alpha}}+\frac{\tau_q^{2\alpha}}{(2\alpha)!}\frac{\partial^{2\alpha}}{\partial t^{2\alpha}}\right)\left[\ddot{\theta}+\varepsilon\left(1+\gamma_0\frac{\partial}{\partial t}\right)\operatorname{div}\ddot{\vec{u}}-Q_0\right]=C_T^2\left(1+\frac{\tau_\nu^{\alpha}}{\alpha!}\frac{\partial^{\alpha}}{\partial t^{\alpha}}\right)\frac{\partial^2\theta}{\partial x^2}+C_K^2\left(\frac{\partial}{\partial t}+\frac{\tau_T^{\alpha}}{\alpha!}\frac{\partial^{\alpha+1}}{\partial t^{\alpha+1}}\right)\frac{\partial^2\theta}{\partial x^2}, \tag{25}$$

Let us define Laplace-Fourier double transform of a function $g(x,t)$ by

$$\begin{aligned}\bar{f}(x,p)&=\int_0^{\infty}e^{-pt}f(x,t)\,dt,\quad \operatorname{Re}(p)>0,\\ \hat{\bar{f}}(\xi,p)&=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{i\xi x}\bar{f}(x,p)\,dx.\end{aligned} \tag{26}$$

Applying Laplace-Fourier double integral transform to the Eqs. (22)-(25), we have

$$\hat{\bar{\tau}}_{xx}(\xi,p) = -i\xi\left[1+\left\{\alpha_0+\left(\alpha_1-\alpha_0\right)\frac{2C_S^2}{C_P^2}\right\}p\right]\hat{\bar{u}}(\xi,p)-\left(1+\gamma_0 p\right)\hat{\bar{\theta}}(\xi,p), \tag{27}$$

$$\hat{\bar{e}}_{xx}(\xi,p) = -i\xi\beta_1\hat{\bar{u}}(\xi,p), \tag{28}$$

$$\hat{\bar{u}}(\xi,p) = \frac{i\xi C_P^2(1+\gamma_0 p)}{\left[p^2+\left\{C_P^2+\left\{\alpha_0\left(C_P^2-2C_S^2\right)+2\alpha_1 C_S^2\right\}p\right\}\xi^2\right]}\hat{\bar{\theta}}(\xi,p), \tag{29}$$

$$\begin{aligned}&\left[\left(1+\frac{\tau_q^\alpha}{\alpha!}p^\alpha+\frac{\tau_q^{2\alpha}}{(2\alpha)!}p^{2\alpha}\right)p^2+\xi^2 C_T^2\left(1+\frac{\tau_\nu^\alpha}{\alpha!}p^\alpha\right)+\xi^2 C_K^2\left(p+\frac{\tau_T^\alpha}{\alpha!}p^{\alpha+1}\right)\right]\hat{\bar{\theta}}(\xi,p)=\\&-i\xi\varepsilon p^2(1+\gamma_0 p)\left(1+\frac{\tau_q^\alpha}{\alpha!}p^\alpha+\frac{\tau_q^{2\alpha}}{(2\alpha)!}p^{2\alpha}\right)\hat{\bar{u}}(\xi,p)=\left(1+\frac{\tau_q^\alpha}{\alpha!}p^\alpha+\frac{\tau_q^{2\alpha}}{(2\alpha)!}p^{2\alpha}\right)\hat{\bar{Q}}_0,\end{aligned} \tag{30}$$

Solving Eqs. (29) and (30) for $\hat{\bar{u}}(\xi,p)$ and $\hat{\bar{\theta}}(\xi,p)$ we get

$$\hat{\bar{u}}(\xi\, p) = \frac{i\xi C_P^2(1+\gamma_0 p)\left(1+\frac{\tau_q^\alpha}{\alpha!}p^\alpha+\frac{\tau_q^{2\alpha}}{(2\alpha)!}p^{2\alpha}\right)\hat{\bar{Q}}_0}{M^\phi(\xi)}, \tag{31}$$

$$\hat{\bar{\theta}}(\xi\, p) = \frac{\hat{\bar{Q}}_0\left[p^2+\left\{C_P^2+\left\{\alpha_0\left(C_P^2-2C_S^2\right)+2\alpha_1 C_S^2\right\}p\right\}\xi^2\right]\left(1+\frac{\tau_q^\alpha}{\alpha!}p^\alpha+\frac{\tau_q^{2\alpha}}{(2\alpha)!}p^{2\alpha}\right)}{M^\phi(\xi)}, \tag{32}$$

where

$$\begin{aligned}M^\phi(\xi) &= M_1(p)\xi^4+M_2(p)\xi^2+M_3(p)\\&= M_1(p)(\xi-\xi_1)(\xi-\xi_2)(\xi-\xi_3)(\xi-\xi_4),\end{aligned} \tag{33}$$

and $M_1(p),\ M_2(p),\ M_3(p)$ are given by

$$M_1(p) = \left\{C_T^2\left(1+\frac{\tau_\nu^\alpha}{\alpha!}p^\alpha\right)+C_K^2\left(p+\frac{\tau_T^\alpha}{\alpha!}p^{\alpha+1}\right)\right\}\left\{C_P^2+\left\{\alpha_0\left(C_P^2-2C_S^2\right)+2\alpha_1 C_S^2\right\}p\right\}, \tag{34}$$

$$\begin{aligned}M_2(p) = p^2\left\{C_T^2\left(1+\frac{\tau_\nu^\alpha}{\alpha!}p^\alpha\right)+C_K^2\left(p+\frac{\tau_T^\alpha}{\alpha!}p^{\alpha+1}\right)\right\}+\left\{C_P^2+\left\{\alpha_0\left(C_P^2-2C_S^2\right)+2\alpha_1 C_S^2\right\}p+\varepsilon p^2(1+\gamma_0 p)^2\right\}\\ \times\left(1+\frac{\tau_\nu^\alpha}{\alpha!}p^\alpha+\frac{\tau_\nu^{2\alpha}}{(2\alpha)!}p^{2\alpha}\right),\end{aligned} \tag{35}$$

$$M_3(p) = p^4\left(1+\frac{\tau_\nu^\alpha}{\alpha!}p^\alpha+\frac{\tau_\nu^{2\alpha}}{(2\alpha)!}p^{2\alpha}\right). \tag{36}$$

The expression for strain and stress in Laplace-Fourier transform domain can be obtained from Eqs. (27) and (28) using (31) and (32) as follows

$$\hat{\bar{e}}_{xx}(\xi\, p) = \frac{\xi^2 \beta_1 C_P^2 \hat{\bar{Q}}_0 (1+\gamma_0 p)\left(1+\frac{\tau_q^{\alpha}}{\alpha!}p^{\alpha}+\frac{\tau_q^{2\alpha}}{(2\alpha)!}p^{2\alpha}\right)}{M^{\phi}(\xi)}, \tag{37}$$

$$\hat{\bar{\tau}}_{xx}(\xi\, p) = \frac{-\hat{\bar{Q}}_0 (1+\gamma_0 p)\left(p^2+2\alpha_1 p\xi^2 C_S^2\right)\left(1+\frac{\tau_q^{\alpha}}{\alpha!}p^{\alpha}+\frac{\tau_q^{2\alpha}}{(2\alpha)!}p^{2\alpha}\right)}{M^{\phi}(\xi)}. \tag{38}$$

Inverse Fourier transforms of the Eqs. (31), (32), (37) and (38) give the following solutions for displacement, temperature, strain and stress in Laplace transform domain

$$\bar{u}(x,p) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{i\xi \hat{\bar{Q}}_0 C_P^2 (1+\gamma_0 p)\left(1+\frac{\tau_q^{\alpha}}{\alpha!}p^{\alpha}+\frac{\tau_q^{2\alpha}}{(2\alpha)!}p^{2\alpha}\right)}{M^{\phi}(\xi)} e^{-i\xi x} d\xi \tag{39}$$

$$\bar{\theta}(x,p) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{\hat{\bar{Q}}_0 \left[p^2+\left\{C_P^2+\left\{\alpha_0\left(C_P^2-2C_S^2\right)+2\alpha_1 C_S^2\right\}p\right\}\xi^2\right]\left(1+\frac{\tau_q^{\alpha}}{\alpha!}p^{\alpha}+\frac{\tau_q^{2\alpha}}{(2\alpha)!}p^{2\alpha}\right)}{M^{\phi}(\xi)} e^{-i\xi x} d\xi \tag{40}$$

$$\bar{e}_{xx}(x,p) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{\xi^2 \beta_1 C_P^2 \hat{\bar{Q}}_0 (1+\gamma_0 p)\left(1+\frac{\tau_q^{\alpha}}{\alpha!}p^{\alpha}+\frac{\tau_q^{2\alpha}}{(2\alpha)!}p^{2\alpha}\right)}{M^{\phi}(\xi)} e^{-i\xi x} d\xi \tag{41}$$

$$\bar{\tau}_{xx}(x,p) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{-\hat{\bar{Q}}_0 (1+\gamma_0 p)\left(p^2+2\alpha_1 p\xi^2 C_S^2\right)\left(1+\frac{\tau_q^{\alpha}}{\alpha!}p^{\alpha}+\frac{\tau_q^{2\alpha}}{(2\alpha)!}p^{2\alpha}\right)}{M^{\phi}(\xi)} e^{-i\xi x} d\xi \tag{42}$$

2.1 Periodically varying heat source

Now we assume that the heat sources are distributed over the plane $x = 0$ in the following form

$$\begin{aligned} Q_0 &= Q_0^{\phi}\delta(x)\sin\left(\frac{\pi t}{\tau}\right) && \text{for } 0 \le t \le \tau, \\ &= 0 && \text{for } t > \tau, \end{aligned} \tag{43}$$

Then

$$\hat{\bar{Q}}_0 = \frac{Q_0^{\phi}\pi\tau\left(1+e^{-p\tau}\right)}{\sqrt{2\pi}\left(\pi^2+p^2\tau^2\right)}. \tag{44}$$

Thus, the expressions for thermal displacement, temperature, strain and thermal stress in Laplace transform domain take the following form

$$\bar{u}(x,p)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{i\xi C_P^2 Q_0^{\phi}\tau\left(1+e^{-p\tau}\right)(1+\gamma_0 p)\left(1+\frac{\tau_q^{\alpha}}{\alpha!}p^{\alpha}+\frac{\tau_q^{2\alpha}}{(2\alpha)!}p^{2\alpha}\right)}{2\left(\pi^2+p^2\tau^2\right)M^{\phi}(\xi)}e^{-i\xi x}d\xi \tag{45}$$

$$\bar{\theta}(x,p)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{Q_0^{\phi}\tau\left(1+e^{-p\tau}\right)\left[p^2+\left\{C_P^2+\left\{\alpha_0\left(C_P^2-2C_S^2\right)+2\alpha_1 C_S^2\right\}p\right\}\xi^2\right]\left(1+\frac{\tau_q^{\alpha}}{\alpha!}p^{\alpha}+\frac{\tau_q^{2\alpha}}{(2\alpha)!}p^{2\alpha}\right)}{2\left(\pi^2+p^2\tau^2\right)M^{\phi}(\xi)}e^{-i\xi x}d\xi \tag{46}$$

$$\bar{e}_{xx}(x,p)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\xi^2\beta_1 C_P^2 Q_0^{\phi}\tau\left(1+e^{-p\tau}\right)(1+\gamma_0 p)\left(1+\frac{\tau_q^{\alpha}}{\alpha!}p^{\alpha}+\frac{\tau_q^{2\alpha}}{(2\alpha)!}p^{2\alpha}\right)}{2\left(\pi^2+p^2\tau^2\right)M^{\phi}(\xi)}e^{-i\xi x}d\xi \tag{47}$$

$$\bar{\tau}_{xx}(x,p)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{-Q_0^{\phi}\tau\left(1+e^{-p\tau}\right)(1+\gamma_0 p)\left(p^2+2\alpha_1 p\xi^2 C_S^2\right)\left(1+\frac{\tau_q^{\alpha}}{\alpha!}p^{\alpha}+\frac{\tau_q^{2\alpha}}{(2\alpha)!}p^{2\alpha}\right)}{2\left(\pi^2+p^2\tau^2\right)M^{\phi}(\xi)}e^{-i\xi x}d\xi \tag{48}$$

Applying contour integration to Eqs. (45)-(48), we obtain

$$\begin{aligned}\bar{u}(x,p)&=Q_0^{\phi}\pi\tau N(p)\sum_{\substack{j=1\\ \mathrm{Im}(\xi_j)<0}}^{4}A_j\xi_j C_P^2 e^{-i\xi_j x} && \text{for } x>0\\ &=-Q_0^{\phi}\pi\tau N(p)\sum_{\substack{j=1\\ \mathrm{Im}(\xi_j)>0}}^{4}A_j\xi_j C_P^2 e^{-i\xi_j x} && \text{for } x<0,\end{aligned} \tag{49}$$

$$\begin{aligned}\bar{\theta}(x,p)&=-\frac{iQ_0^{\phi}\pi\tau N(p)}{(1+\gamma_0 p)}\sum_{\substack{j=1\\ \mathrm{Im}(\xi_j)<0}}^{4}A_j\Im(\xi_j)e^{-i\xi_j x} && \text{for } x>0\\ &=\frac{iQ_0^{\phi}\pi\tau N(p)}{(1+\gamma_0 p)}\sum_{\substack{j=1\\ \mathrm{Im}(\xi_j)>0}}^{4}A_j\Im(\xi_j)e^{-i\xi_j x} && \text{for } x<0,\end{aligned} \tag{50}$$

$$\begin{aligned}\bar{e}_{xx}(x,p)&=-iQ_0^{\phi}\pi\tau N(p)C_P^2\sum_{\substack{j=1\\ \mathrm{Im}(\xi_j)<0}}^{4}A_j\beta_1\xi_j^2 e^{-i\xi_j x} && \text{for } x>0\\ &=iQ_0^{\phi}\pi\tau N(p)C_P^2\sum_{\substack{j=1\\ \mathrm{Im}(\xi_j)>0}}^{4}A_j\beta_1\xi_j^2 e^{-i\xi_j x} && \text{for } x<0,\end{aligned} \tag{51}$$

$$\begin{aligned}\bar{\tau}_{xx}(x,p)&=iQ_0^{\phi}\pi\tau N(p)\sum_{\substack{j=1\\ \mathrm{Im}(\xi_j)<0}}^{4}A_j\left(p^2+2\alpha_1 p\xi_j^2 C_S^2\right)e^{-i\xi_j x} && \text{for } x>0\\ &=-iQ_0^{\phi}\pi\tau N(p)\sum_{\substack{j=1\\ \mathrm{Im}(\xi_j)>0}}^{4}A_j\left(p^2+2\alpha_1 p\xi_j^2 C_S^2\right)e^{-i\xi_j x} && \text{for } x<0,\end{aligned} \tag{52}$$

where A_j 's are given by

$$A_j = \prod_{\substack{n=1 \\ n \neq j}}^{4} \frac{1}{\left(\xi_j - \xi_n\right)}, \quad j = 1,2,3,4, \tag{53}$$

$$N(p) = \frac{\left(1+e^{-p\tau}\right)\left(1+\gamma_0 p\right)\left(1+\frac{\tau_q^{\alpha}}{\alpha!}p^{\alpha}+\frac{\tau_q^{2\alpha}}{(2\alpha)!}p^{2\alpha}\right)}{M_1(p)\left(\pi^2+p^2\tau^2\right)}, \tag{54}$$

And

$$\Im(\xi_j) = \left[p^2 + \left\{C_P^2 + \left\{\alpha_0\left(C_P^2 - 2C_S^2\right) + 2\alpha_1 C_S^2\right\}p\right\}\xi_j^2\right]. \tag{55}$$

3 NUMERICAL INVERSION OF LAPLACE TRANSFORM

Let $\bar{f}(x,p)$ be the Laplace transform of a function $f(x,t)$. Then the inversion formula for Laplace transform can be written as:

$$f(x,t) = \frac{1}{2\pi i}\int_{d-i\infty}^{d+i\infty} e^{pt}\bar{f}(x,p)dp, \tag{56}$$

where d is an arbitrary small real number greater than the real part of all the singularities of $\bar{f}(x,p)$.

Taking $p = d + iw$, the preceding integral takes the form

$$f(x,t) = \frac{e^{dt}}{2\pi}\int_{-\infty}^{\infty} e^{itw}\bar{f}(x,d+iw)dw, \tag{57}$$

Expanding the function $h(x,t) = e^{-dt}f(x,t)$ in a Fourier series in the interval $[0,2T]$ we obtain the approximate formula [45],

$$f(x,t) = f_{\infty}(x,t) + E_D, \tag{58}$$

where

$$f(x,t) = \frac{1}{2}c_0 + \sum_{k=1}^{\infty} c_k \qquad \text{for } 0 \le t \le 2T \tag{59}$$

And

$$c_k = \frac{e^{dt}}{T}\left[e^{\frac{ik\pi t}{T}} f\left(x, d + \frac{ik\pi t}{T}\right)\right]. \tag{60}$$

The discretization error E_D can be made arbitrary small by choosing d large enough [45]. Since the infinite series in Eq. (59) can be summed up to a finite number N of terms, the approximate value of $f(x,t)$ becomes

$$f_N(x,t)=\frac{1}{2}c_0+\sum_{k=1}^{N}c_k \qquad \text{for} \quad 0\le t\le 2T. \tag{61}$$

Using the preceding formula to evaluate $f(x,t)$ we introduce a truncation error E_T that must be added to the discretization error. Next the $\varepsilon-$algorithm is used to accelerate the convergence [45].

The Korrecktur method uses the following formula to evaluate the function $f(x,t)$

$$f(x,t)=f_\infty(x,t)+e^{-2dT}f_\infty(x,2T+t)+E'_D. \tag{62}$$

where the discretization error $|E'_D|<<|E_D|$. Thus the approximate value of $f(x,t)$ becomes

$$f_{NK}(x,t)=f_N(x,t)+e^{-2dT}f_{N'}(x,2T+t), \tag{63}$$

where N' is an integer such that $N'<N$.

We shall now describe the $\varepsilon-$algorithm that is used to accelerate the convergence of the series in Eq. (61). Let $N=2q+1$, where q is a natural number and let $s_m=\sum_{k=1}^{m}c_k$ be the sequence of partial sum of the series in (61).

We define the $\varepsilon-$sequence by $\varepsilon_{0,m}=0,\ \varepsilon_{1,m}=s_m$ and $\varepsilon_{r+1,m}=\varepsilon_{r-1,m+1}+\dfrac{1}{\varepsilon_{r,m+1}-\varepsilon_{r,m}};\ r=1,2,3,\ldots$

It can be shown that [45] the sequence $\varepsilon_{1,1},\varepsilon_{3,1},\varepsilon_{5,1},\ldots,\varepsilon_{N,1}$ converges to $f(x,t)+E_D-\dfrac{c_0}{2}$ faster than the sequence of partial sums $s_m, m=1,2,3,\ldots$

The actual procedure used to invert the Laplace transform consists of using Eq. (63) together with the $\varepsilon-$algorithm. The values of d and T are chosen so according to the criterion outlined in [45].

4 NUMERICAL RESULTS AND DISCUSSIONS

To get the solution for thermal displacement, temperature, thermal stress in space-time domain we have to apply Laplace inversion formula to the Eqs. (49)-(52) respectively. This has been done numerically using a method based on Fourier series expansion technique. To get the roots of the polynomial $M(\xi)$ in complex domain we have used Laguerre's method. For computational purpose, copper like material has been taken into consideration. The values of the material constants are taken as follows [42]

$$\varepsilon=0.0168,\ \lambda_e=1.387\times10^{11}\,\text{N/m}^2,\ \mu_e=0.448\times10^{11}\,\text{N/m}^2,\ \alpha_t=1.67\times10^{-8}/\text{K},\theta_0=1\text{K}$$

and the hypothetical values of the phase-lag parameters are taken as:

$$\alpha_0=0.05s,\quad \alpha_1=0.1s,\quad \tau_q=0.001s,\quad \tau_T=0.05s,\quad \tau_\nu=0.05s.$$

Also, we have taken $Q_0^\phi=1,\ \tau=1,\ \ C_P=1, C_T=2, C_K=0.6,$ so that the faster wave is the thermal wave. In order to study the effect of the non-local fractional parameter α on thermal displacement, temperature and stress field for a viscous material, we now present the results in form of the graphical representation for three different models (3P lag model, GN II, GN III).

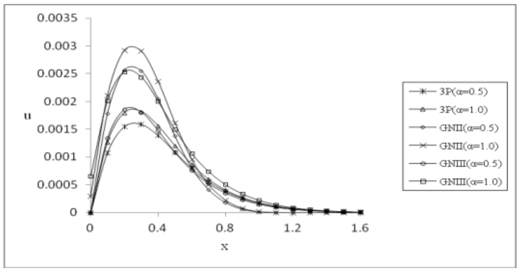

Fig.1
Variation of u versus x for $t = 0.6$ for viscous material.

Fig. 1 depicts the variation of the displacement component u versus the distance x when t = 0.6 for different values of the non-local fractional parameter (α = 0.5,1.0) for three different models. From the figure, it is observed that with the increase of x , the displacement component will increase its maximum near x = 0.3 and then it falls to zero for all the models. Also, the magnitude of the profile of u is larger for α = 1.0 than that of α = 0.5 for $0 \prec x \prec 0.6$ and beyond this, u falls to zero. Furthermore, the decay of the magnitude is faster for GN II model than that of GN III model which is again faster than that of 3P lag model.

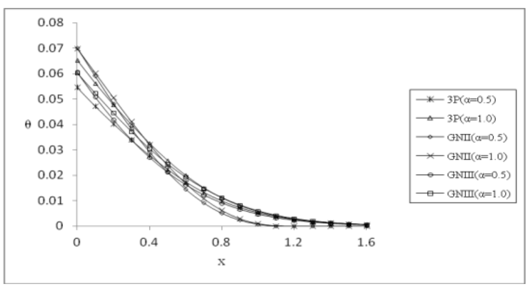

Fig.2
Variation of θ versus x for $t = 0.6$ for viscous material.

Fig. 2 is plotted to study the effect of the fractional parameter α on the temperature field θ when $t = 0.6$. It is seen that the temperature is maximum near the plane $x = 0$ where the heat source is applied, and, with the increase of the distance, θ diminishes to zero. This can also be verified from the expression of $\bar{\theta}$ given in Eq. (50) involving $e^{-i\xi_j x}, \mathrm{Im}(\xi_j) < 0$ for $x > 0$. Further, the increase of the fractional parameter α will increase the profile of θ in between $0 < x < 0.5$.

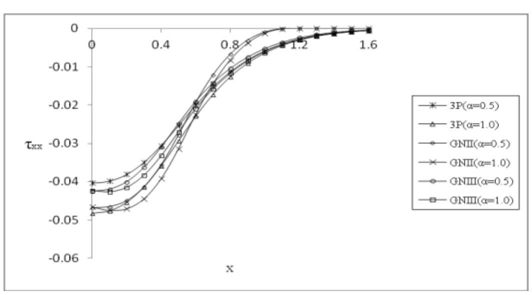

Fig.3
Variation of τ_{xx} versus x for $t = 0.6$ for viscous material.

Fig. 3 is plotted to study the effect of fractional parameter α on the thermal stress τ_{xx} for a viscous material when $t = 0.6$. It is seen that near the plane of application of the heat source, the thermal stress is compressive in nature for different α, which is quite plausible. Also, the magnitude of τ_{xx} decays sharply after $x = 0.8$ for GN II model compared to GN III model, decay of which is again faster than 3P lag model.

Figs. 4-6 depict the variation of the thermophysical quantities versus x for different values of the fractional parameters $(\alpha = 0.5, 1.0)$ for 3P lag model.

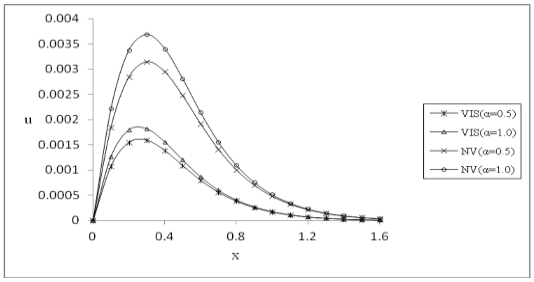

Fig.4
Variation of u versus x for $t = 0.6$ and $\alpha = 0.5, 1.0$ for 3P-lag model.

Fig. 4 depicts the variation of displacement u versus distance x when $t = 0.6$ for viscous and nonviscous materials. It is observed that due to the presence of the viscosity term, the peak of the thermal displacement is larger for non-viscous material compared to that of a viscous one. Further, it is observed that for both viscous and non viscous material, the peak of the thermal displacement becomes larger with the increase of the nonlocal fractional parameter α.

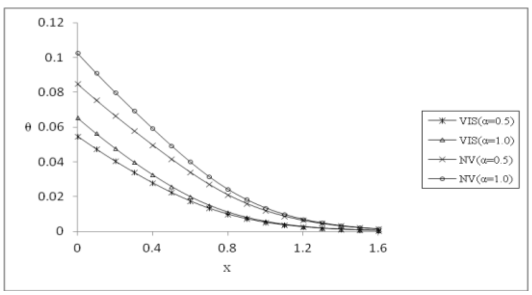

Fig.5
Variation of θ versus x for $t = 0.6$ and $\alpha = 0.5, 1.0$ for 3P-lag model.

Fig. 5 depicts the variation of θ against the distance x for different fractional parameter α when $t = 0.6$. A similar qualitative behavior in the nature of θ is seen.

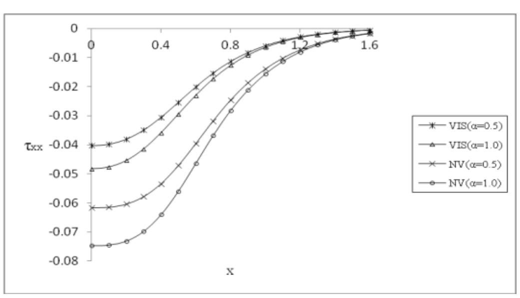

Fig.6
Variation of τ_{xx} versus x for $t = 0.6$ and $\alpha = 0.5, 1.0$ for 3P-lag model.

Fig. 6 depicts the variation of τ_{xx} against the distance x for $t = 0.6$. It is observed from the figure that the stress is compressive in nature near the plane of application of the heat source. Here also, the increase of the fractional parameter α will increase of the magnitude of τ_{xx} for both viscous and non viscous materials.

Figs. 7-9 are plotted to study the effect of viscosity for fractional parameter $\alpha = 0.5$ for three different models when $t = 0.6$.

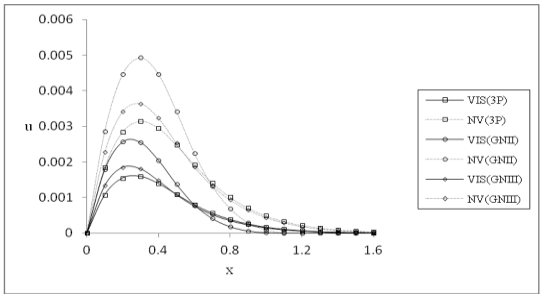

Fig.7
Variation of u versus x for $t = 0.6$ and $\alpha = 0.5$.

Fig. 7 depicts the variation of u against the distance x for both viscous and non viscous materials for three different models. It is seen that with the increase of x, the magnitude of the displacement will increase and finally diminishes to zero. This is because the heat source varies periodically with time for a short duration. This obtained result for GN II model for a non viscous material complies with the results of Roychoudhuri and Dutta [42] which were obtained analytically.

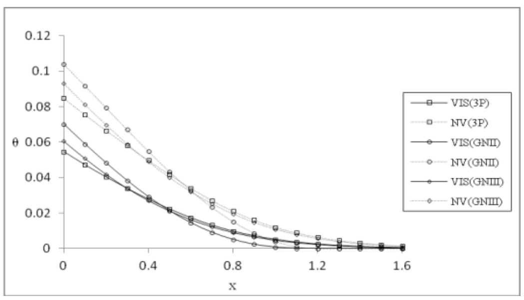

Fig.8
Variation of θ versus x for $t = 0.6$ and $\alpha = 0.5$.

Fig. 8 depicts the variation of θ versus x for both viscous and non viscous materials. From the figure, it is seen that the effect of the viscosity will decrease the peak of the magnitude of the thermoelastic material and for a viscous material, the temperature diminishes to zero faster compared to that of non viscous material.

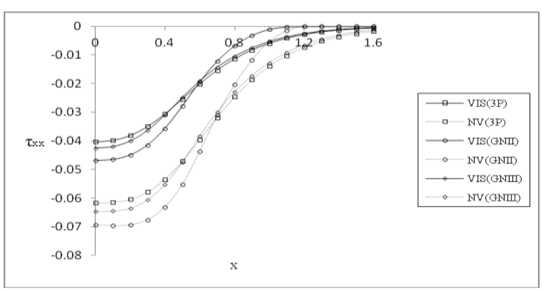

Fig.9
Variation of τ_{xx} versus x for $t = 0.6$ and $\alpha = 0.5$.

Fig. 9 shows the effect of viscosity on τ_{xx} for $t = 0.6$. Here also, effect of thermal stress is more prominent in non viscous material than that of a viscous material.

5 CONCLUSIONS

The present problem of investigating the thermophysical quantities in an isotropic Kelvin-Voigt viscoelastic material subjected to a periodically varying heat source is studied in the light of fractional order generalized thermoelasticity theories with three different models (GN II, GNIII, 3P lag model). The analysis of the results permits some concluding remarks.

1. It is seen that the increase of the non-local fractional order parameter will increase the magnitude of the thermophysical quantities.
2. Here all the results for a 3P lag model, for fractional parameter $\alpha = 1$, agree with the existing literature [47].

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