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Stress Analysis in Thermosensitive Elliptical Plate with Simply Supported Edge and Impulsive Thermal Load

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ABSTRACT

The paper concerns the thermoelastic problems in a thermosensitive elliptical plate subjected to the activity of a heat source which changes its place on the plate surface with time. The solution of conductivity equation and the corresponding initial and boundary conditions is obtained by employing a new integral transform technique. In addition, the intensities of bending moments, resultant force, etc. are formulated involving the Mathieu and modified functions and their derivatives. The analytical solution for the thermal stress components is obtained in terms of resultant forces and resultant moments.

Keywords: Elliptical plate; Thermosensitive; Temperature distribution; Thermal stresses; Mathieu function; Thermal moment.

1 INTRODUCTION

THE mechanical properties, such as thermal conductivity, the coefficient of linear thermal expansion and the Young's Modulus in materials subjected to thermal loads due to high temperature, high gradient temperature, and cyclical changes of temperature, vary with temperature. Touloukian [1-4] investigated the thermo-physical properties of material in an elevated thermal environment which largely changes with increased temperature. Further few papers were found in which Lee [5] studied the effect of temperature within piezoelectric treatment, Zhu and Chao [6] reported the effect on welding simulation, Shariyat [7] investigated thermal buckling analysis based on a layer wise theory, respectively. Several authors reported investigations concerning the thermomechanical behaviour of temperature-dependent structural elements on the basis of uncoupled or quasi-coupled theories of thermoelasticity. For example, Sugano [8-9] analysed temperature-dependent orthotropic rectangular plate and a perforated plate of variable thickness under unsteady state conditions. Two highly cited literature reviews on thermal stresses in materials with temperature-dependent properties were presented by Noda [10-12]. The main focus of his review was to consider on the papers published after 1980 to the early 1990s and established different analytical procedures to solve the governing differential equations. Neglecting the temperature dependence of material properties during the selection of structural components which are exposed to high-temperature changes have resulted in significant errors. Therefore, some theoretical studies concerning them have been reported so far. For example, Tang [13-14] presented a simple thermal stress analysis for thin plates which includes the temperature-dependent thermal-mechanical properties. The assumptions of the uncoupled theory of thermoelasticity and small deflections of thin plates were used. Tanigawa et al. [15] obtained the heat conduction and thermal stresses integral

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form solution of a temperature-dependent nonhomogeneous infinitely long plate formulated under the mechanical of the traction-free condition. The heat conduction analyses with temperature-sensitive body convective heat transfer taking into account the temperature-dependence of thermos-physical characteristics of materials were carried out by Popovych and Harmatiy [16-17]. In this regard, Rakocha and Popovych [18] investigated the influence of the thermos-sensitivity of materials on the temperature distribution and the components of stress-strain state in a three-layer infinite hollow cylinder subjected to the steady temperature. Similarly, Kushnir and Protsiuk [19] studied nonstationary heat conduction problems for thermosensitive bodies with simple nonlinearity (the coefficients of thermal conductivity and the heat capacity per unit volume depend on temperature, but the coefficient of thermal diffusivity is constant) heated by convective heat exchange from the surrounding medium. Recently, Yevtuchenko [20] considered the linear dependence of the thermo-physical properties of the disk and pad materials to study the boundary-value heat conduction problem by the Kirchhoff transformation and numerical–analytical solution was found by using the integral Laplace transform and the Newton–Raphson methods. Kushnir and Popovych [21] has investigated the thermal stresses in a stressed state thermosensitive body in rotation state taking Titanium alloy Ti-6Al-4V. Later on in [22] they have determined the heat conductivity problem by linearizing it with Kirchhoff's variable for the case of complex heat exchange in which nonlinear boundary conditions dependence on the expression of temperature. Further in Harmatij et al. [23] has succeeded in determining the quasi-static nonlinear nonstationary heat conduction problem and its thermal stresses for a thermosensitive infinite circular cylinder of complex heat exchange and solved numerically by using the method of lines. However, the aforesaid reviewed literature lack in the thermoelastic analysis for an elliptical plate made from temperature-dependent materials with internal heat generation within the body and also taking thermal moments and thermal forces into consideration has been hardly solved. Bhad et al. [24-28] investigated the thermoelastic problems on elliptical objects in which internal heat sources are generated within the solid, with compounded effect due to sectional heating and boundary conditions of the Dirichlet type based on the theory of integral transformations. This is precisely the theme taken up in the present paper.

The aims of this paper are threefold as (i) Using Kirchhoff transform the conductivity equation and the corresponding initial and boundary conditions are transformed to be linear in part and solved it using integral transform technique. (ii) Thermal deflection is obtained based on the small-deflection theory of the plate in the elliptical coordinate system. In addition, the intensities of thermal bending moments and resultant forces are formulated involving the Mathieu and modified functions and their derivatives. (iii) The analytical solution for the thermal stress components is obtained in terms of small deflection and resultant moments. The success of this novel research mainly lies with the new mathematical procedures which present a much simpler approach for optimisation of the design in terms of material usage and performance in engineering problem. Due to the usage of thermosensitive materials in aerodynamic ventures, submarines, car building, games equipment's, and so on in view of their versatility to various circumstances and the relative simplicity of mix with different materials to serve specific purposes and exhibit desirable properties is preferred compared to isotropic materials. In particular for the determination of the thermoelastic behaviour in elliptical plate engaged as the foundation of reactor vessels, turbine engines, space vehicles, furnaces, and refractory industries, etc.

2 FORMULATION OF THE PROBLEM

It is assumed that a thin elliptical plate is occupying the space $D:\{(\xi,\eta,z)\in R^3 : 0<\xi<a, 0<\eta<2\pi, 0<z<\ell\}$ under unsteady-state temperature field due to internal heat source within it. The curves $\eta =$ constant represent a family of confocal hyperbolas while the curves $\xi =$ constant represent a family of confocal ellipses (refer Fig. 1). Both sets of curves intersect each other orthogonally at every point in space. The parameter ξ varies from 0 where it defines the interfocal line, to *a*, the coordinate η is an angular coordinate taking the range $\eta\in[0,2\pi)$, and $z\in(0,\ell)$.

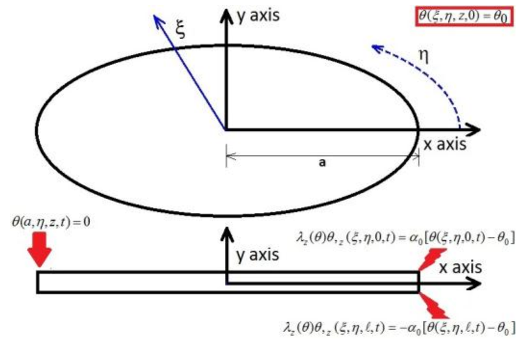


Fig.1
Physical domain and coordinate system.

The heat conduction equation and boundary conditions are given as:

$$h^2\left\{\frac{\partial}{\partial\xi}\left(\lambda_\xi(\theta)\frac{\partial\theta}{\partial\xi}\right)+\frac{\partial}{\partial\eta}\left(\lambda_\xi(\theta)\frac{\partial\theta}{\partial\eta}\right)\right\}+\frac{\partial}{\partial z}\left(\lambda_z(\theta)\frac{\partial\theta}{\partial z}\right)+q(\xi,\eta,z,t)=c_v(\theta)\;\rho\;\frac{\partial\theta}{\partial t} \tag{1}$$

In which $\theta(\xi,\eta,z,t)$ is the temperature of the plate at the point (ξ,η,z) at time $t,\lambda_\xi(\theta)$ and $\lambda_z(\theta)$ are the coefficient of thermal conductivity along the respective directions, $c_v(\theta)$ are the three-dimensional heat capacity and $q(\xi,\eta,z,t)$ represents an energy generation term. The heat generation term is assumed in the form

$$q(\xi,\eta,z,t)=Q\delta(\xi-a_0)\delta(\eta-2\pi)\,\delta(z-\ell_0) \tag{2}$$

In which Q characterizes the stream of heat, $\delta(\;)$ is the Dirac delta function in which $\xi\neq a_0, a_0\in[0,a]$ and $z\neq\ell_0, z\in[0,\ell]$. The temperature distribution in the elliptical plate is obtained as a solution of the Eq. (1) with the following initial and boundary conditions

$$\theta(\xi,\eta,z,0)=\theta_0(\xi,\eta,z) \tag{3}$$

$$\theta(a,\eta,z,t)=0 \tag{4}$$

$$\lambda_z(\theta)\frac{\partial\theta}{\partial z}(\xi,\eta,0,t)=\alpha_0[\theta(\xi,\eta,0,t)-\theta_0] \tag{5}$$

$$\lambda_z(\theta)\frac{\partial\theta}{\partial z}(\xi,\eta,\ell,t)=-\alpha_0[\theta(\xi,\eta,\ell,t)-\theta_0] \tag{6}$$

In which α_0 is the heat transfer coefficient, θ_0 is the known temperature of the surrounding medium of the plate. The most general form of differential equation for normal deflection $\omega\;(\xi,\,\eta,t)$ of the plate is given as:

$$D\nabla^4\omega=-\frac{\nabla^2 M_\theta}{1-\upsilon} \tag{7}$$

with initial and simply supported boundary conditions as:

$$[\omega]_{t=0}=f_1(\xi,\eta),\;[d\omega/dt]_{t=0}=g_1(\xi,\eta) \tag{8}$$

In which ∇^2 denotes the two-dimensional Laplacian operator in $(\xi,\,\eta),\upsilon$ denotes the Poisson's ratio and D is the flexural rigidity of the plate given as:

$$D = \int_0^{\ell} \frac{E(\theta)}{1-\upsilon^2} z^2 \, dz \tag{9}$$

and $M_\theta(\xi, \eta, t)$ is bending of the plate due to change of temperature is given as:

$$M_\theta = \int_0^{\ell} \frac{\alpha(\theta)E(\theta)}{1-\upsilon} z\,\theta \, dz \tag{10}$$

In which $\alpha(\theta)$ and $E(\theta)$ denoting coefficient of linear thermal expansion and Young's Modulus of the material of the plate respectively. The thermally induced resultant forces $N_\theta(\xi, \eta, t)$ can be defined as:

$$N_\theta = \int_0^{\ell} \alpha(\theta) E(\theta) \theta \, dz \tag{11}$$

The maximum normal stresses acting on those sections are parallel to ξz or ηz planes. Furthermore, the thermal stress components can be determined using small deflection and resultant moment as:

$$\begin{aligned}
(\sigma_{\xi\xi})_{\max} &= \frac{6}{\ell^2}\left\{ D h^2 \left[\left(\frac{\partial^2 \omega}{\partial \xi^2} + \upsilon \frac{\partial^2 \omega}{\partial \eta^2} \right) - \frac{(1-\upsilon)\sinh 2\xi}{(\cosh 2\xi - \cos 2\eta)} \frac{\partial \omega}{\partial \xi} + \frac{(1-\upsilon)\sin 2\eta}{(\cosh 2\xi - \cos 2\eta)} \frac{\partial \omega}{\partial \eta} \right] + \frac{M_\theta}{1-\upsilon} \right\} \\
(\sigma_{\eta\eta})_{\max} &= \frac{6}{\ell^2}\left\{ D h^2 \left[\left(\upsilon \frac{\partial^2 \omega}{\partial \xi^2} + \frac{\partial^2 \omega}{\partial \eta^2} \right) + \frac{(1-\upsilon)\sinh 2\xi}{(\cosh 2\xi - \cos 2\eta)} \frac{\partial \omega}{\partial \xi} - \frac{(1-\upsilon)\sin 2\eta}{(\cosh 2\xi - \cos 2\eta)} \frac{\partial \omega}{\partial \eta} \right] + \frac{M_\theta}{1-\upsilon} \right\} \\
(\sigma_{\xi\eta})_{\max} &= \frac{6}{\ell^2}\left\{ D h^2 \left[\frac{\partial \omega}{\partial \xi} \sin 2\eta + \frac{\partial \omega}{\partial \eta} \sinh 2\xi - \frac{\partial^2 \omega}{\partial \xi \partial \eta} (\cosh 2\xi - \cos 2\eta) \right] \right\}
\end{aligned} \tag{12}$$

In order to complete the formulation of the problem, it is necessary to introduce suitable boundary conditions. The plate edge, $\xi = a$ is here assumed to be simply supported, that is

$$[\omega]_{\xi=a} = 0 \tag{13}$$

The Eqs. (1) to (13) constitute the mathematical formulation of the problem under consideration.

3 SOLUTION TO THE PROBLEM

The change of variable is assumed as $\theta_c = \theta - \theta_0$ for mathematical simplification. Similarly with the assumption that the coefficient of thermal diffusivity as constant, the coefficient of thermal conductivity in the form $\lambda_\xi(\theta) = \lambda_o \lambda_\xi^*(\theta)$, $\lambda_z(\theta) = \lambda_o \lambda_z^*(\theta)$, and the heat capacity per unit per unit volume of the material of the body in the form $c_v(\theta) = c_o c_v^*(\theta)$. Here for convenience, we consider the mechanical material comprised of two factors. The first one with subscript naught is a dimensional invariant and second one with a superscript asterisk is a dimensionless function of the dimensionless thermosensitive variant. We further assume that the temperature dependence of the material of thermal conductivities $\lambda_\xi(\theta)$ and $\lambda_z(\theta)$, and heat capacity $c_v(\theta)$ are of same characteristic i.e. $\lambda_z(\theta)/\lambda_\xi(\theta) = k_z \approx$ constant and $c_v(\theta)/\lambda_\xi(\theta) = a \approx$ constant, it is possible to partially linearize the Eqs. (3)-(6) by introducing the Kirchhoff variable

$$\Theta = \frac{1}{\lambda_0} \int_{\theta_0}^{\theta} \lambda_\xi(\zeta) d\zeta \tag{14}$$

After transformation, we obtain the following problem in the variable Θ as:

$$h^2\left\{\frac{\partial^2\Theta}{\partial\xi^2}+\frac{\partial^2\Theta}{\partial\eta^2}\right\}+k_z\frac{\partial^2\Theta}{\partial z^2}+\frac{Q(\xi,\eta,z,t)}{\lambda_0}=\frac{1}{\kappa}\frac{\partial\Theta}{\partial t} \tag{15}$$

$$\Theta(\xi,\eta,z,0)=\Theta_0(\xi,\eta,z) \tag{16}$$

$$\Theta(a,\eta,z,t)=0 \tag{17}$$

$$\frac{\partial\Theta}{\partial z}(\xi,\eta,0,t)-h_1\,\Theta_c(\xi,\eta,0,t)=0 \tag{18}$$

$$\frac{\partial\Theta}{\partial z}(\xi,\eta,\ell,t)+h_2\,\Theta_c(\xi,\eta,\ell,t)=0 \tag{19}$$

Here $h_i=\alpha_i/\lambda_{0},_i\,(i=1,2),\Theta_c$ is the temperature determined in terms of the Kirchhoff variable from relations (14). To complete the linearization of the conditions of convective heat transfer (18)-(19), we carry out the change of variable

$$\Theta_c(\xi,\eta,z,t)=(1+\upsilon)\Theta(\xi,\eta,z,t) \tag{20}$$

where υ is an unknown parameter and when $\upsilon=0$, condition (18) coincides with coincides (18) at $z=0$ and (19) at $z=\ell$. Thus Eqs. (18) and (19) can be represented as:

$$\frac{\partial\Theta}{\partial z}(\xi,\eta,0,t)-h_1\,\Theta(\xi,\eta,0,t)=0 \tag{21}$$

$$\frac{\partial\Theta}{\partial z}(\xi,\eta,\ell,t)+h_2\,\Theta(\xi,\eta,\ell,t)=0 \tag{22}$$

In order to solve the fundamental Eq. (15), we firstly introduce the extended integral transform of Gupta [29] of order *n* and *m* over the variable ξ and η as:

$$\begin{Bmatrix}\bar f(\ q_{2n,m})\\ \bar f(-q_{2n,m})\end{Bmatrix}=\int_0^a\int_0^{2\pi}(\cosh 2\xi-\cos 2\eta)\begin{Bmatrix}Ce_{2n}(\xi,\ \ q_{2n,m})\\ Ce_{2n}(\xi,-q_{2n,m})\end{Bmatrix}\times\begin{Bmatrix}ce_{2n}(\eta,\ \ q_{2n,m})\\ ce_{2n}(\eta,-q_{2n,m})\end{Bmatrix}f(\xi,\eta)\,d\xi\,d\eta \tag{23}$$

Inversion theorem of (23) at any point within the range as:

$$f(\xi,\eta)=\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\begin{Bmatrix}\bar f(\ q_{2n,m})\\ \bar f(-q_{2n,m})\end{Bmatrix}\begin{Bmatrix}Ce_{2n}(\xi,\ \ q_{2n,m})\\ Ce_{2n}(\xi,-q_{2n,m})\end{Bmatrix}\times\begin{Bmatrix}ce_{2n}(\eta,\ \ q_{2n,m})\\ ce_{2n}(\eta,-q_{2n,m})\end{Bmatrix}/\,C_{2n,m} \tag{24}$$

In which $\pm q_{2n,m}$ is the root of the transcendental equation $Ce_{2n}(a,\pm q)=0,\ \ ce_{2n}(\eta,\pm q)=\sum_{r=0}^{\infty}\begin{Bmatrix}A\\B\end{Bmatrix}_{2r}^{(2n)}\cos 2r\eta$ is a Mathieu function, $Ce_{2n}(\xi,\pm q)=\sum_{r=0}^{\infty}\begin{Bmatrix}A\\B\end{Bmatrix}_{2r}^{(2n)}\cosh 2r\xi$ is a modified Mathieu function, $q=k^2=\lambda c^2/4,$

$$C_{2n,m}=\pi\int_0^a(\cosh 2\xi-\Theta_{2n,m})Ce_{2n}^2(\xi,\pm q_{_{2n,m}})\,d\xi \tag{25}$$

and

$$\Theta_{2n,m} = \frac{1}{\pi}\int_0^{2\pi} \cos 2\eta\, ce_{2n}^2(\eta, \pm q_{2n,m})\, d\eta$$
$$= \begin{Bmatrix} A \\ B \end{Bmatrix}_0^{(2n)} \begin{Bmatrix} A \\ B \end{Bmatrix}_2^{(2n)} + \sum_{r=0}^{\infty} \begin{Bmatrix} A \\ B \end{Bmatrix}_{2r}^{(2n)} \begin{Bmatrix} A \\ B \end{Bmatrix}_{2r+2}^{(2n)} \tag{26}$$

In these series A's and B's being the functions of q. The kernel of above transform are given in elliptical function, and it removes the two-dimensional Laplacian operator in (ξ, η) as shown in (15) for the boundary conditions of the third kind as illustrated in Eq. (18) and (19). Applying integral transformation (23) for the variables (ξ, η) and using (17) we obtain

$$k_z \frac{\partial^2 \overline{\Theta}}{\partial z^2} - \alpha_{2n,m}^2 \overline{\Theta} + \frac{\overline{Q}(q_{2n,m}, z, t)}{\lambda_0} = \frac{1}{\kappa}\frac{\partial \overline{\Theta}}{\partial t} \tag{27}$$

$$\frac{\partial \overline{\Theta}}{\partial z}(q_{2n,m}, 0, t) - h_1 \overline{\Theta}(q_{2n,m}, 0, t) = 0 \tag{28}$$

$$\frac{\partial \overline{\Theta}}{\partial z}(q_{2n,m}, \ell, t) + h_2 \overline{\Theta}(q_{2n,m}, \ell, t) = 0 \tag{29}$$

where $\alpha_{2n,m}^2 = 4q_{n,m}/c^2$. We introduce another finite integral transform formula for the axial variable that is derived from results given in [30]

$$\overline{\overline{\Theta}}(q_{2n,m}, \beta_k, t) = \int_0^{\ell} \overline{\Theta}(q_{2n,m}, z, t)[\sin(\beta_k z) - \beta_k \sin(\beta_k z)/h_3]dz \tag{30}$$

$$\overline{\Theta}(q_{2n,m}, z, t) = \sum_k \overline{\overline{\Theta}}(q_{2n,m}, \beta_k, t)[\sin(\beta_k z) - \beta_k \sin(\beta_k z)/h_3]/N_k \tag{31}$$

In which the β_k is the set of positive roots of

$$(\beta_k^2 - h_1 h_2)\sin(\beta_k \ell) - \beta_k (h_1 + h_2)\cos(\beta_k \ell) = 0 \tag{32}$$

and

$$N_k = \frac{(\beta_k^2 + h_1^2)[\ell(\beta_k^2 + h_2^2) + h_2] + h_1(\beta_k^2 + h_2^2)}{\beta_k^2(\beta_k^2 + h_2^2)} \tag{33}$$

$$T\left\{\frac{\partial^2 \Theta}{\partial z^2}\right\} = \left(\frac{\partial \Theta}{\partial z} - h_1 \Theta\right)\Bigg|_{z=0} + \frac{\beta_k}{h_2}\left[\sin(\beta_k \ell) - \frac{h_1}{\beta_\ell}\cos(\beta_k \ell)\right]\left(\frac{\partial \overline{\Theta}}{\partial z} + h_2 \Theta\right)\Bigg|_{z=\ell} - \beta_k^2 \overline{\Theta} \tag{34}$$

On applying (30) on Eq. (27) satisfying boundary conditions (28) and (29), one obtains

$$\frac{d\overline{\overline{\Theta}}}{dt} + \kappa(\alpha_{2n,m}^2 + k_z \beta_k^2)\overline{\overline{\Theta}} = \frac{\kappa \overline{\overline{Q}}(q_{2n,m}, \beta_k, t)}{\lambda_0} \tag{35}$$

which gives

$$\bar{\bar{\Theta}}(q_{2n,m},\beta_k,t)=\bar{\bar{\Theta}}_0\exp[-\kappa\Lambda t]+\frac{\kappa}{\lambda_0}\int_0^t\exp[-\kappa\Lambda t]\,\bar{\bar{Q}}(q_{2n,m},\beta_k,t-\tau)d\tau \tag{36}$$

where $\Lambda=\alpha_{2n,m}^2+k_z\ \beta_k^2$.

Using inversion theorem defined in Eq. (31) and then by (24), we get the solution as:

$$\begin{aligned}\Theta(\xi,\eta,z,t)=\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\Bigg\{\sum_{k=1}^{\infty}\Bigg[\Bigg(\bar{\bar{\Theta}}_0\exp[-\kappa\Lambda t]+\frac{\kappa}{\lambda_0}\int_0^t(\exp[-\kappa\Lambda t]\times\bar{\bar{Q}}(q_{2n,m},\beta_k,t-\tau))d\tau\Bigg)[\sin(\beta_k z)-\beta_k\sin(\beta_k z)/h_3]/N_k\Bigg]\\ \times Ce_{2n}(\xi,q_{2n,m})ce_{2n}(\eta,q_{2n,m})/C_{2n,m}\Bigg\}\end{aligned} \tag{37}$$

To concretize further calculations, as an example, we consider the most extensively used linear dependence of the thermal conductivity coefficient on temperature, namely $\lambda_\xi^*(\theta)=1+\lambda_0(\theta-\theta_0)$, where λ_0 is a given constant. Then, on the basis of Eq. (14), the temperature is expressed in terms of the Kirchhoff variable by the formula

$$\theta=(\Theta+A)^{1/2}-1/(2\lambda_0) \tag{38}$$

which, after the expansion of square root in the Eq. (38), takes the series form as:

$$\theta=A^{1/2}[1+(1/2)(\Theta/A)-(1/8)(\Theta/A)^2+(1/16)(\Theta/A)^3+...]-1/(2\lambda_0) \tag{39}$$

In which $A=[\theta_0+1/(2\lambda_0)]/\lambda_0$. For mathematical simplification, we carry out the computation of the temperature θ by ignoring higher variant of Eq. (39) and verify that the both conditions (28) and (29) hold; Thus from Eqs. (37) and (39), we obtain final temperature equation $\theta(\xi,\eta,z,t)$ as:

$$\begin{aligned}\theta=\frac{(\lambda_0\Omega-1)}{2\lambda_0\Omega}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\Bigg\{\sum_{k=1}^{\infty}\Bigg[\Bigg(\bar{\bar{\Theta}}_0\exp[-\kappa\Lambda t]+\frac{\kappa}{\lambda_0}\int_0^t(\exp[-\kappa\Lambda t]\times\bar{\bar{Q}}(q_{2n,m},\beta_k,t-\tau))d\tau\Bigg)\times[\sin(\beta_k z)-\beta_k\sin(\beta_k z)/h_3]/N_k\Bigg]\\ \times Ce_{2n}(\xi,q_{2n,m})ce_{2n}(\eta,q_{2n,m})/C_{2n,m}\Bigg\}\end{aligned} \tag{40}$$

In which $\Omega=2A^{1/2}$. Substituting Eq. (40) in Eq. (9), one obtains the thermal moment as:

$$\begin{aligned}M_\theta=&\frac{1}{72\lambda_0(1-\upsilon)}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\Bigg\{\sum_{k=1}^{\infty}\frac{1}{h_3N_kC_{2n,m}}\Bigg[A_1+\frac{(h_3-\beta_k)}{\Omega}\Bigg(\bar{\bar{\Theta}}_0\exp[-\kappa\Lambda t]\\ &+\frac{\kappa}{\lambda_0}\int_0^t(\exp[-\kappa\Lambda t]\,\bar{\bar{Q}}(q_{2n,m},\beta_k,t-\tau))d\tau\Bigg)\Bigg]Ce_{2n}(\xi,q_{2n,m})ce_{2n}(\eta,q_{2n,m})\\ &+\Bigg[A_2+A3\Bigg(\bar{\bar{\Theta}}_0\exp[-\kappa\Lambda t]+\frac{\kappa}{\lambda_0}\int_0^t(\exp[-\kappa\Lambda t]\,\bar{\bar{Q}}(q_{2n,m},\beta_k,t-\tau))d\tau\Bigg)\Bigg]\\ &\times Ce_{2n}(\xi,q_{2n,m})ce_{2n}(\eta,q_{2n,m})-A_4\Bigg[\Bigg(\bar{\bar{\Theta}}_0\exp[-\kappa\Lambda t]+\frac{\kappa}{\lambda_0}\int_0^t(\exp[-\kappa\Lambda t]\\ &\times\bar{\bar{Q}}(q_{2n,m},\beta_k,t-\tau))d\tau\Bigg)Ce_{2n}(\xi,q_{2n,m})ce_{2n}(\eta,q_{2n,m})\Bigg]^2\end{aligned} \tag{41}$$

Substituting Eq. (40) in Eq. (13), one obtains the resultant force as:

$$N_\theta = \frac{1}{12\lambda_0(1-\upsilon)}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\left\{\sum_{k=1}^{\infty}\frac{1}{h_3^3\,\beta_k N_k^3\,C_{2n,m}^3}\left\langle\left[\frac{A_1}{3\ell\beta_k}+\frac{(h_3-\beta_k)}{\Omega}\left(\overline{\overline{\Theta}}_0\exp[-\kappa\Lambda t]\right.\right.\right.\right.$$
$$\left.\left.+\frac{\kappa}{\lambda_0}\int_0^t(\exp[-\kappa\Lambda t]\,\overline{\overline{Q}}(q_{2n,m},\beta_k,t-\tau))d\tau\right)\right]Ce_{2n}(\xi,q_{2n,m})\,ce_{2n}(\eta,q_{2n,m})$$
$$\times 12B_1(1-\cos\ell\beta_k)+\frac{16}{\Omega^2}E_0\alpha_0(h_3-\beta_k)^2\left[\left(\overline{\overline{\Theta}}_0\exp[-\kappa\Lambda t]+\frac{\kappa}{\lambda_0}\int_0^t(\exp[-\kappa\Lambda t]\right.\right. \tag{42}$$
$$\left.\left.\times\overline{\overline{Q}}(q_{2n,m},\beta_k,t-\tau))d\tau\right)Ce_{2n}(\xi,q_{2n,m})\,ce_{2n}(\eta,q_{2n,m})\right]^2(2+\cos\ell\beta_k)$$
$$\times\sin^4\left(\frac{\beta_k\ell}{2}\right)+3B_2\left[\left(\overline{\overline{\Theta}}_0\exp[-\kappa\Lambda t]+\frac{\kappa}{\lambda_0}\int_0^t(\exp[-\kappa\Lambda t]\right.\right.$$
$$\left.\left.\left.\left.\times\overline{\overline{Q}}(q_{2n,m},\beta_k,t-\tau))d\tau\right)Ce_{2n}(\xi,q_{2n,m})\,ce_{2n}(\eta,q_{2n,m})\right](2\ell\beta_k-\sin 2\ell\beta_k)\right\rangle\right\}$$

In which

$$A_1=\frac{36C_{2n,m}^3h_3^2\,\ell^2N_k^2\,\beta_k^2\,(\lambda_0\Omega-1)}{2\lambda_0}\left\{1+\left(\frac{\lambda_0\Omega-1}{2\lambda_0}-\theta_0\right)\left[E_0-\alpha_0+\left(\frac{\lambda_0\Omega-1}{2\lambda_0}-\theta_0\right)E_0\alpha_0\right]\right\}$$
$$A_2=72\,B_1\,(\ell\beta_k\cos\ell\beta_k+\sin\ell\beta_k)$$
$$A_3=9B_2(1+2\ell^2\beta_k^2-\cos 2\ell\beta_k-2\ell\beta_k\sin\ell\beta_k)$$
$$A_4=\frac{6\ell\beta_k}{\Omega^2}E_0\alpha_0(h_3-\lambda_n)^2(9\cos\ell\beta_k-\cos 3\ell\beta_k+27\sin\ell\beta_k-\sin 3\ell\beta_k)$$
$$B_1=C_{2n,m}^2h_3^2N_k^2\left\{1+\frac{3(\lambda_0\Omega-1)^2E_0\alpha_0}{4\lambda_0^2}+\theta_0\,(\alpha_0-E_0+\theta_0E_0\alpha_0)\right.$$
$$\left.-\frac{(\lambda_0\Omega-1)}{\lambda_0}(\alpha_0-E_0+2\theta_0E_0\alpha_0)\right\}$$
$$B_2=\frac{C_{2n,m}h_3\,N_k}{\Omega}(E_0-\alpha_0+3\frac{(\lambda_0\Omega-1)}{2\lambda_0}E_0\alpha_0-2\theta_0E_0\alpha_0)(h_3-\beta_k)$$

Substituting Eq. (41) in Eq. (7), one obtains the expression for thermal deflection as:

$$\omega=-\frac{c^2}{288\lambda_0(1-\upsilon)^2}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\left\{\sum_{\ell=1}^{\infty}\frac{1}{q_{2n,m}\,D\,h_3\,N_k\,C_{2n,m}}\right.$$
$$\times\left\langle\left[A_1+\frac{(h_3-\beta_k)}{\Omega}\left(\overline{\overline{\Theta}}_0\exp[-\kappa\Lambda t]+\frac{\kappa}{\lambda_0}\int_0^t(\exp[-\kappa\Lambda t]\right.\right.\right.$$
$$\left.\left.\times\overline{\overline{Q}}(q_{2n,m},\beta_k,t-\tau))d\tau\right)\right]Ce_{2n}(\xi,q_{2n,m})\,ce_{2n}(\eta,q_{2n,m})$$
$$+\left[(A_2+A3\left(\overline{\overline{\Theta}}_0\exp[-\kappa\Lambda t]+\frac{\kappa}{\lambda_0}\int_0^t(\exp[-\kappa\Lambda t]\;\overline{\overline{Q}}(q_{2n,m},\beta_k,t-\tau))d\tau\right)\right] \tag{43}$$
$$\times Ce_{2n}(\xi,q_{2n,m})\,ce_{2n}(\eta,q_{2n,m})-A_4\left[\left(\overline{\overline{\Theta}}_0\exp[-\kappa\Lambda t]\right.\right.$$
$$\left.+\frac{\kappa}{\lambda_0}\int_0^t(\exp[-\kappa\Lambda t]\;\overline{\overline{Q}}(q_{2n,m},\beta_k,t-\tau))d\tau\right)$$
$$\left.\left.\left.\times\;Ce_{2n}(\xi,q_{2n,m})\,ce_{2n}(\eta,q_{2n,m})\right]^2\right\rangle\right\}$$

The resulting equations of stress components can be obtained by substituting the Eqs. (41) and (43) in Eqs. (12). The equations of stresses are rather lengthy, and consequently the same have been omitted here for the sake of brevity, but have been considered during the graphical discussion using MATHEMATICA software.

4 TRANSITION TO CIRCULAR PLATE

When the elliptical plate tends to a circular plate of radius a, the semi-focal $c \to 0$ and therefore α_m is the roots of the transcendental equation $J_0(\alpha_m) = 0.$
Also $e \to 0\left[as\xi \to \infty\right], \cosh 2\xi\, d\xi \to 2\cosh 2\xi \sinh 2\xi\, d\xi \to 2rdr/\ell^2, \sinh\xi \to \cosh\xi, \mathrm{h}\cosh\xi \to r$ [as $h \to 0$], $\cosh\xi\, d\xi \to r\,dr$, $\mathrm{h}\sinh\xi\, d\xi \to dr.$ Using results from [31]

$$Ce_0(\xi, q_{0,m}) \to p_0' J_0\left(\lambda_m r\right), Ce_0'(\xi, q_{0,m}) \to p_0' J_0'\left(\lambda_m r\right), Ce_0''(\xi, q_{_{0,m}}) \to p_0' J_0''\left(\lambda_m r\right), ce_0(\eta, q_{_m}) \to 1/\sqrt{2},$$
$$A_0^{(0)} \to 1/\sqrt{2}, A_2^{(0)} \to 0, \Theta_{2m} \to 0,\ \lambda_{0,m}^2 = \alpha_{0,m}^2/a^2 = \alpha_m^2/a^2 = \lambda_m^2, p_0' = Ce_0(0, q_{_{0,m}})ce_0(2\pi, q_{_{0,m}})/A_0^{(0)}.$$

Eq. (40) degenerates into temperature distribution into a circular plate

$$\theta = \frac{(\lambda_0\Omega - 1)}{2\lambda_0\Omega}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\left\{\sum_{k=1}^{\infty}\left[\left(\overline{\overline{\Theta}}_0 \exp[-\kappa\Lambda t] + \frac{\kappa}{\lambda_0}\int_0^t (\exp[-\kappa\Lambda t]\times\overline{\overline{Q}}(q_{2n,m}, \beta_k, t-\tau))d\tau\right)\right.\right.$$
$$\left.\left.[\sin(\beta_k z) - \beta_k \sin(\beta_k z)/h_3]/N_k\right]\times Ce_{2n}\left(\xi, q_{2n,m}\right)ce_{2n}\left(\eta, q_{2n,m}\right)/C_{2n,m}\right\} \tag{44}$$

In which $C_m = (\pi/\ell^2)\int_0^a 2r[p_0' J_0\left(\lambda_m r\right)]^2\, dr$. The aforementioned degenerated results agrees with the results [32].

5 NUMERICAL RESULTS DISCUSSION AND REMARKS

For the sake of simplicity of calculation, we introduce the following dimensionless values

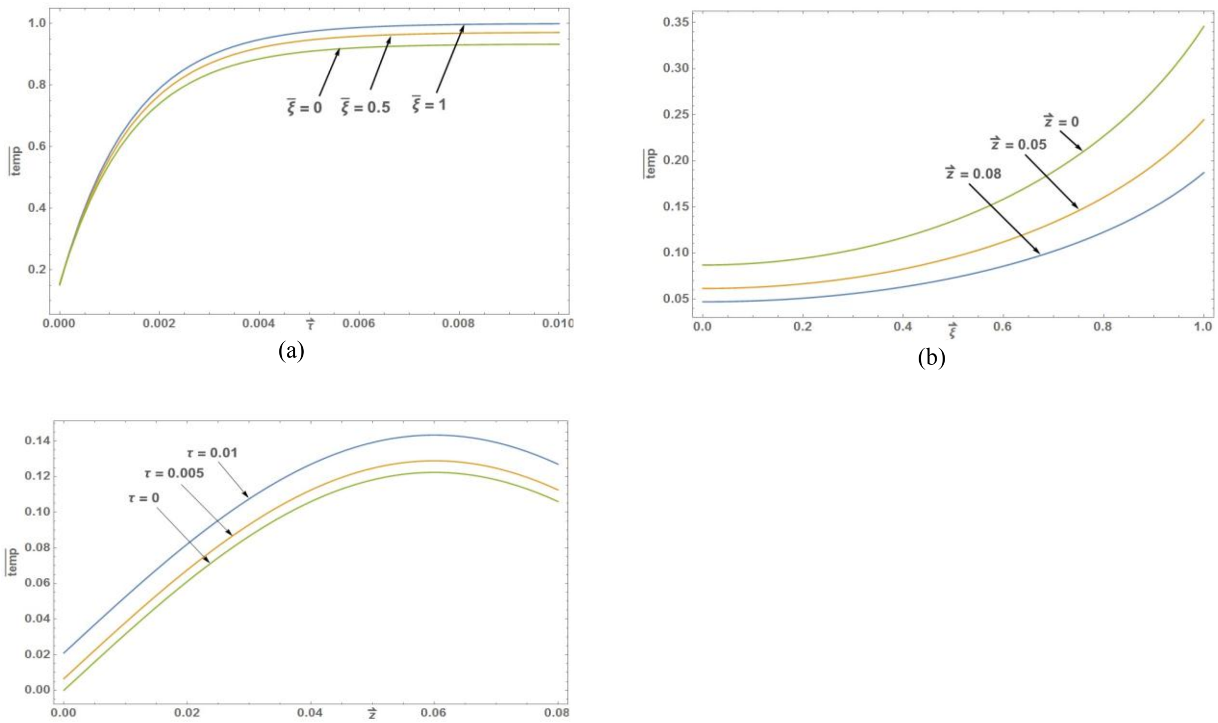
$$\left.\begin{aligned}&\bar{\xi} = \xi/a,\ \bar{z} = z/a,\ e = c/a,\ \tau = \kappa t/a^2,\\&\bar{\theta} = \theta/\theta_0, \bar{\omega} = \omega/\alpha\theta_0\xi_0, \bar{\sigma}_{ij} = \sigma_{ij}/E\alpha\theta_0\ \ (i,j = \xi,\eta)\end{aligned}\right\} \tag{45}$$

Substituting the value of Eq. (45) in temperature Eq. (40), deflection Eq. (43) and components of stresses, we obtained the expressions for the temperature, deflection and thermal stresses respectively for our numerical discussion. The numerical computations have been carried out for an elliptical plate from steel C12 with physical parameter as $\xi_0 = 1m, \ell = 0.08m$ and reference temperature as $273K$. The thermo-mechanical properties take the form as :

- Modulus of elasticity $E(\bar{\theta}) = 2(1+\upsilon)G(\bar{\theta})$,
- Bulk modulus $G(\bar{\theta}) = 0.8265\times 10^{11}(1 - 0.243\bar{\theta})[\mathrm{Pa}]$,
- Poisson's ratio $\upsilon = 0.30$,
- Thermal expansion coefficient $\alpha(\bar{\theta}) = 10.42\times 10^{-6}(1 + 0.786\bar{\theta})[/K]$,
- Thermal conductivity $G(\bar{\theta}) = 47.7\ (1 - 0.366\bar{\theta})[W/mK]$.

The $q_{2n,m} =$ 0.0986, 0.3947, 0.8882, 1.5791, 2.4674, 3.5530, 4.8361, 6.3165, 7.9943, 9.8696, 11.9422, 14.2122, 16.6796, 19.3444, 22.2066, 25.2661, 28.5231, 31.9775, 35.6292, 39.4784 are the positive & real roots of the transcendental equation $Ce_{2n}(a,q) = 0$. In order to examine the influence of heating on the plate, the numerical calculations were performed for all the variables and numerical calculations are depicted in the following figures with the help of MATHEMATICA software. Figs. 2–4 illustrate the temperature distribution, thermally induced deflection and its associated bending stresses.

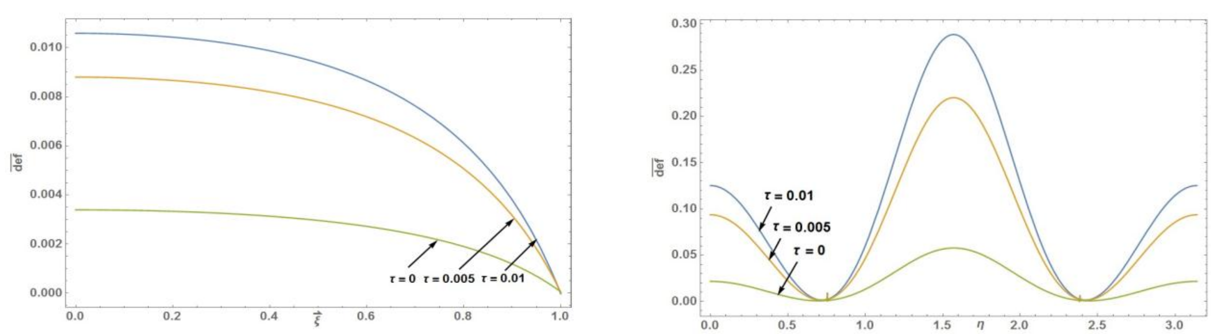
Fig. 2(a) depicts the temperature distribution along time series for various values of $\bar{\xi}$. The temperature gradually increases with time and attains maximum after certain interval then stabilise itself due to constant tensile force. Fig. 2(b) illustrates the temperature profile along radial direction gradually towards the end of the plate. This temperature increment may be due to the constant supply of sectional heat. Initially, the temperature starts with a certain value due to the availability of internal heat generation and varies for different points of $\bar{z}$. Fig. 2(c) depicts shows that the temperature distribution along the $\bar{z}$ -direction for different values of known time which maximises its magnitude towards outer edge may be due to the energised heat supply.



(a) (b) (c)

Fig.2
a) Temperature distribution along τ for different values of $\bar{\xi}$. b) Temperature distribution along $\bar{\xi}$ for different values of $\bar{z}$. c) Temperature distribution along $\bar{z}$ for fixed value of τ .

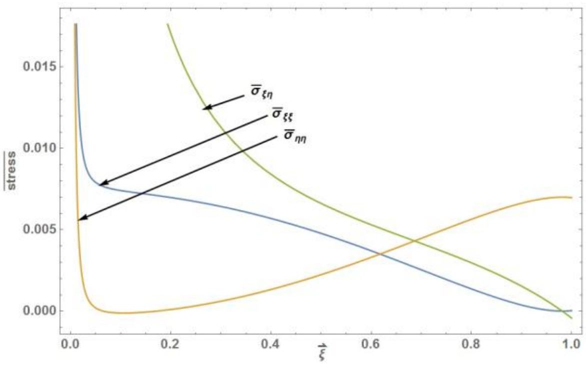
Fig. 3(a) shows the relationship between the deflections along $\bar{\xi}$ -direction for different time. The deflection is gradual increases from the outer edge to the inner core. The deflection is maximum at plate centre and variation of $\bar{\omega}(\bar{\xi},\eta,\tau)$ for various instant, but it was found that it always tends to zero at $\bar{\xi}=1$, thus satisfies the boundary condition (13). Fig. 3(b) depicts thermally induced deflection along the angular direction for different time. The deflection profile is sinusoidal in nature due to the periodicity.



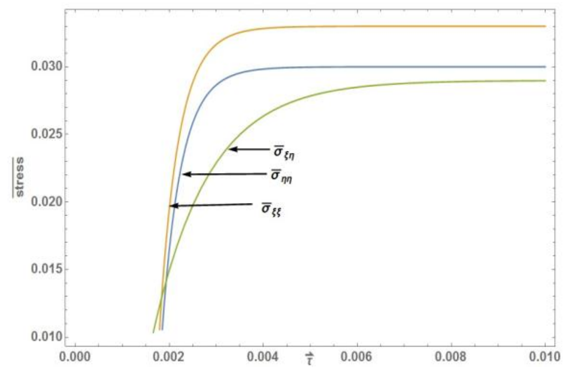
(a) (b)

Fig.3
a) Thermally induced deflection along $\bar{\xi}$ for different values of τ . b) Thermally induced deflection along η for different values of τ .

Fig. 4(a) depicts the radial, tangential and shear stress along the radial direction of the fixed position at a particular time. The stress is maximum in the inner core due to accumulation heat energy. Initially, the stress is low at the outer edge, and it may be due to the internal heat source. The stresses increase gradually towards the inner core due to the applied sectional heat. With time the stresses increase gradually over time as shown in Fig. 4(b). The stresses attain certain maxima and stabilise itself after certain duration for fixed space position.



(a)



(b)

Fig.4
a) Dimensionless stresses along $\bar{\xi}$ for fixed values of $(\bar{\xi},\eta,\bar{z})$. b) Dimensionless stresses along τ for fixed values of position.

6 CONCLUSIONS

In this article, we have described the theoretical treatment of stress analysis distribution for the thermosensitive elliptic plate with simply supported edge. The temperature distribution and the deflection in the form of ordinary and modified Mathieu functions are used to determine thermal stresses by proposed operational methods. The analytical technique proposed here is relatively simple and widely applicable compared with methods proposed by other researchers. The mentioned results were obtained while carrying out during our research are illustrated as follows:

- The advantage of this approach is its generality and its mathematical power to handle different types of mechanical and thermal boundary conditions during small deflection under thermal loading.
- The maximum tensile stress shifting in the inner surface due to maximum expansion at the inner part of the plate and its absolute value increases towards inner core with radius may be due to heat, stress, concentration or available internal heat sources under known temperature field.
- Finally, the maximum tensile stress occurs in the circular core on the major axis is compared to elliptical central part indicates the distribution of weak heating. It might be due to insufficient penetration of heat through the elliptical inner surface.
- The aforementioned thermo-sensitivity concept can be very well applicable in the field of hybridising metallurgy, ceramics, solid state physics and chemistry. They have a lot of application in the biomedical field and so forth.

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